KYBERNETIKA — VOLUME *50* (2014), NUMBER 6, PAGES 1032–1048

# DYNAMIC PORTFOLIO OPTIMIZATION WITH RISK MANAGEMENT AND STRATEGY CONSTRAINTS

Csilla Krommerová and Igor Melicherčík

We investigate the problem of power utility maximization considering risk management and strategy constraints. The aim of this paper is to obtain admissible dynamic portfolio strategies. In case the floor is guaranteed with probability one, we provide two admissible solutions, the option based portfolio insurance in the constrained model, and the alternative method and show that none of the solutions dominate the other. In case the floor is guaranteed partially, we provide one admissible solution, the portfolio insurance with spreads.

*Keywords:* power utility maximization, risk management, convex constraints

*Classification:* 49L20, 60J65, 91G10, 91G20

## 1. INTRODUCTION

The problem of maximizing the expected utility over a given time horizon is one of the most frequently examined problems in financial mathematics. One can achieve the maximum expected utility by choosing the proper portfolio strategy, i. e. by optimal allocation of the available funds among risky and risk-free assets.

The problem was first examined by P. A. Samuelson. In his work [14], the model is presented in a discrete form and interpreted as a problem of dynamic stochastic programming, solving the Bellman equation. Samuelson states that for power utility functions, the optimal portfolio strategy is constant over time. Merton [9] confirms the results of Samuelson for a continuous-time case. In his work, Nutz [10] expands the power utility maximization problem by a special case in which the portfolio strategy is constrained by a fixed convex set and shows that in such case the optimal portfolio strategy is also constant.

No portfolio with risky assets guarantees any return. The aim of the portfolio insurance is to limit the losses and simultaneously to allow the participation on the rising market. The idea of insuring the portfolio against losses was first introduced by H. Leland and M. Rubinstein [5] in 1976. They developed the option based portfolio insurance, also referred to as OBPI, which consists of a risky asset and a put option written on it. The strike price of the put option represents the floor such that the value of the investment at the maturity is higher than the floor with 100% probability.

DOI: 10.14736/kyb-2014-6-1032

There is a possibility that the required put option is not available on the market. By Leland and Rubinstein [6], in such case one can synthesize the put option with a replication portfolio that consists of the underlying asset and a risk-free bond. Using the replication portfolio, the OBPI becomes dynamic so that one can guarantee the discounted level of the floor at any time from the beginning until the maturity.

While the OBPI method requires a guaranteed floor with probability one, the Value-at-Risk based risk management guarantees the floor with a given probability less than one. Basak and Shapiro [1] introduced the power utility optimization model using the Value-at-Risk based risk management (also called VaR-RM).

Even though both the optimal portfolio selection and the portfolio insurance were examined by many scientists, both problems still offer many research opportunities. The aim of this paper is to bring together these two areas, specifically, we investigate how to insure the portfolio when convex constraints are imposed on the portfolio strategy. We intend to provide either optimal or admissible solutions for the problem of dynamic portfolio optimization with risk management and strategy constraints.

We specify the convex constraints representing the case when short-selling of both the risky and risky-free assets is prohibited. Our goal is to investigate the portfolio insurance with a guaranteed floor in the constrained model and the portfolio insurance with a partially guaranteed floor in the constrained model. We provide different methods of solution and compare them based on their certainty equivalents.

This paper is organized as follows. Section 2 specifies the economic settings used in this paper and describes the main results of the power utility maximization problem - first with no strategy constraints, then with convex constraints. In Section 3, we introduce the portfolio insurance with guaranteed floor. In case when convex constraints apply, we provide two admissible methods of calculation: the option based portfolio insurance in the constrained model and the alternative method in the constrained model. As we show in the example, none of the methods dominate the other. In Section 4, we examine the question of the portfolio insurance with partially guaranteed floor. Since the Value-at-Risk based risk management is not admissible in the constrained model, we provide an admissible alternative to it, the portfolio insurance with spreads. Section 5 concludes the paper.

## 2. ECONOMIC SETTINGS

Let $T > 0$ represent the time horizon and let the triplet $(\Omega, \mathcal{F}, P)$ represent the probability space. We use $d$ risky assets and one risk-free bond to construct our portfolio.

For a given quantity, we use the upper index $i = 1, 2, \dots, d$ to represent a particular asset and the lower index $t \in \langle 0, T \rangle$ to express the time dependence.

We denote the expected return on the asset $i$ by $\mu^i$, the volatility matrix by $\sigma = \{\sigma^{ij}, i = 1, \dots, d, j = 1, \dots, d\}$, the covariance matrix by $c^R = \sigma\sigma^\top$ and the risk-free interest rate by $r$. We consider these parameters to be constant over the time.

