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DRIVE NETWORK TO A DESIRED ORBIT BY PINNING CONTROL

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The primary objective of the present paper is to develop an approach for analyzing pinning synchronization stability in a complex delayed dynamical network with directed coupling. Some simple yet generic criteria for pinning such coupled network are derived analytically. Compared with some existing works, the primary contribution is that the synchronization manifold could be chosen as a weighted average of all the nodes states in the network for the sake of practical control tactics, which displays the different influences and contributions of the various nodes in synchronization seeking processes of the dynamical network. Furthermore, it is shown that in order to drive a complex network to a desired synchronization state, the coupling strength should vary according to the controller. In addition, the theoretical results about the time-invariant network is extended to the time-varying network, and the result on synchronization problem can also be extended to the consensus problem of networked multi-agent systems. Subsequently, the theoretic results are illustrated by a typical scale-free (SF) neuronal network. Numerical simulations with three kinds of the homogenous solutions, including an equilibrium point, a periodic orbit, and a chaotic attractor, are finally given to demonstrate the effectiveness of the proposed control methodology.

Keywords: complex dynamical network, pinning control, directed coupling, time delay, DCN oscillator

Classification: 74H65, 70K40

1. INTRODUCTION

During the last few decades, the synchronization of complex dynamical networks has been the subject of extensive research from various fields such as biology or ecology, physics, and control engineering. Especially, special interest has been focused on synchronization and control in large-scale complex dynamical networks composing of coupled chaotic oscillators with variety networks topologies such as small-world, scale-free features and random graph, etc., [1, 2, 18, 27, 28]. Meanwhile, a great many synchronization criteria (conditions) have also been addressed for different networks coupling such as time delays, time varying and impulsive characters, etc, (see [9, 11, 15, 16, 19, 20, 21, 38] and relevant references therein).

As is well known now, the real-world complex networks generally have a large number of nodes, which lead to the fact that it is usually difficult to control such complex network

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by adding the controllers to all nodes. To reduce the number of controlled nodes, some feedback injections have been added to a fraction of network nodes, which is known as pinning control. As a result, some researchers have focused on the investigations different pinning control strategies for various complex dynamical networks [5, 8, 10, 14, 22, 23, 24, 25, 26, 29, 30, 31, 32, 33, 34, 35, 36, 39]. For example, Wang and Chen [29] revealed that, it is much more effective to pin some most-highly connected nodes than to pin randomly selected nodes since the extremely inhomogeneous connectivity distribution of scale-free networks. Li et al. [10] proposed the concept of virtual control to display that the control actions applied to the pinned nodes can be propagated to the rest of network nodes through the couplings in the network, and eventually, result in the synchronization of the whole network. While Chen et al. [5] proved that, if the coupling strength is large enough, even one single pinning controller is able to control a large network. Furthermore, Lu [14] introduced an adaptive dynamical network by integrating the complex network model and adaptive technique, and discussed the synchronization of such network. Sorrentino et al. [25] presented a scheme for the numerical exploration of pinning-controllability by the master-stability function. In addition, Zhou et al. [39] addressed an adaptive pinning control law with the property that a sufficient number of controlled nodes are selected arbitrarily. And later on, Yu et al. [36] concerned with pinning performance of complex dynamical network. Other research works about pinning control of complex networks can be seen in [8, 12, 17, 22, 23, 24, 26, 30, 31, 32, 33, 34, 35, 40, 41] and many references cited therein.

However, among the above control approaches, the controlled synchronization manifold is only taken into account to be a special solution of the homogenous systems [1, 5, 8, 10, 18, 19, 20, 22, 23, 24, 27, 29, 31, 30, 34, 35, 33, 32, 36, 38, 39, 40, 41]. To solve these problems, recently we developed a new and effective control approach to synchronize an arbitrary given complex delayed dynamical network with directionally coupling to a desired synchronization orbit, which could be chosen as a weighted average of all the nodes states in the network in the aim of practical control tactics [42]. In view of the reality, here the proposed control tactics considers the different influences and contributions of all the nodes in synchronization seeking processes of the complex delayed dynamical network. On the other hand, most previous studies focused mainly on pinning synchronization for time-invariant dynamical networks, i. e., the coupling configuration matrix is invariable. However, time-varying networks exist in a wide variety of natural and synthetic system [20], such as telecom communication networks, interpersonal relationship networks, and social economic networks. Hence, it is necessary to study pinning synchronization issue in time-varying complex dynamical networks. In addition, a topic closely related to synchronization is consensus [6, 7], and it is clear that the consensus is a special case of synchronization. Thus, all the results concerning synchronization can be applied to consensus problem. This paper is an attempt toward these goals. Therefore, these studies are especially important to understand dynamical behaviors and practical architectures of more realistic complex networks.

Motivated by the aforementioned comments, the main contribution of this work is to propose an approach for analyzing pinning stability in a complex dynamical complex delayed dynamical network comprised of linearly coupled identical oscillator systems. We intend to design an appropriate feedback controller such that the states of the dynamical network exponentially synchronize to a desired synchronization orbit, which is presented in terms of a weighted average of all the nodes states in the network. In addition, it is found that in order to make all of the nodes in the dynamical network synchronize to a desired synchronization orbit, the choice of the coupling strength changes with the choice of the controller. And some sufficiency conditions for synchronization stability of such directed network are presented. Furthermore, the theoretic results are extended to the time-varying complex networks, as well as the consensus of complex networks. Finally, simulations of a typical scale-free (SF) coupled neuronal network are given to verify the correctness of the theoretical results.

The organization of the paper is as follows. In Section 2, some mathematical preliminaries relevant to complex delayed dynamical network are first introduced, and then the problem formulations for pinning synchronization seeking in a general mode of complex dynamical network with directionally coupling are presented. In Section 3, some simple yet generic sufficient conditions for pinning synchronization are derived analytically based on the stability theory of dynamical system. These criteria can be easily used to synchronize an arbitrary given delayed dynamical network to a desired synchronization orbit. In Section 4, a typical scale-free (SF) coupled network composing of delayed cellular neuron (DCN) oscillators is used as an example to illustrate the developed pinning control strategy, and numerical simulations are also done to demonstrate the effectiveness of the proposed control techniques. Finally, some concluding remarks are given in Section 5.

2. PRELIMINARIES AND FORMULATIONS

Throughout this paper, the following notations and definitions will be used.

Let $R = (-\infty, +\infty)$ be the set of real numbers, $R^+ = [0, +\infty)$ be the set of nonnegative real numbers, and $Z^+ = \{1, 2, \ldots\}$ be the set of positive integer numbers. For the vector $u \in R^n$, $u^\top$ denotes its transpose. The norm of the vector u is defined as $\|u\| = (u^\top u)^{1/2}$. $R^{n\times n}$ stands for the set of $n\times n$ real matrices. $I_n \in R^{n\times n}$ is the identity matrix of order n. For the matrices $A \in R^{n\times n}$, $A^s = (A + A^\top)/2$ is its symmetric part, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote its minimum and maximum eigenvalue, respectively. For the symmetric matrices $A \in R^{n\times n}$, $A > 0$ ($A \geq 0$) means that A is positive definite (semi-positive definite). The spectral norm of A is defined as $\|A\| = [\lambda_{\max}(A^\top A)]^{1/2}$. Matrix dimensions, if not explicitly stated, are assumed to be compatible for algebraic operations. For $A = (a_{ij}) \in R^{n\times m}$, $B = (b_{ij}) \in R^{p\times q}$, the Kronecker product of A and B is defined as

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix}.$$

For the later use, we need the following lemmas.

Lemma 2.1. (See Chen, Liu, and Lu [5]) If $A = (a_{ij}) \in R^{n\times n}$ is an irreducible matrix with $\mathrm{Rank}(A) = n - 1$ and satisfying $a_{ij} = a_{ji}$, if $i \neq j$, and $\sum_{j=1}^{n} a_{ij} = 0$, for

$i = 1, 2, \ldots, n$. Then, all eigenvalues of the matrix

$$\tilde{A} = \begin{bmatrix} a_{11} - \varepsilon & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

are negative for any positive constant ε.

Lemma 2.2. (See Zhou and Chen [37]) Let $v(t) > 0$ for $t \in R, \tau \in [0, +\infty)$ and $t_0 \in R$. Suppose that

$$\dot{v}(t) \leq -av(t) + b\Big(\sup_{t-\tau \leq s \leq t} v(s))\Big),$$

for $t > t_0$. If $a > b > 0$, then there exist constants $\gamma > 0$ and $k > 0$, such that

$$v(t) \leq ke^{-\gamma(t-t_0)}, \quad \text{for} \quad t > t_0.$$

Now we consider a general model of time-varying complex dynamical network consisting of N linearly coupled identical nodes, where each node of the network is an n-dimensional nonautonomous dynamical system with time-varying delays. The state equations of the whole network are described by the following differential equations:

$$\dot{x}_i(t) = f(t, x_i(t), x_i(t - \tau(t))) + c\sum_{j=1}^{N} b_{ij}(t)\Gamma(t)x_j(t), \quad i = 1, 2, \ldots, N, \tag{1}$$

in which $x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))^\top \in R^n$ are the state variables of the ith dynamical node, $f : R \times R^n \times R^n \to R^n$ is continuously vector-valued function governing the dynamics of isolated nodes, the time delay $\tau(t)$ may be unknown but is bounded by a known constant, i.e., $0 \leq \tau(t) \leq \tau$, the positive constant c is the coupling strength, $\Gamma(t) = (\gamma_{kl}(t))_{n\times n} \in R^{n\times n}$ is the inner-connecting matrix of the network at time t, $B(t) = (b_{ij}(t))_{N\times N}$ is the coupling configuration matrix representing the coupling strength and the topological structure of the network at time t, in which $b_{ij}(t)$ is defined as follows: If there is a connection from node i to node j ($j \neq i$) at time t, then $b_{ij}(t) \neq 0$; otherwise, $b_{ij}(t) = 0$ ($j \neq i$). Without loss of generality, we further assume that the coupling matrix B possesses the following properties [3, 4, 42]:

$$\sum_{j=1}^{N} b_{ij}(t) = 0, \quad b_{ij}(t) \geq 0 \quad (i \neq j,\, i = 1, 2 \ldots, N). \tag{2}$$

As stated in literature [5, 9, 15], the coupling matrix $B(t)$ can be regarded as the Laplacian matrix of a weighted graph with a directed spanning tree, it has an eigenvalue 0 with multiplicity 1, and all the other eigenvalues have negative real parts at any time t. It should be noticed that, the coupling configuration matrix $B(t)$ is not necessarily a symmetrical or irreducible matrix, and the inner connecting matrix Γ is not assumed to be diagonal or positive definite. Obviously, the network (1) is a generalization of the model discussed in [5, 10, 27, 28]. As usual, the model (1) is also called as the general time-varying complex delayed dynamical network.

Next we consider an isolated dynamical node in the model (1), which is described by the following form of n-dimensional differential equations with time-varying delays [3]:

$$\dot{x}(t) = f(t, x(t), x(t-\tau(t))) = Ax(t) + g(t, x(t), x(t-\tau(t))), \tag{3}$$

in which $x(t) = (x_1(t), \ldots, x_n(t))^\top \in R^n$, $A \in R^{n\times n}$, and the vector-valued function $g(t, x(t), x(t-\tau(t))) = [g_1(t, x(t), x(t-\tau(t))), \ldots, g_n(t, x(t), x(t-\tau(t)))]^\top \in R^n$ satisfies uniform Lipschitz condition with respect to the time t, i. e.,

(A_1) For any $x(t) = (x_1(t), \ldots, x_n(t))^\top \in R^n$, $y(t) = (y_1(t), \ldots, y_n(t))^\top \in R^n$, there exist constants $k_{ij} > 0$, and $l_{ij} > 0$ satisfying

$$\begin{aligned} &\Big|g_i(t, x(t), x(t-\tau(t))) - g_i(t, y(t), y(t-\tau(t)))\Big| \\ \leq\ &\sum_{j=1}^{n} \Big(k_{ij}|x_j(t) - y_j(t)| + l_{ij}|x_j(t-\tau(t)) - y_j(t-\tau(t))|\Big), \quad i = 1, \ldots, n. \end{aligned} \tag{4}$$

Remark 2.3. In general, we are mainly interested in control and synchronization of complex networks composing of coupled dissipative dynamical systems, where the uncoupled dissipative dynamical system may possess an equilibrium point, a periodic orbit, or even a chaotic attractors [29]. Therefore, we always assume that all the solutions of (3) are bounded. And it is not difficult to check that the class of systems in the form of Eqs. (3) – (4) includes almost all the well-known chaotic systems without or with time delays such as the Lorenz system, Chen system, Chua's circuit, Rössler system as well as the delayed Hopfied neural networks, delayed Mackey–Glass system or delayed Ikeda equations and delayed cellular neural networks (CNNs), and so on (see [3, 42] and their references).

Now we shall address pinning synchronization in complex delayed dynamical network (1) by feedback controllers. The primary objective here is to design and implement an appropriate pinning strategy on a small fraction ε $(0 < \varepsilon \ll 1)$ of the nodes in network (1) such that all the states of the controlled network will exponentially synchronize to a desired synchronization state $s(t)$, which is presented in terms of a weighted average of all the states in the network. More concretely, this is to say,

$$\lim_{t\to+\infty} \|x_i(t) - s(t)\| = 0, \qquad s(t) = \sum_{k=1}^{N} \xi_k x_k(t), \quad i = 1, 2 \ldots, N, \tag{5}$$

where $x_i(t)$ $(i = 1, 2, \ldots, N)$ are the solutions of the delayed dynamical network (1), and ξ_k $(k = 1, 2, \ldots, N)$ can be chosen such that $\sum_{k=1}^{N} \xi_k = 1$ and $\xi_k > 0$ for the purpose of practical control strategy [3, 42]. In this case, the controlled network can be described as

$$\dot{x}_i(t) = Ax_i(t) + g(t, x_i(t), x_i(t-\tau(t))) + c\sum_{j=1}^{N} b_{ij}(t)\Gamma(t)x_j(t) + U_i, \quad i = 1, \ldots, N. \tag{6}$$

Remark 2.4. Distinctly, it turns out that the developed control tactics consider both the different influences and contributions of all nodes in synchronization seeking processes. In general, among most previous studies, the controlled synchronization state is only taken into account to be a special solution of the homogenous systems, that is, $\dot{s}(t) = f(s(t))$ [1, 4, 5, 8, 10, 18, 19, 20, 22, 23, 24, 27, 29, 31, 30, 34, 35, 33, 32, 36, 38, 39, 40, 41]. Or consider an average of all the nodes states in the networks, namely, $s(t) = \frac{1}{N}\sum_{k=1}^{N} x_k(t)$ [9, 37]. However, it is well known that, the synchronization of the networks not only depends on the topology of the entire networks, but also heavily depends on the dynamic of each node [3, 42]. In this paper, we present an effective control approach to synchronize an arbitrary given complex delayed dynamical network with directed coupling to a desired synchronization orbit, which could be chosen as a weighted average of all the nodes states in the network in the aim of practical control tactics, that is to say, $s(t) = \sum_{k=1}^{N} \xi_k x_k(t)$ with $\sum_{k=1}^{N} \xi_k = 1$. In addition, the controlled synchronization state $s(t)$ here may be an equilibrium point, a periodic orbit, or a chaotic attractor, among the others [3, 42].