Let $w_t = (w_t^1, w_t^2, \dots, w_t^d)^\top$ be an $\mathbb{R}^d$-valued Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Then the prices of the risky assets and the non-risky bond follow

$$dS_t^i = S_t^i[\mu_i dt + \sigma_i dw_t], \qquad \text{for } i = 1, 2, \dots, d, \tag{1}$$

$$dB_t = B_t r dt. \tag{2}$$

We define the portfolio strategy as $\beta_t = (\beta_t^1, \beta_t^2, \ldots, \beta_t^d)^\top$, where $\beta_t^i$ represents the proportion of the total wealth invested in the $i$th asset at time $t$. For simplicity we fix the initial capital $X_0$. The wealth process then follows

$$dX_t = X_t[r + \beta_t^\top(\mu - r\mathbf{1})]dt + X_t\beta_t^\top \sigma dw_t, \tag{3}$$

where $\mathbf{1} = (1, 1, \ldots, 1)^\top$.

The existence of the unique state price density process $\xi_t$ ensures the market completeness (under no-arbitrage). The stochastic differential equation for $\xi_t$ is given as

$$d\xi_t = -\xi_t[rdt + \kappa^\top dw_t],$$

where $\kappa = \sigma^{-1}(\mu - r\mathbf{1})$ is the market price of the risk process and is also considered to be constant over time. In all cases we consider the portfolio to be self-financing

$$E[\xi_T X_T] \leq \xi_0 X_0,$$

i. e. after the initial investment, no further investments are needed (the assumption of zero net investments), and buying or selling one type of asset is balanced by selling or buying other assets (the principle of self-financing).

The agent strives to utilize the expected terminal wealth $U(X_T)$. The utility function $U$ is assumed to be increasing, concave and twice continuously differentiable. In our work, we focus on the power utility functions of the form

$$U(X) = \frac{X^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0. \tag{4}$$

We exclude the case when $\gamma = 1$ as in this case the utility function is logarithmic.

By Prigent [13], the power utility functions have a constant Arrow-Pratt measure of relative risk-aversion in the form

$$R(W_T) = -W_T \frac{U(W_T)''}{U(W_T)'} = \gamma.$$

Mehra and Prescott [8] state that a reasonable relative risk-aversion takes values between $\gamma \in \langle 2, 10 \rangle$. The higher the parameter of the risk aversion is, the more conservative the agent is.

Note that in the literature, the power utility function can also be referred to as isoelastic function or CRRA (Constant Relative Risk Aversion) function.

### 2.1. Power utility maximization

Our aim is to find the optimal portfolio strategy that maximizes the expected utility from the terminal wealth using the power utility function and assuming no strategy constraints, i. e.

$$\max_\beta E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right], \tag{5}$$

where we maximize through all dynamic strategies $\beta$. By Nutz [10], the optimal portfolio strategy is the argmax of a deterministic function

$$\eta(\beta) := r + \beta^T(\mu - r\mathbf{1}) - \frac{\gamma}{2}\beta^T c^R \beta \tag{6}$$

and can be expressed as

$$\hat{\beta} = \frac{1}{\gamma}(c^R)^{-1}(\mu - r\mathbf{1}). \tag{7}$$

Let $\mathcal{S} \subseteq \mathbb{R}^d$ be the set of constraints imposed on the agent. Then the set of admissible strategies according to the initial wealth $X_0$ is

$$\mathcal{A}(X_0) := \{\beta : X_t > 0 \quad \text{and} \quad \beta_t \in \mathcal{S} \quad \text{for all} \quad t \in \langle 0, T\rangle\}.$$

In case of fixed $X_0$, we simply write $\mathcal{A}$ instead of $\mathcal{A}(X_0)$ and we optimize

$$\max_{\beta \in \mathcal{A}} E\left[\frac{X_T^{1-\gamma}}{1-\gamma}\right]. \tag{8}$$

**Theorem 2.1.** (Nutz [10], Theorem 3.2.) Assume that $\mathcal{S}$ is convex and there is no arbitrage on the market. Then, there exists an optimal strategy $\hat{\beta}$ such that $\hat{\beta}$ is a constant vector and is characterized by

$$\hat{\beta} \in \arg\max_{\beta \in \mathcal{S}} \eta(\beta), \tag{9}$$

where $\eta(.)$ is given by (6).

## 3. PORTFOLIO INSURANCE WITH GUARANTEED FLOOR

The main idea of insuring the portfolio against losses is to guarantee a minimum return and simultaneously allow the portfolio to participate on the rising market.

The OBPI strategy consist of a portfolio covered by a put option written on it. The put option has the same maturity $T$ as the portfolio and its strike price $\underline{W}$ is the predefined floor. The basic overview of OBPI can be found in [3].