3. PINNING SYNCHRONIZATION

3.1. Time-invariant dynamical network

When $B(t)$, $\Gamma(t)$ are constant matrices, network (1) becomes a time-invariant dynamical network

$$\dot{x}_i(t) = f(t, x_i(t), x_i(t-\tau(t))) + c\sum_{j=1}^{N} b_{ij}\Gamma x_j(t), \quad i = 1, 2, \ldots, N, \tag{7}$$

Without loss of generality, let the nodes $i_1, i_2, \ldots, i_l$ be selected as the pinned nodes and rearrange the order of the nodes in the network (7) such that the first $l = [\varepsilon N]$ nodes are pinned or controlled [29]. Let

$$\begin{cases} U_i = -d_i(x_i(t) - s(t)), & 1 \le i \le l, \\ U_i = 0, & (l+1) \le i \le N, \end{cases} \tag{8}$$

where d_i are constants called the control gains. Substituting (8) into (7), we have

$$\begin{cases} \dot{x}_i(t) = Ax_i(t) + g(t, x_i(t), x_i(t-\tau(t))) + c\sum_{j=1}^{N} b_{ij}\Gamma x_j(t) \\ \qquad\quad -d_i(x_i(t) - s(t)), & 1 \le i \le l, \\ \dot{x}_i(t) = Ax_i(t) + g(t, x_i(t), x_i(t-\tau(t))) + c\sum_{j=1}^{N} b_{ij}\Gamma x_j(t), \\ & (l+1) \le i \le N. \end{cases} \tag{9}$$

By introducing the synchronization error $e_i(t) = x_i(t) - s(t)\,(i = 1, 2, \ldots, N)$, then the error dynamical system is governed as follows:

$$\begin{cases} \dot{e}_i(t) = Ae_i(t) + \Big(g(t, x_i(t), x_i(t-\tau(t))) - g(t, s(t), s(t-\tau(t)))\Big) \\ \qquad\quad +c\sum_{j=1}^{N} b_{ij}\Gamma e_j(t) - d_i e_i(t) + J, & 1 \le i \le l, \\ \dot{e}_i(t) = Ae_i(t) + \Big(g(t, x_i(t), x_i(t-\tau(t))) - g(t, s(t), s(t-\tau(t)))\Big) \\ \qquad\quad +c\sum_{j=1}^{N} b_{ij}\Gamma e_j(t) + J, & (l+1) \le i \le N, \end{cases} \tag{10}$$

where

$$J = g\Big(t, s(t), s(t-\tau(t))\Big) - \sum_{k=1}^{N} \xi_k g(t, x_k(t), x_k(t-\tau(t))) - c\sum_{j=1}^{N}\sum_{k=1}^{N} \xi_k b_{kj} \Gamma e_j(t).$$

We define

$$e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_N(t) \end{bmatrix}, \qquad \tilde{J} = \begin{bmatrix} J \\ J \\ \vdots \\ J \end{bmatrix},$$

and

$$\tilde{g}(t) = \begin{bmatrix} g(t, x_1(t), x_1(t-\tau(t))) - g(t, s(t), s(t-\tau(t))) \\ g(t, x_2(t), x_2(t-\tau(t))) - g(t, s(t), s(t-\tau(t))) \\ \vdots \\ g(t, x_N(t), x_N(t-\tau(t))) - g(t, s(t), s(t-\tau(t))) \end{bmatrix}.$$

Let

$$\mathbf{A} = (I_N \otimes A), \quad \mathbf{B} = (B \otimes \Gamma), \quad \mathbf{I} = (I_N \otimes I_n), \quad \mathbf{D} = (D \otimes I_n),$$

where the matrix $D = \mathrm{diag}(d_1, \ldots, d_l, \underbrace{0, \ldots, 0}_{N-l})$, and the notation $\otimes$ indicates the Kronecker product of both matrices. Thus, the error dynamical system (10) can be given by

$$\dot{e}(t) = \mathbf{A}e(t) + \tilde{g}(t) + c\mathbf{B}e(t) + \tilde{J}. \tag{11}$$

Suppose that $\lambda_{\min}(\Gamma^s) \neq 0$ and $\|\Gamma\| \geq 0$. In this section, let the matrix $\tilde{B}^\Lambda$ be defined as $\tilde{B}^\Lambda \overset{\mathrm{def}}{=} \Lambda^{\frac{1}{2}} \tilde{B} \Lambda^{-\frac{1}{2}} + \Lambda^{-\frac{1}{2}} \tilde{B}^\top \Lambda^{\frac{1}{2}} - \Delta$, where the weight matrix $\Lambda = \mathrm{diag}(\xi_1, \ldots, \xi_N)$, $\Lambda^{\frac{1}{2}} = \mathrm{diag}(\sqrt{\xi_1}, \ldots, \sqrt{\xi_N})$, $\Lambda^{-\frac{1}{2}} = (\Lambda^{\frac{1}{2}})^{-1}$, $\tilde{B} = (\tilde{b}_{ij})_{N\times N}$, $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_N)$ with

$$\tilde{b}_{ij} = \begin{cases} b_{ij}\|\Gamma\|, & i \neq j, \\ b_{ii}\lambda_{\min}(\Gamma^s), & i = j, \end{cases} \qquad \delta_i = \sum_{j=1}^{N} \Big[\sqrt{\frac{\xi_i}{\xi_j}} b_{ij} + \sqrt{\frac{\xi_j}{\xi_i}} b_{ji}\Big].$$

Then the matrix $\tilde{B}^\Lambda$ is a symmetrical irreducible matrix with zero-sum and nonnegative off-diagonal elements. This implies that zero is an eigenvalue of $\tilde{B}^\Lambda$ with multiplicity 1, and all the other eigenvalues of $\tilde{B}^\Lambda$ are strictly negative. Its eigenvalues can be ordered as $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$. Furthermore, the eigenvalues of the matrix $(\tilde{B}^\Lambda - \frac{2}{c}D)$ can be ordered as $0 > \mu_1 \geq \mu_2 \geq \ldots \geq \mu_N$.

Theorem 3.1. Consider the controlled complex delayed dynamical network (9). Assume that, the following condition is satisfied in addition to (A_1),

(A_2) There exists a number $\varepsilon \in [0, 1]$, such that $\alpha + c\mu_1 + \beta < 0$, where

$$\begin{aligned}\alpha &= \lambda_{\max}(2A^s + P) + c\lambda_{\max}(\Delta),\\ \beta &= \max_{1\le r\le n}\textstyle\sum_{s=1}^{n} l_{sr}^{2(1-\varepsilon)},\\ P &= \mathrm{diag}(p_1, p_2, \ldots, p_n) \text{ with } p_r = \textstyle\sum_{s=1}^{n}(k_{rs}^{2\varepsilon} + l_{rs}^{2\varepsilon} + k_{sr}^{2(1-\varepsilon)}).\end{aligned}$$

Then the controlled complex delayed dynamical network (9) is exponentially synchronized to $s(t) = \sum_{k=1}^{N}\xi_k x_k(t)$ with $\sum_{k=1}^{N}\xi_k = 1$ ($\xi_k > 0$).

P r o o f. Let $v(t) = (v_1^\top(t), v_2^\top(t), \ldots, v_N^\top(t))^\top = (\sqrt{\xi_1}e_1^\top(t), \sqrt{\xi_2}e_2^\top(t), \ldots, \sqrt{\xi_N}e_N^\top(t))^\top$, construct a Lyapunov function as

$$\begin{aligned}V(t) &= \frac{1}{2}e^\top(t)(\Lambda\otimes I_n)e(t) = \frac{1}{2}\sum_{i=1}^{N}\xi_i e_i^\top(t)e_i(t)\\ &= \frac{1}{2}\sum_{i=1}^{N}v_i^\top(t)v_i(t) = \frac{1}{2}v^\top(t)(I_N\otimes I_n)v(t). \qquad (12)\end{aligned}$$

Differentiating $V(t)$ with respect to time along the solution of Eqs. (10), we can get

$$\begin{aligned}\dot V(t) &= \sum_{i=1}^{N}\xi_i e_i^\top(t)\dot e_i(t)\\ &= \sum_{i=1}^{N}\xi_i e_i^\top(t)\Big\{Ae_i(t) + \Big(g(t, x_i(t), x_i(t-\tau(t))) - g(t, s(t), s(t-\tau(t)))\Big)\\ &\quad + c\sum_{j=1}^{N}b_{ij}\Gamma e_j(t) + J\Big\} - e^\top(t)\Big((\Lambda D)\otimes I_n\Big)e(t)\\ &\le \sum_{i=1}^{N}\xi_i e_i^\top(t)\Big(\frac{A+A^\top}{2}\Big)e_i(t) + \sum_{i=1}^{N}\sum_{r=1}^{n}\xi_i|e_{ir}(t)|\cdot|g_r(t, x_i(t), x_i(t-\tau(t)))\\ &\quad - g_r(t, s(t), s(t-\tau(t)))| + c\sum_{i=1}^{N}\xi_i\sum_{j=1, j\ne i}^{N}b_{ij}e_i^\top(t)\Gamma e_j(t)\\ &\quad + c\sum_{i=1}^{N}\xi_i b_{ii}e_i^\top(t)\Big(\frac{\Gamma+\Gamma^\top}{2}\Big)e_i(t) - v^\top(t)(D\otimes I_n)v(t).\end{aligned}$$

By employing Condition (A_1), we have

$$\begin{aligned}\dot V(t) &\le \sum_{i=1}^{N}\xi_i e_i^\top(t)A^s e_i(t) + \sum_{i=1}^{N}\sum_{r=1}^{n}\xi_i|e_{ir}(t)|\sum_{s=1}^{n}\Big(k_{rs}|e_{is}(t)| + l_{rs}|e_{is}(t-\tau(t))|\Big)\\ &\quad + c\sum_{i=1}^{N}\sum_{j=1, j\ne i}^{N}\xi_i b_{ij}\|\Gamma\|\cdot\|e_i(t)\|\cdot\|e_j(t)\| + c\sum_{i=1}^{N}\xi_i b_{ii}\lambda_{\min}(\Gamma^s)e_i^\top(t)e_i(t)\\ &\quad - v^\top(t)(D\otimes I_n)v(t)\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{N}\xi_i e_i^\top(t)A^s e_i(t)+\sum_{i=1}^{N}\sum_{r=1}^{n}\sum_{s=1}^{n}\xi_i\Big[(k_{rs}^{\varepsilon}|e_{ir}(t)|)\cdot(k_{rs}^{1-\varepsilon}|e_{is}(t)|)\\
&\quad+(l_{rs}^{\varepsilon}|e_{ir}(t)|)\cdot(l_{rs}^{1-\varepsilon}|e_{is}(t-\tau(t))|)\Big]+c\sum_{i=1}^{N}\xi_i\sum_{j=1}^{N}\tilde{b}_{ij}\|e_i(t)\|\cdot\|e_j(t)\|\\
&\quad-v^\top(t)(D\otimes I_n)v(t).
\end{aligned}
$$

By using the inequality $xy \leq \frac{1}{2}(x^2+y^2)$, we can obtain

$$
\begin{aligned}
\dot{V}(t) &\leq \sum_{i=1}^{N}\xi_i e_i^\top(t)A^s e_i(t)+\frac{1}{2}\sum_{i=1}^{N}\sum_{r=1}^{n}\sum_{s=1}^{n}\xi_i\Big[\Big(k_{rs}^{2\varepsilon}e_{ir}^2(t)+k_{rs}^{2(1-\varepsilon)}e_{is}^2(t)\Big)\\
&\quad+\Big(l_{rs}^{2\varepsilon}e_{ir}^2(t)+l_{rs}^{2(1-\varepsilon)}e_{is}^2(t-\tau(t))\Big)\Big]+c\sum_{i=1}^{N}\sum_{j=1}^{N}\xi_i\tilde{b}_{ij}\|e_i^\top(t)\|\cdot\|e_j(t)\|\\
&\quad-v^\top(t)(D\otimes I_n)v(t)\\
&\leq \sum_{i=1}^{N}\xi_i e_i^\top(t)A^s e_i(t)+\frac{1}{2}\sum_{i=1}^{N}\sum_{r=1}^{n}\sum_{s=1}^{n}\xi_i k_{rs}^{2\varepsilon}e_{ir}^2(t)+\frac{1}{2}\sum_{i=1}^{N}\sum_{r=1}^{n}\sum_{s=1}^{n}\xi_i k_{rs}^{2(1-\varepsilon)}e_{is}^2(t)\\
&\quad+\frac{1}{2}\sum_{i=1}^{N}\sum_{r=1}^{n}\sum_{s=1}^{n}\xi_i l_{rs}^{2\varepsilon}e_{ir}^2(t)+\frac{1}{2}\sum_{i=1}^{N}\sum_{r=1}^{n}\sum_{s=1}^{n}\xi_i l_{rs}^{2(1-\varepsilon)}e_{is}^2(t-\tau(t))\\
&\quad+ce^\top(t)\Big((\Lambda\tilde{B})\otimes I_n\Big)e(t)-v^\top(t)(D\otimes I_n)v(t)\\
&\leq \sum_{i=1}^{N}\xi_i e_i^\top(t)A^s e_i(t)+\frac{1}{2}\sum_{i=1}^{N}\sum_{r=1}^{n}\sum_{s=1}^{n}\xi_i k_{rs}^{2\varepsilon}e_{ir}^2(t)+\frac{1}{2}\sum_{i=1}^{N}\sum_{s=1}^{n}\sum_{r=1}^{n}\xi_i k_{sr}^{2(1-\varepsilon)}e_{ir}^2(t)\\
&\quad+\frac{1}{2}\sum_{i=1}^{N}\sum_{r=1}^{n}\sum_{s=1}^{n}\xi_i l_{rs}^{2\varepsilon}e_{ir}^2(t)+\frac{1}{2}\sum_{i=1}^{N}\sum_{s=1}^{n}\sum_{r=1}^{n}\xi_i l_{sr}^{2(1-\varepsilon)}e_{ir}^2(t-\tau(t))\\
&\quad+\frac{1}{2}ce^\top(t)\Big((\Lambda\tilde{B}+\tilde{B}^\top\Lambda)\otimes I_n\Big)e(t)-v^\top(t)(D\otimes I_n)v(t).
\end{aligned}
$$