Let the risky portfolio $X$, invested in $d$ risky assets and a non-risky bond, follow the process

$$dX_t = X_t \mu_X dt + X_t\, \sigma_X dw_t, \tag{10}$$

where $\mu_X = r + \beta^\top(\mu - r)$ is the drift of the portfolio, $\sigma_X = \sqrt{\beta^\top c^R \beta}$ is the volatility of the portfolio and $w_t$ is a one-dimensional Brownian motion.

Let $V_t^{put}$ be the price of the put option written on the asset $X$ and $V_t^{call}$ be the price of the call option written on the asset $X$ with maturity $T$ and strike price $\underline{W}$ at time $t \in \langle 0, T\rangle$. The value of the insured portfolio $W_t$ at time $t$ is given as

$$W_t = X_t + V_t^{put} = \underline{W}e^{-r(T-t)} + V_t^{call},$$

due to the put-call parity. One can see that the value of the insured portfolio $W_t$ is always above the deterministic level $\underline{W}e^{-r(T-t)}$ at any time $t$.

Using the Black–Scholes pricing, the prices of $V_t^{put}$ and $V_t^{call}$ at time $t$ can be calculated as (see e. g. [2])

$$V_t^{put} = \underline{W} e^{-r(T-t)} \Phi\left(-d_2(\underline{W})\right) - X_t \Phi\left(-d_1(\underline{W}\right),$$
$$V_t^{call} = X_t \Phi\left(d_1(\underline{W})\right) - \underline{W} e^{-r(T-t)} \Phi\left(d_2(\underline{W})\right),$$

with

$$d_1(\underline{W}) = \frac{\ln \frac{X_t}{\underline{W}} + \left(r + \frac{\sigma_X^2}{2}\right)(T-t)}{\sigma_X \sqrt{T-t}} \quad \text{and} \quad d_2(\underline{W}) = d_1 - \sigma_X \sqrt{T-t},$$

where $\Phi(.)$ is the standard normal distribution function.

Possible difficulties might occur when the desired put option cannot be found on the market. In such case the put option can be synthesized by a replication portfolio invested in the risk-free asset and the underlying portfolio. The replication portfolio should have the same characteristics as the put option (e. g. the value, payoff and risk).

The replication portfolio at time $t$ can be expressed as

$$V_t = \varphi_t X_t + \psi_t B_t, \tag{11}$$

where $\varphi_t = \dfrac{\partial V_t}{\partial X_t}$ is the so-called delta of the option, in other words the sensitivity of the option-value on the value of the underlying portfolio. The delta of the put option $\varphi_t$ can be computed as

$$\varphi_t = \Phi(d_1(\underline{W})) - 1 \tag{12}$$

and one can easily see that $-1 < \varphi_t < 0$ for every $t$. When using the Black–Scholes formula for calculating $V_t$, $\psi_t$ can be calculated from (11). For details about the replicating strategy, we refer to [2].

The value of the insured portfolio can be expressed as

$$W_t = (1 + \varphi_t) X_t + \psi_t B_t.$$

Since the portfolio weights are calculated as

$$\text{weight}^i = \frac{\text{money invested in the asset } i}{\text{total money invested}},$$

the new portfolio strategy can be expressed as

$$\theta_t^i = \frac{(1+\varphi_t)\beta^i X_t}{W_t}, \qquad i = 1, \ldots, d.$$

Subsequently, the portfolio process follows

$$dW_t = W_t \mu_W dt + W_t \sigma_W dw_t, \tag{13}$$

where the drift is $\mu_W = r + \theta_t^\top(\mu - r\mathbf{1})$, the volatility is $\sigma_W = \sqrt{\theta_t^\top c^R \theta_t^\top}$ and $w_t$ is a one-dimensional Brownian motion. Both $\mu_W$ and $\sigma_W$ are time-dependent, however, for the sake of simplicity, we drop the lower index $t$.

### 3.1. OBPI in the unconstrained model

The portfolio manager aims to maximize the utility from the expected terminal wealth of the insured portfolio

$$\max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \qquad (14)$$
$$s.t. \qquad W_T \geq \underline{W},$$

where the maximum is taken through all dynamic strategies $\theta$. Note that in order to avoid immediate arbitrage situations, the floor must satisfy the condition $\underline{W} < W_0 e^{rT}$, where $W_0 > 0$ is the initial amount invested in the portfolio insured with OBPI. The following theorem is a special case of Proposition 3 from [1] (page 380).