Based on the definition of the matrix $\tilde{B}^\Lambda$, we further have

$$
\begin{aligned}
\dot{V}(t) &\leq \sum_{i=1}^{N}\xi_i e_i^\top(t)A^s e_i(t)+\frac{1}{2}\sum_{i=1}^{N}\sum_{r=1}^{n}\xi_i\Big\{\Big[\sum_{s=1}^{n}(k_{rs}^{2\varepsilon}+l_{rs}^{2\varepsilon}+k_{sr}^{2(1-\varepsilon)})\Big]e_{ir}^2(t)\\
&\quad+\sum_{s=1}^{n}l_{sr}^{2(1-\varepsilon)}e_{ir}^2(t-\tau(t))\Big\}+\frac{1}{2}ce^\top(t)(\Lambda^{\frac{1}{2}}\otimes I_n)^\top\Big((\tilde{B}^\Lambda+\Delta)\otimes I_n\Big)\\
&\quad\cdot(\Lambda^{\frac{1}{2}}\otimes I_n)e(t)-v^\top(t)(D\otimes I_n)v(t)
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \sum_{i=1}^{N}\xi_i e_i^{\top}(t)A^s e_i(t) + \frac{1}{2}\sum_{i=1}^{N}\sum_{r=1}^{n}\xi_i\Big\{p_r e_{ir}^2(t) + \sum_{s=1}^{n} l_{sr}^{2(1-\varepsilon)} e_{ir}^2(t-\tau(t))\Big\} \\
& +\frac{1}{2}cv^{\top}(t)(\tilde{B}^{\Lambda}\otimes I_n)v(t) + \frac{1}{2}c\lambda_{\max}(\Delta)v^{\top}(t)(I_N\otimes I_n)v(t) \\
& -v^{\top}(t)(D\otimes I_n)v(t) \\
\leq\ & \frac{1}{2}\sum_{i=1}^{N}\xi_i e_i^{\top}(t)(2A^s+P)e_i(t) + \beta V(t-\tau(t)) + \frac{1}{2}cv^{\top}(t)(\tilde{B}^{\Lambda}\otimes I_n)v(t) \\
& +\frac{1}{2}c\lambda_{\max}(\Delta)v^{\top}(t)(I_N\otimes I_n)v(t) - v^{\top}(t)(D\otimes I_n)v(t) \\
\leq\ & \Big[\lambda_{\max}(2A^s+P) + c\lambda_{\max}(\Delta)\Big]V(t) + \beta V(t-\tau(t)) \\
& +\frac{1}{2}cv^{\top}(t)(\tilde{B}^{\Lambda}\otimes I_n)v(t) - v^{\top}(t)(D\otimes I_n)v(t).
\end{aligned}
$$

Therefore, we can get

$$
\begin{aligned}
\dot{V}(t) \ \leq\ & \alpha V(t) + \beta V(t-\tau(t)) + \frac{1}{2}cv^{\top}(t)(\tilde{B}^{\Lambda}\otimes I_n)v(t) - v^{\top}(t)(D\otimes I_n)v(t) \\
=\ & \frac{1}{2}cv^{\top}(t)\Big[\Big(\frac{\alpha}{c}I_N + \tilde{B}^{\Lambda} - \frac{2}{c}D\Big)\otimes I_n\Big]v(t) + \beta V(t-\tau(t)).
\end{aligned} \tag{13}
$$

Based on Lemma 2.1, we obtain $0 > \mu_1 \geq \mu_2 \geq \ldots \geq \mu_N$. Since $(\tilde{B}^{\Lambda} - \frac{2}{c}D)$ is a symmetrical matrix, there exists a unitary matrix $U = (u_1, \ldots, u_N)$ with $UU^{\top} = I_N$ such that

$$
(\tilde{B}^{\Lambda} - \frac{2}{c}D) = U\mathrm{diag}(\mu_1, \mu_2, \ldots, \mu_N)U^{\top}. \tag{14}
$$

Let $z(t) = (z_1^{\top}(t), z_2^{\top}(t), \ldots, z_N^{\top}(t))^{\top}$, and introduce a transformation $z(t) = (U^{\top}\otimes I_n)v(t)$. Then

$$
z^{\top}(t)z(t) = v^{\top}(t)(U^{\top}\otimes I_n)^{\top}(U^{\top}\otimes I_n)v(t) = v^{\top}(t)(UU^{\top}\otimes I_n) = \frac{1}{2}V(t). \tag{15}
$$

According to Eqs. (14) – (15), and the property of the Kronecker product of the matrices, we obtain

$$
\begin{aligned}
& \frac{1}{2}cv^{\top}(t)\Big[\Big(\frac{\alpha}{c}I_N + \tilde{B}^{\Lambda} - \frac{2}{c}D\Big)\otimes I_n\Big]v(t) \\
=\ & \frac{1}{2}cv^{\top}(t)(U\otimes I_n)\Big\{\mathrm{diag}\Big[\frac{\alpha}{c}+\mu_1, \frac{\alpha}{c}+\mu_2, \ldots, \frac{\alpha}{c}+\mu_N\Big]\otimes I_n\Big\}(U^{\top}\otimes I_n)v(t) \\
=\ & \frac{1}{2}cz^{\top}(t)\Big\{\mathrm{diag}\Big[\frac{\alpha}{c}+\mu_1, \frac{\alpha}{c}+\mu_2, \ldots, \frac{\alpha}{c}+\mu_N\Big]\otimes I_n\Big\}z(t) \\
\leq\ & \frac{1}{2}c(\frac{\alpha}{c}+\mu_1)z^{\top}(t)z(t) \\
=\ & (\alpha + c\mu_1)V(t).
\end{aligned} \tag{16}
$$

Thus, by (13) and (16), we have

$$\dot{V}(t) \le (\alpha + c\mu_1)V(t) + \beta V(t-\tau(t)) \le (\alpha + c\mu_1)V(t) + \beta \sup_{t-\tau\le s\le t} V(s). \tag{17}$$

It follows from Lemma 2.2 and Condition (A_2) that the proof of Theorem 3.1 is complete. □

Remark 3.2. Theorem 3.1 reveals that, for a given dynamical network (7) and the selected weight matrix Λ, exponential synchronization of the controlled delayed dynamical network (9) can be achieved by a suitable design of the control gain d_i such that the condition of Theorem 3.1 is satisfied. It turns out that synchronization of the controlled delayed dynamical network is heavily dependent on control gain. This point will be further illustrated through the numerical simulations in the next section.

Finally, specify the weighted matrix $\Lambda = \mathrm{diag}(\frac{1}{N}, \dots, \frac{1}{N})$, the inner connecting matrix $\Gamma = I_n$, and the coupling matrix B be a symmetrical matrix, then its eigenvalues can be ordered as $0 = \lambda_1 > \lambda_2 \ge \cdots \ge \lambda_N$. It follows that $\tilde{B}^{\Lambda} = 2B$, $\Delta = \mathrm{diag}(0, 0, \dots, 0)$, and $\tilde{B}^{\Lambda} - \frac{2}{c}D = 2(B - \frac{1}{c}D)$. Furthermore, it is easy to know the eigenvalues of the matrix $2(B - \frac{1}{c}D)$ satisfy $0 > \mu_1 \ge \mu_2 \ge \ldots \ge \mu_N$. Thus, we can obtain the following result can be derived from Theorem 3.1 immediately:

Corollary 3.3. Let P and β be precisely the same as those of Theorem 3.1, and assume that, Condition (A_1) holds, we have the following condition:

$(\tilde{A}_2)$ $\lambda_{\max}(2A^s + P) + c\mu_1 + \beta < 0$.

Then the controlled complex delayed dynamical network (9) is globally exponentially synchronized to $s(t) = \frac{1}{N}\sum_{i=1}^{N} x_i(t)$.