**Theorem 3.1.** The optimal portfolio strategy for the problem (14) is

$$\hat{\theta}_t = \frac{1}{\gamma}[c^R]^{-1}(\mu - r\mathbf{1})\frac{(1+\varphi_t)X_t}{W_t},$$

where $X_t$ is given by (10) and $W_t$ follows (13). The fraction of wealth invested in stocks can be expressed as

$$\hat{\theta}_t = q_t\hat{\beta}, \qquad (15)$$

where $\hat{\beta}$ is the optimal portfolio strategy of the uninsured model without constraints (5), calculated by (7) and

$$q_t = \frac{(1+\varphi_t)X_t}{W_t}. \qquad (16)$$

### 3.2. OBPI in the constrained model

Let us consider a portfolio with convex constraints on the portfolio strategy that is insured by a put option. Mathematically, our model can be written as

$$\max_{\theta\in\mathcal{C}} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \qquad (17)$$
$$s.t. \qquad W_T \geq \underline{W},$$
$$\mathcal{C} = \left\{\theta^i \geq 0,\ i = 1, 2, \ldots, d; \sum \theta^i \leq 1\right\}.$$

**Theorem 3.2.** Let $\hat{\beta}$ be the optimal portfolio strategy for the portfolio with convex constraints, computed as

$$\hat{\beta} = \arg\max_{\beta\in\mathcal{C}} \beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\gamma\beta^\top c^R\beta, \qquad (18)$$

where $\mathcal{C}$ is given in problem (17). Let $X_t$ follow (10) and $W_t$ follow (13). Let $\varphi_t$ be the delta of the put option calculated by (12). Then the portfolio strategy

$$\theta_t = (1+\varphi_t)\hat{\beta}\frac{X_t}{W_t} \qquad (19)$$

is admissible for problem (17).

P r o o f . The optimal portfolio strategy $\hat{\beta}$ for the model with convex constraints and no insurance satisfies

$$\hat{\beta}^i \geq 0, \qquad \text{for } i = 1, 2, \dots d,$$
$$\sum_{i=1}^{d} \hat{\beta}^i \leq 1.$$

Since $-1 < \varphi_t < 0$, $\hat{\beta}^i \geq 0$ for $i = 1, 2, \dots d$ and $0 \leq \frac{X_t}{W_t} \leq 1$, then $\theta_t^i \geq 0$ for $i = 1, 2, \dots d$ and

$$\sum_{i=1}^{d} \theta_t^i = \sum_{i=1}^{d} (1 + \varphi_t) \hat{\beta}^i \frac{X_t}{W_t} \leq (1 + \varphi_t) \sum_{i=1}^{d} \hat{\beta}^i \leq \sum_{i=1}^{d} \hat{\beta}^i \leq 1.$$

Therefore the strategy $\theta_t$ is admissible for problem (17). □

**Corollary 3.3.** Let the solution $\hat{\beta}$ computed by (7) be optimal for the problem (5). In case that the optimal solution $\hat{\beta}$ with no constraints on the portfolio strategy satisfies $\hat{\beta}^i \geq 0$ for $i = 0, \dots, d$ and $\sum_{i=1}^{d} \hat{\beta}^i \leq 1$, the portfolio strategy $\theta_t = \frac{(1 + \varphi_t) X_t}{W_t} \hat{\beta}$ is optimal for problem (17).

### 3.3. Alternative method in the constrained model

In this section, we provide an alternative strategy for problem (17). One can observe that using the set $\mathcal{C}$ from (17) to calculate (18) could be too restrictive. The idea behind the alternative strategy is to relax the restriction set to be $\mathcal{C}a = \{\, \beta a^i \geq 0,\ i = 1, 2, \dots, d\}$ and calculate the corresponding weights as

$$\hat{\beta}a = \arg \max_{\beta \in \mathcal{C}a} \ \beta^\top (\mu - r\mathbf{1}) - \frac{1}{2} \gamma \beta^\top c^R \beta. \tag{20}$$

Then we calculate the final portfolio strategy using $\hat{\beta}a$ in (19). In case the sum of the final portfolio weights, $\sum_{i=1}^{d} \theta a$, is greater than 1, we divide each weight by the sum, i. e. we normalize them. Note that using $\mathcal{C}a$ instead of $\mathcal{C}$ implies generally different mutual proportions of the risky assets in the final dynamic portfolio strategy. Therefore the alternative strategy is different from the OBPI strategy (19).

Suppose that at time $t$ the value of our portfolio is $Wa_t \geq \underline{W} e^{-r(T-t)}$. We construct the insured portfolio by investing a value of $Xa_t$ to the asset with constant portfolio strategy $\hat{\beta}a$ and investing a value of $Va_t^{put}$ to a corresponding put option. The volatility of the asset is given as $\sigma_{Xa} = \sqrt{\hat{\beta}a^\top c^R \hat{\beta}a}$ and the put option guarantees the level $\underline{W}$ at time $T$. Then, $Xa_t$ is the solution of the equation

$$Wa_t = Xa_t + Va_t^{put}. \tag{21}$$

Note that (21) has a solution since $Va_t^{put} = \underline{W} e^{-r(T-t)} < Wa_t$ for $Xa_t = 0$, $Wa_t < Xa_t + Va_t^{put}$ for $Wa_t < Xa_t$ and the right-hand side of (21) is continuous with respect to $Xa_t$.