Remark 3.4. Theorem 3.1 and Corollary 3.3 have an advantage that simple and effective feedback controllers can be provided to synchronize a given delayed dynamical network to a desired synchronization state. Furthermore the synchronization state considered here may be not a special solution of the homogenous systems. From condition $(\tilde{A}_2)$ in Corollary 3.3, it is easy to verify that if

$$-c\mu_1 > \lambda_{\max}(2A^s + P) + \beta \tag{18}$$

then condition $(\tilde{A}_2)$ is satisfied with properly coupling strength c. It should be noted that, here μ_1 is determined by the topological structure of the network, the coupling strength and the control gains. Therefore, it is clear from Corollary 3.3 that if the coupling strength c is selected properly, then the coupled delayed network dynamical network (9) is exponentially synchronized to a desired synchronization state.

Remark 3.5. If the controller is revise as [5, 36]:

$$U_i = -cd_i(x_i(t) - s(t)),$$

then rewrite (18) as

$$c > -\frac{\lambda_{\max}(2A^s + P) + \beta}{\mu_1}, \tag{19}$$

where μ_1 is the maximum eigenvalue of the matrix $\tilde{B}^\Lambda - 2D$, which is irrelevant to the coupling strength c. Therefore, it is clear from (19) that if the coupling strength c is sufficiently large, then the network (9) is synchronized to a desired orbit, which is in accordance with the conclusion in [5]. It should be noted that the theoretical coupling strength given in (19) is too conservative, usually much smaller than the needed values. In general, it is desirable to make the coupling strength as small as possible in practice, while it is quite large in theory. Furthermore, from (18) and (19), we can see that the choice of the coupling strength c is related to the choice of the controller U_i.

Remark 3.6. In particular, if $l = 1$, it can provide a simple and effective strategy to pin a given delayed dynamical network to a desired orbit by a single feedback controller. In addition, the conditions of Theorem 3.1 and Corollary 3.3 are all sufficient conditions but not necessary, i. e., error system may remain exponentially stable, although the conditions of Theorem 3.1 and Corollary 3.3 may fail.

Consider the time-invariant dynamical network

$$\dot{x}_i(t) = f(t, x_i(t), x_i(t - \tau(t))) + c\sum_{j=1}^{N} b_{ij}\Gamma x_j(t), \quad i = 1, 2, \ldots, N. \tag{20}$$

A topic closely related to synchronization is consensus. The basic idea of consensus is that each agent updates its state based on the states of its neighbors and its own such that the states of all agents will converge to a common value. The interaction rule that specifies the information exchange between an agent and its neighbors is called the consensus protocol. Let $(f = 0, \Gamma = I_n, n = 1)$, then network (20) can be reduced as

$$\dot{x}_i(t) = c\sum_{j=1}^{N} b_{ij}x_j(t), \quad i = 1, 2, \ldots, N. \tag{21}$$

It is clear that the consensus is a special case of synchronization. Therefore, all the results concerning synchronization can apply to consensus. Choose the controllers as (8), substituting (8) into (21), we have

$$\begin{cases} \dot{x}_i(t) = c\sum_{j=1}^{N} b_{ij}x_j(t) - d_i(x_i(t) - s(t)), & 1 \le i \le l, \\ \dot{x}_i(t) = c\sum_{j=1}^{N} b_{ij}x_j(t), & (l+1) \le i \le N. \end{cases} \tag{22}$$

It was shown in [15] that under some assumptions, we can obtain

$$\lim_{t\to\infty} \|x_i(t) - \sum_{k=1}^{N} \xi_k x_k(t)\| = 0, \quad i = 1, 2, \ldots, N, \tag{23}$$

where $[\xi_1, \ldots, \xi_N]$ is the left eigenvector of B corresponding to the eigenvalue 0 satisfying $\sum_{k=1}^{N} \xi_k = 1$.

As in the consensus problem

$$\sum_{k=1}^{N} \xi_k x_k(t) = \sum_{k=1}^{N} \xi_k x_k(0), \tag{24}$$

for all $t > 0$, we get

$$\lim_{t\to\infty} |x_i(t) - \sum_{k=1}^{N} \xi_k x_k(0)| = 0, \quad i = 1, 2, \ldots, N. \tag{25}$$

When $\xi_1 = \xi_2 = \cdots = \xi_N = \frac{1}{N}$, $\sum_{k=1}^{N} x_k(0)/N = Ave(x) = s$ is an invariant quantity, then in this case the consensus should be called as the average consensus problem. Let the matrix $\tilde{B}$ be defined as $\tilde{B} = B + B^\top - \Delta$, where $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_N)$ with $\delta_i = \sum_{j=1}^{N} [b_{ij} + b_{ji}]$. Then the matrix $\tilde{B}$ is a symmetrical irreducible matrix with zero-sum and nonnegative off-diagonal elements. This implies that zero is an eigenvalue of $\tilde{B}$ with multiplicity 1, and all the other eigenvalues of $\tilde{B}$ are strictly negative. Its eigenvalues can be ordered as $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$. Furthermore, the eigenvalues of the matrix $(\tilde{B} - \frac{2}{c} D)$ can be ordered as $0 > \mu_1 \geq \mu_2 \geq \ldots \geq \mu_N$, where $D = \mathrm{diag}(d_1, \ldots, d_l, \underbrace{0, \ldots, 0}_{N-l})$. Then we have the following results, which is an immediate consequence of Theorem 3.1.

Theorem 3.7. Consider the controlled dynamical network (21). If the following condition is satisfied

$$\lambda_{\max}(\Delta) + \mu_1 < 0$$

then the controlled complex delayed dynamical network (8) is exponentially synchronized to $s(t) = \sum_{k=1}^{N} \xi_k x_k(t)$ with $\sum_{k=1}^{N} \xi_k = 1\,(\xi_k > 0)$.

Then the controlled network (21) achieve average consensus globally exponentially to $Ave(x) = \sum_{k=1}^{N} x_k(0)/N$.

Proof. The proof is omitted here since it can easily be deduced from Theorem 3.1. □

3.2. Time-varying dynamical network

In this section, we consider a time-varying complex dynamical network without delays for the sake of simplicity, which is described as

$$\dot{x}_i(t) = f(x_i(t)) + c \sum_{j=1}^{N} b_{ij}(t) \Gamma(t) x_j(t), \quad i = 1, 2, \ldots, N. \tag{26}$$

When the network achieve synchronization, the coupling term $c\sum_{j=1}^{N} b_{ij}(t)\Gamma(t)x_j(t)$ in Eq. (26) will vanish, it then follows that the synchronization state can be translated into

$$\dot{s}(t) = \sum_{k=1}^{N} \xi_k \dot{x}_k(t) = \sum_{k=1}^{N} \xi_k f(x_k(t)) = f(s(t)). \tag{27}$$

Similar to the former section, choose the controllers as

$$\begin{cases} U_i = -d_i(x_i(t) - s(t)), & 1 \leq i \leq l, \\ U_i = 0, & (l+1) \leq i \leq N. \end{cases} \tag{28}$$

Substituting (28) into (26), we have

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + c\sum\limits_{j=1}^{N} b_{ij}(t)\Gamma(t)x_j(t) - d_i(x_i(t) - s(t)), & 1 \leq i \leq l, \\ \dot{x}_i(t) = f(x_i(t)) + c\sum\limits_{j=1}^{N} b_{ij}(t)\Gamma(t)x_j(t), & (l+1) \leq i \leq N. \end{cases} \tag{29}$$