Because the particular put option might not be available on the market, we synthesize it. The delta of the put option can be calculated as

$$\varphi a_t = \Phi\left(\frac{\ln \frac{Xa_t}{\underline{W}} + \left(r + \frac{\sigma_{Xa}}{2}\right)(T-t)}{\sigma_{Xa}\sqrt{T-t}}\right) - 1$$

and the candidate for the portfolio strategy is

$$h_t = (1+\varphi a_t)\,\hat{\beta a}\frac{Xa_t}{Wa_t}.$$

Problem (17) requires that the sum of the portfolio weights does not access the upper bound 1, therefore we define the new portfolio strategy as

$$\theta a_t = \begin{cases} (1+\varphi a_t)\,\hat{\beta a}\dfrac{Xa_t}{Wa_t} & \text{if} \quad \sum_{i=1}^d h_t^i \le 1, \\ \dfrac{(1+\varphi a_t)\,\hat{\beta a}\dfrac{Xa_t}{Wa_t}}{\sum_{i=1}^d h_t^i} & \text{if} \quad \sum_{i=1}^d h_t^i \ge 1. \end{cases}$$

The portfolio value $Wa_t$ then follows the equation

$$dWa_t = Wa_t[r + \theta a_t^\top(\mu - r\mathbf{1})]dt + Wa_t\sqrt{\theta a_t^\top c^R\,\theta a_t}dw_t. \tag{22}$$

Note that since $\hat{\beta a} \in Ca$, the portfolio strategy satisfies $\theta a_t \ge \mathbf{0}$.

**Theorem 3.4.** Let $Ca = \{\beta a^i \ge 0,\ i = 1,2,\ldots,d\}$ and $\hat{\beta a}$ be calculated as

$$\hat{\beta a} = \arg\max_{\beta a \in Ca}\ \beta a^\top(\mu - r\mathbf{1}) - \frac{1}{2}\gamma\,\beta a^\top c^R\,\beta a.$$

Let $Wa_t$ be the value of the portfolio at time $t$. For $Wa_t \ge \underline{W}e^{-r(T-t)}$, we define the portfolio strategy $\theta a_t$ as

$$\theta a_t = \begin{cases} (1+\varphi a_t)\,\hat{\beta a}\frac{Xa_t}{Wa_t} & \text{if} \quad \sum_{i=1}^d h_t^i \le 1, \\ \dfrac{(1+\varphi a_t)\,\hat{\beta a}\frac{Xa_t}{Wa_t}}{\sum_{i=1}^d h_t^i} & \text{if} \quad \sum_{i=1}^d h_t^i > 1, \end{cases} \tag{23}$$

where $h_t^i = (1+\varphi a_t)\,\hat{\beta a}^i\frac{Xa_t}{Wa_t}$. If $Wa_0 \ge \underline{W}e^{-rT}$, then $\theta a_t$ is admissible for the problem (17) and $Wa_t$ satisfies (22). Moreover, $Wa_t \ge \underline{W}e^{-r(T-t)}$ for all $t \ge 0$ with probability 1.

*Proof.* First, we show that the portfolio strategy $\theta a_t$ is admissible for problem (17), i.e. we show that $\theta a_t \in \mathcal{C}$ for all $t \in \langle 0, T\rangle$. It is clear that $\theta a_t \ge \mathbf{0}$, since $\hat{\beta a} \in Ca$ implies $\hat{\beta a} \ge \mathbf{0}$. Moreover, $Xa_t \ge 0$, $Wa_t \ge 0$ and $(1+\varphi a_t) \ge 0$.

If

$$\theta a_t = (1 + \varphi a_t)\, \hat{\beta} a \frac{Xa_t}{Wa_t},$$

then we have $\sum_{i=1}^{d} \theta a_t^i \leq 1$. Otherwise, by the definition of $\theta a_t$ in (23), it holds that $\sum_{i=1}^{d} \theta a_t^i = 1$.

Second, we show that the dynamic portfolio strategy $\theta a_t$ guarantees the floor $\underline{W}$. From (21) and the put-call parity we have

$$Wa_t = Xa_t + Va_t^{put}(Xa_t, \underline{W}) = \underline{W} e^{-r(T-s)} + Va_t^{call}(Xa_t, \underline{W}), \tag{24}$$

where $Va_t^{put}(Xa_t, \underline{W})$ and $Va_t^{call}(Xa_t, \underline{W})$ denote the values of the put and call options written on the asset with constant portfolio strategy $\hat{\beta} a$ having the value of $Xa_t$ at time t. The maturity of the options is $T$ and their strike price is $\underline{W}$.