By introducing the synchronization error $e_i(t) = x_i(t) - s(t)\,(i = 1, 2, \ldots, N)$, then the error dynamical system is governed as follows:

$$\begin{cases} \dot{e}_i(t) = f(e_i(t) + s(t)) - f(s(t)) + c\sum\limits_{j=1}^{N} b_{ij}(t)\Gamma(t)e_j(t) - d_i e_i(t), & 1 \leq i \leq l, \\ \dot{e}_i(t) = f(e_i(t) + s(t)) - f(s(t)) + c\sum\limits_{j=1}^{N} b_{ij}(t)\Gamma(t)e_j(t), & (l+1) \leq i \leq N. \end{cases} \tag{30}$$

We define

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}, \quad e(t) = \begin{bmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_N(t) \end{bmatrix}, \quad S(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_N(t) \end{bmatrix}.$$

Then, (30) can be rewritten as

$$\dot{e}(t) = \Sigma(t, e(t)), \tag{31}$$

and the Jacobian of $\Sigma(t, e)$ at $e = 0$ is shown as

$$D\Sigma(t, 0)$$

$$= \begin{pmatrix} Df(s(t)) + b_{11}(t)\Gamma(t) & b_{12}(t)\Gamma(t) & \cdots & b_{1N}(t)\Gamma(t) \\ b_{21}(t)\Gamma(t) & Df(s(t)) + b_{22}(t)\Gamma(t) & \cdots & b_{2N}(t)\Gamma(t) \\ \vdots & \vdots & \ddots & \vdots \\ b_{N1}(t)\Gamma(t) & Df(s(t)) + b_{N2}(t)\Gamma(t) & \cdots & Df(s(t)) + b_{NN}(t)\Gamma(t) \end{pmatrix}$$

$$-diag\{d_1, d_2, \ldots, d_l, 0, \ldots, 0\}.$$

Then we have the following result.

Theorem 3.8. Suppose that $\Sigma : \Phi \to R^{nN}$ is continuously differentiable on $\Phi = \{e \in R^{nN} | \|e\| < \rho\}$. The controlled time-varying complex dynamical network (29) is exponentially synchronized to $S(t)$ if there exist two symmetric positive definite matrices, F, $P \in R^{nN \times nN}$, such that

$$P\Big(D\Sigma(t,0)\Big) + \Big(D\Sigma(t,0)\Big)^\top P \le -F \le -l_1 I_{nN},$$

and

$$\begin{aligned} & diag\{df(y_1(t) - df(s(t))), \dots, df(y_N(t) - df(s(t)))\}^\top P \\ & + P diag\{df(y_1(t) - df(s(t))), \dots, df(y_N(t) - df(s(t)))\} \quad \le \quad l_2 I_{nN} \quad < \quad l_1 I_{nN}, \end{aligned}$$

where $l_1 > 0$, $y(t) - S(t) \in \Phi$.

Proof. The proof of Theorem 3.8 is similar to the proof of Theorem 2 in [20] and is omitted from this paper due to the limitation of space. □

4. ILLUSTRATIVE EXAMPLE

Recently, the collective dynamics analysis of complex networks has led to a host of interesting effects. In particular, the study on controlling the dynamics of a network and guiding it to a desired state, such as, an equilibrium point, a periodic orbit, or a chaotic attractor of the network has become an interesting and important direction in this research field.

In order to verify the effectiveness of the theoretical results, some feedback controllers designed in Section 4 are applied to dealing with the problem of pinning synchronization in a prototype composed of the representative neuronal models with scale-free (SF) character [1, 2], where numerical examples are given to verify and also visualize the theoretical results.

For convenience of explicit statement on the theoretical results, we consider a scale-free (SF) network with 30 delayed cellular neuron oscillators. It is well known that the connectively degree distribution in the scale-free networks obeys the power law form [10, 29]. We here take the parameters $N = 30$, $m = m_0 = 3$ and $\gamma_{BA} = 3$, then the coupling matrix $A = A_{sf}$ of the network can be randomly generated by the SF model.

As is well known, the delayed neural network model has become a classic and popular model for analysis of neuronal activity and has also been intensively investigated in the last decade due to its intrinsic and complex dynamical behavior such as the period-doubling bifurcation and chaos [13]. The delayed neural network with two neurons are described by the following differential equations:

$$\dot{x}(t) = Ax(t) + g(t, x(t), x(t - \tau(t))), \tag{32}$$

where $x(t) = (x_1(t), x_2(t))^\top$, $g(t, x(t), x(t - \tau(t))) = M\tilde{g}(x(t)) + N\tilde{g}(x(t - \tau(t)))$ with $\tilde{g}(x(t)) = (\tanh(x_1(t), \tanh(x_2(t)))^\top$, $\tau(t) = 1$, and

$$A = \begin{pmatrix} -1.0 & 0 \\ 0 & -1.0 \end{pmatrix}, \quad M = \begin{pmatrix} 2.0 & -0.1 \\ -5.0 & m_{22} \end{pmatrix}, \quad N = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & n_{22} \end{pmatrix}.$$

For simplicity, we always consider the weighted matrix $\Lambda = \mathrm{diag}(\frac{1}{N}, \ldots, \frac{1}{N})$, the inner connecting matrix $\Gamma = I_2$, and the coupling matrix B be a symmetrical matrix. Three pinning cases will be given below:

Example 4.1. Pinning a network to its equilibrium point. It is easy to know that with the parameters $m_{22} = 0.2$ and $n_{22} = 0.3$, Eq. (32) has three equilibrium points [13]: $x_{-1} = (-0.37614, 2.35978)$, $x_0 = (0, 0)$, $x_1 = (0.37614, -2.35978)$. Without loss of generality, the simulation is pinning the network to its equilibrium state $s = (-0.37614, 2.35978)^\top$. Here g satisfies condition (A_1) with $k_{11} = 2$, $k_{12} = 0.1$, $k_{21} = -5.0$, $k_{22} = 0.2$, $l_{11} = 1.5$, $l_{12} = 0.1$, $l_{21} = 0.2$, $l_{22} = 0.3$, and hence we can get $P = \mathrm{diag}(10.7, 6)$, $\beta = 1.7$, $\lambda_{\max}(2A^s + P) = 8.7$. It is easy to verify that if the following condition hold,

$$10.4 + c\mu_1 < 0, \tag{33}$$

then the condition of Corollary 3.3 is satisfied, which means the controlled delayed dynamical network (9) is exponentially synchronized to $s(t)$.

We select feedback controllers to pin three nodes chosen randomly, and we can get $\mu_1 = -2.132$. Here take the coupling strength $c = 5.0$, the pinned nodes' control gains $d_i = d = 0.5$. Figure 1 visualizes the change process of the state variables of the controlled delayed dynamical network (9), which satisfies the conditions of Corollary 3.3.

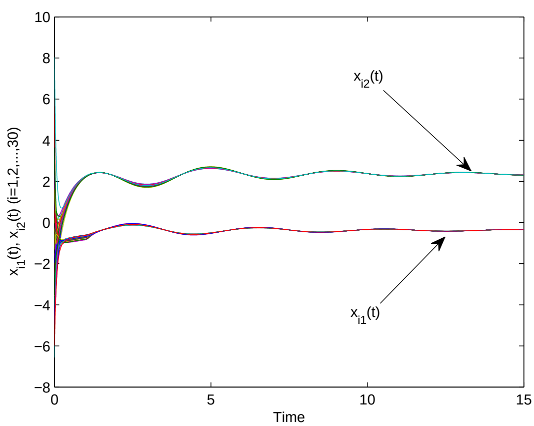

Fig. 1. Pin 30 delayed cellular neuron oscillators to its equilibrium.