If $\sum_{i=1}^{d} h_t^i \leq 1$, then the strategy $\theta a_t = (1 + \varphi a_t)\, \hat{\beta} a \frac{Xa_t}{Wa_t}$ replicates the portfolio with $\underline{W} e^{-r(T-t)}$ invested in the risk-free asset. The remaining part of the portfolio is the is the call option, which value cannot fall below 0. If $\sum_{i=1}^{d} h_t^i > 1$, then the strategy is expressed as

$$\theta a_t = \frac{(1 + \varphi a_t)\, \hat{\beta} a \frac{Xa_t}{Wa_t}}{\sum_{i=1}^{d} h_t^i} = \frac{\Phi\left(\frac{\ln \frac{Xa_t}{\underline{W}} + \left(r + \frac{1}{2}\sigma_{Xa}^2\right)(T-t)}{\sigma_{Xa}\sqrt{T-t}}\right) \hat{\beta} a \frac{Xa_t}{Wa_t}}{\sum_{i=1}^{d}, h_t^i}.$$

One can write (24) in the form

$$Va_t^{call}(Xa_t, \widetilde{\underline{W}}) + [\underline{W} e^{-r(T-s)} + (Va_t^{call}(Xa_t, \underline{W}) - Va_t^{call}(Xa_t, \widetilde{\underline{W}}))],$$

where $\widetilde{\underline{W}}$ represents a level of the floor for which the delta of the call option with the strike $\widetilde{\underline{W}}$ is

$$\Phi\left(\frac{\ln \frac{Xa_t}{\widetilde{\underline{W}}} + \left(r + \frac{1}{2}\sigma_{Xa}^2\right)(T-t)}{\sigma_{Xa}\sqrt{T-t}}\right) = \frac{\Phi\left(\frac{\ln \frac{Xa_t}{\underline{W}} + \left(r + \frac{1}{2}\sigma_{Xa}^2\right)(T-t)}{\sigma_{Xa}\sqrt{T-t}}\right)}{\sum_{i=1}^{d} h_t^i}.$$

It is clear that $\widetilde{\underline{W}} > \underline{W}$ and hence $Va_t^{call}(Xa_t, \underline{W}) > Va_t^{call}(Xa_t, \widetilde{\underline{W}})$. Therefore, in this case we invest a higher amount in the risk-free asset, namely $\underline{W} e^{-r(T-t)} + (Va_t^{call}(Xa_t, \underline{W}) - Va_t^{call}(Xa_t, \widetilde{\underline{W}}))$. The remaining part is invested in the call option with strike price $\widetilde{\underline{W}}$ which value cannot fall below 0.

One can conclude that in any case, the strategy $\theta a_t$ invests at least $\underline{W} e^{-r(T-t)}$ in the risk-free asset and the remaining part of the portfolio does not fall below 0. Therefore it holds that $Wa_t \geq \underline{W} e^{-r(T-t)}$ at every time $t \geq 0$ with probability 1. □

### 3.4. Example

Let us now construct a portfolio from three risky assets and one risk-free bond. We examine the portfolio performance of the OPBI in the constrained model and the alternative method in the constrained model. We investigate whether one method dominates the other one.

We compare the two methods by changing the values of the risk-free interest rate $r$, the parameter of the power utility function $\gamma$, the floor $\underline{W}$ and the expected returns on the risky assets $\mu$. We change only one parameter at the time, the remaining variables are kept fixed. By default, we set the risk-free interest rate $r = 2\%$, the power utility parameter $\gamma = 5$ and the floor $\underline{W} = 1$. The default expected returns and the covariance matrix are determined based on data analysis, considering the risky assets of McDonald's Corp. (MCD), Johnson & Johnson (JNJ) and Toyota Motor Corporation (TM). We use the daily data from 4th October 2011 to 2nd October 2012 to estimate the expected yearly returns as $\mu = (0.06626, 0.1113, 0.1625)^\top$ and the covariance matrix as

$$c^R = \begin{pmatrix} 0.02155 & 0.00825 & 0.00749 \\ 0.00825 & 0.01517 & 0.01190 \\ 0.00749 & 0.01190 & 0.05011 \end{pmatrix}.$$

We set the initial wealth $W_0 = 1$, the maturity $T = 1$.