Example 4.2. Pinning a network to a periodic orbit. As is well known, for $m_{22} = 4.5$ and $n_{22} = -2.5$, the numerical integration of Eqs. (32) shows that the neuron oscillator exhibits a periodic behavior, as shown in Figure 2. Figure 3 shows the phase portrait, accordingly. In this simulation, we select randomly three nodes to be pinned, the rest parameters are chosen as the the same as that in Figure 1. Figure 4 is the simulation result corresponding to this situation, which shows that the network can be pinned to periodic orbit $s(t)$ by some feedback controllers.

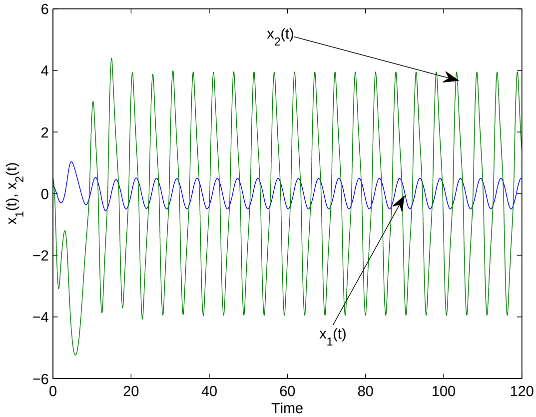

Fig. 2. A periodic orbit of the delayed cellular neuron oscillator.

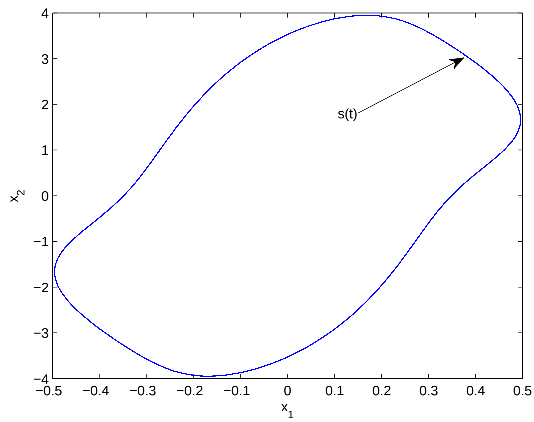

Fig. 3. The phase portrait of the periodic orbit.

Example 4.3. Pinning a network to a chaotic attractor. It is known that with the parameters $m_{22} = 4.5$ and $n_{22} = -4.45$, the solution trajectory $s(t) = (s_1(t), s_2(t))^\top$ of the system (32) approaches a chaotic attractor, as shown in Figure 5, where the initial value is taken as $(x_1(0), x_2(0))^\top = (-0.4, 2.2)^\top$. Here choose three nodes randomly to be controlled, and let the control parameters are the same as that in Figure 1. Figure 6 indicates that the network can be pinned to chaotic orbit $s(t)$ via a few controllers. Figure 7 shows the phase portrait, accordingly.

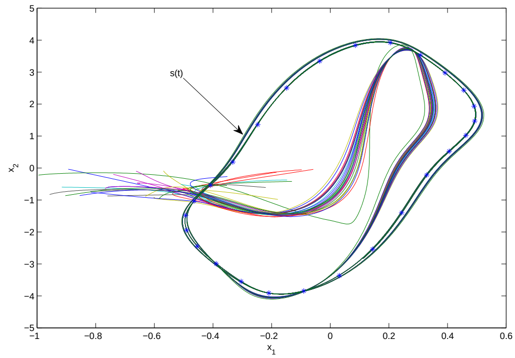

Fig. 4. The phase portrait of pin 30 oscillators to a periodic orbit. Stars denote the snapshots on the periodic orbit.

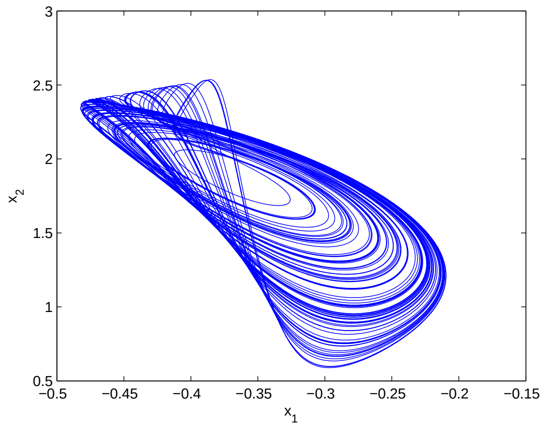

Fig. 5. A chaotic attractor of the isolated delayed neuron oscillator.

5. CONCLUSIONS

In this paper, we have investigated the issue of pinning synchronization in a complex dynamical network comprised of linearly coupled identical dynamical systems with directionally coupling. We established some explicit and conclusive results of pinning synchronization for such a directed network. These results can be used as an effective control scheme to synchronize an arbitrary given dynamical network to a desired synchronization orbit, which is particularly meaningful for some practical applications. More importantly, here the synchronization orbit could be chosen as a weighted average of all the nodes states in the network in the aim of practical control tactics. It should

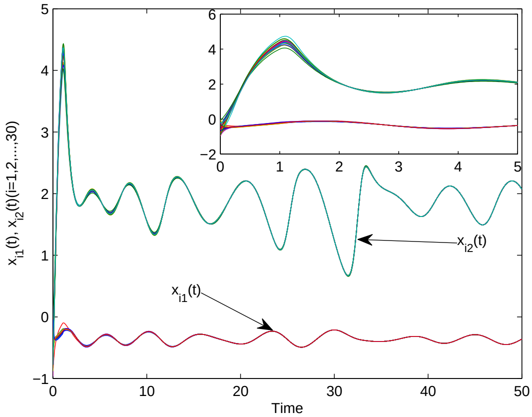

Fig. 6. Pin 30 oscillators to a chaotic attractor.

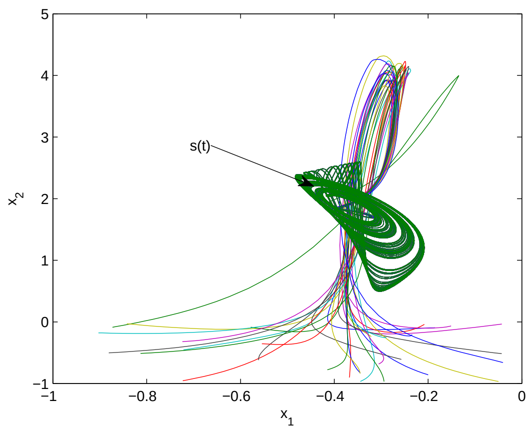

Fig. 7. The phase portrait of pin 30 oscillators to a chaotic attractor.

be noted that in order to make all the states of the nodes in a network synchronize to a desired synchronization state, the coupling strength changes with the controller. Furthermore, the theoretical result on the time-invariant network is extended to the time-varying network, and the result on synchronization problem can be extended to the consensus problem of networked multi-agent systems. Both theoretical analysis and numerical simulations indicate the effectiveness of the proposed methodology. It is believed that the idea and approach developed in this paper could be further generalized to deal with some other problems on chaos control and synchronization for more general complex dynamical networks. In addition, it should be noticed that there exist unknown factors of the real-world complex networks in the mathematical models, which are called uncertainty. Possible uncertainties include unknown parameters, unknown functions, unmodeled dynamics, and disturbances. For uncertain systems, the problem is to devise a control that uses the dynamic equation to govern the trajectory of the system with acceptable performance. The main approach for designing the controller is the adaptive technique, which can deal with structured uncertainties. These are the problems that will be investigated in the future.

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