Table 1 compares the certainty equivalents of the OBPI in the constrained model and those of the alternative method, calculated as

$$C = \left((1-\gamma)E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} \quad \text{and} \quad Ca = \left((1-\gamma)E\left[\frac{Wa_T^{1-\gamma}}{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}},$$

respectively.

| $r$ | $C$ | $Ca$ |
|---|---|---|
| 1% | 1.03372 | 1.03393 |
| 2% | 1.05016 | 1.05028 |
| 4% | 1.07097 | 1.07091 |

| $\gamma$ | $C$ | $Ca$ |
|---|---|---|
| 3 | 1.05437 | 1.05655 |
| 5 | 1.05016 | 1.05028 |
| 8 | 1.04391 | 1.04386 |

| $\underline{W}$ | $C$ | $Ca$ |
|---|---|---|
| 0.98 | 1.06213 | 1.06240 |
| 1 | 1.05016 | 1.05028 |
| 1.01 | 1.04071 | 1.04078 |

| $\mu$ | $C$ | $Ca$ |
|---|---|---|
| $(0.06626, 0.11130, 0.16250)^\top$ | 1.050159 | 1.050284 |
| $(0.06626, 0.09000, 0.16250)^\top$ | 1.044167 | 1.044124 |
| $(0.06626, 0.09000, 0.18000)^\top$ | 1.047472 | 1.047426 |

**Tab. 1.** Certainty equivalents of the OBPI in the constrained model and those of the alternative method.

We see that there is no exact answer whether one should choose the OBPI in the constrained model or the alternative method. In other words, we can say that none of the methods is optimal in the constrained model.

When changing the interest rate $r$, the parameter of the absolute risk-aversion $\gamma$, or the expected returns $\mu$, none of the methods dominate the other one. When changing the floor $\underline{W}$, in our specific settings, the alternative method dominates the OBPI in constrained model.

## 4. PORTFOLIO INSURANCE WITH A PARTIALLY GUARANTEED FLOOR

In this section, we allow the portfolio to fall under the guaranteed floor with a given probability. We show that the Value-at-Risk based risk management in the constrained model is not admissible and we provide an alternative admissible strategy to it, the portfolio insurance with spreads.

### 4.1. Value-at-Risk based risk management

In case of insuring the portfolio with a put option, the terminal value $W_T$ of the portfolio does not fall under the predefined floor, i. e. $W_T \geq \underline{W}$ with the probability of 100%. Now, let us investigate the case of relaxing the condition

$$P(W_T \geq \underline{W}) = 1$$

and consider instead the probability of not falling under the predefined floor to be greater than $1 - \alpha$, i. e.

$$P(W_T \geq \underline{W}) \geq 1 - \alpha. \tag{25}$$

Inequality (25) represents the so-called Value-at-Risk constraint.

Note that for $\alpha = 1$ the investor behaves as a benchmark agent who does not consider any risk management. If $\alpha = 0$, the investor behaves as a portfolio insurer (securing with put options). In such case the terminal wealth will exceed the floor at all states.

### 4.2. VaR-RM in the unconstrained model

Our goal is to maximize the expected utility from the terminal wealth under the VaR-RM, i. e.

$$\begin{aligned} &\max_{\theta} E[U(W_T)] \qquad (26)\\ s.t. \qquad &P(W_T \geq \underline{W}) \geq 1 - \alpha, \end{aligned}$$

where we maximize through all dynamic strategies $\theta$ and the initial wealth is given as $W_0$.

Basak and Shapiro [1] state in Proposition 3 that the optimal portfolio strategy can be expressed as

$$\theta_t^{VaR} = q_t^{VaR} \hat{\beta}, \tag{27}$$

where $\hat{\beta}$ is the portfolio strategy of the benchmark agent, calculated by (7) and $q_t^{VaR} \geq 0$ is deterministic.

The advantage of focusing on power utility functions is that knowing the optimal strategy $\hat{\beta}$ of the benchmark agent and the ratio $q_t^{VaR}$, which can be calculated from the model settings, one can easily determine the optimal strategy $\theta_t^{VaR}$ of the maximizing problem under VaR-RM at each time $t$.

### 4.3. VaR-RM in the constrained model

Basak and Shapiro [1] derived the VaR-RM model for the portfolio with no strategy constraints. However, the VaR-RM is not admissible when the portfolio strategy is constrained.

Let $\mathcal{C}$ be the set of all admissible portfolio strategies where short-selling is prohibited and the agent is not allowed to borrow risk-free bonds or cash to finance the purchase of further risky assets:

$$\mathcal{C} = \left\{ \theta^i \geq 0, i = 1, 2, \ldots d; \sum_1^d \theta^i \leq 1 \right\}.$$

Using the power utility function, we can define the VaR-RM problem in the constrained model as

$$\begin{aligned} & \max_{\theta^{VaR} \in \mathcal{C}} E\left[\frac{(W_T^{VaR})^{1-\gamma}}{1-\gamma}\right] \\ s.t. \quad & P(W_T^{VaR} \geq \underline{W}) \geq 1-\alpha, \\ & \mathcal{C} = \left\{ (\theta_t^{VaR})^i \geq 0, i = 1, 2, \ldots d; \sum_1^d (\theta_t^{VaR})^i \leq 1, \forall t \in (0, T) \right\}, \end{aligned} \tag{28}$$

with a given initial $W_0$.

By Basak and Shapiro [1] Proposition 3 *ii*), under certain conditions ($t \to T$ and $\xi_t \to \bar{\xi}$), the ratio $q_t^{VaR}$ defined in the equation (27) is approaching infinity with a positive probability. In such case the sum of the portfolio strategy of the VaR-RM is greater than one, i. e.

$$\sum_{i=1}^d (\theta_t^{VaR})^i = q_t^{VaR} \sum_{i=1}^d \beta_t^i \geq 1$$

and hence the Value-at-Risk based risk management is not admissible in the constrained model.

### 4.4. Portfolio insurance with spreads

The Value-at-Risk based risk management was developed for portfolios with no constraints on the portfolio strategy. We showed that such a strategy is useless when constraints are required. Insuring the portfolio with a put spread can eliminate this problem.

According to the VaR-constraint (25) we adjust our strategy in a following way:

- in case the risky asset $X_T$ satisfies the condition, we do not insure the portfolio at all,

- in case the risky asset $X_T$ does not satisfy the condition, we modify the portfolio by buying a put option with strike price $\underline{W}$ and selling a put option with strike price $\underline{\underline{W}}$ such that $P(X_T \geq \underline{\underline{W}}) = 1 - \alpha$.

Formally, we can express the above strategy as

$$W = \begin{cases} X + Put(X_T \geq \underline{W}) - Put(X_T \geq \underline{\underline{W}}) & \text{if } P(X_T \geq \underline{W}) < 1-\alpha, \\ X & \text{if } P(X_T \geq \underline{W}) \geq 1-\alpha. \end{cases} \tag{29}$$

According to this strategy, we leave the worst $\alpha\%$ cases uninsured. Figure 1 depicts the payoff diagram of (29).

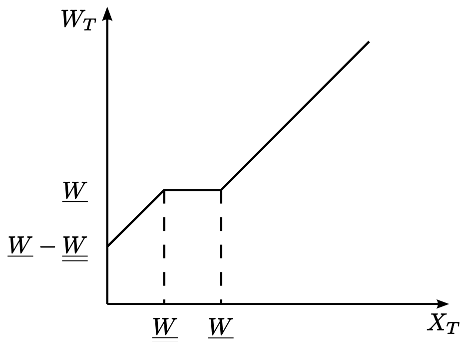


**Fig. 1.** Payoff diagram of the strategy (29).

From Ito's lemma, the condition $P(X_T \geq \underline{W}) \geq 1-\alpha$ can be expressed as $\underline{W} \leq \underline{\underline{W}}$, where $\underline{\underline{W}} = X_0 e^{\Gamma}$ with

$$\Gamma = \left( r + \beta^{\top}(\mu - r\mathbf{1}) - \frac{1}{2}\,\sigma_X^2 \right) T - \,\sigma_X \sqrt{T}\Phi^{-1}(1-\alpha).$$

In case when $\underline{\underline{W}} < \underline{W}$ and the put options are synthesized, we can express the portfolio value as

$$W_t = \left[1 + \varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})\right] X_t + [\psi_t(\underline{W}) - \psi_t(\underline{\underline{W}})]B_t,$$

where $\varphi_t(\underline{W}) = \Phi(d_1(\underline{W})) - 1$ is the delta of the option with strike $\underline{W}$ and $\varphi_t(\underline{\underline{W}}) = \Phi(d_1(\underline{\underline{W}})) - 1$ is the delta of the option with strike $\underline{\underline{W}}$. The difference $\varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})$ is called the hedging ratio.

Note that in case when $\underline{\underline{W}} \geq \underline{W}$, it holds that $W_t = X_t$.

### 4.5. Portfolio insurance with spreads in the unconstrained models

We investigate the problem

$$\begin{aligned} &\max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ s.t. \qquad &P(W_T \geq \underline{W}) \geq 1-\alpha. \end{aligned} \tag{30}$$

In this case, there are no constraints required on the portfolio strategy.

**Theorem 4.1.** Let $\hat{\beta}$ be the optimal portfolio strategy, computed by (7). Then the portfolio strategy defined by

$$\theta_t = \begin{cases} \frac{\left[1+\varphi_t(\underline{W})-\varphi_t(\underline{\underline{W}})\right]\hat{\beta}X_t}{W_t} & \text{if} \quad \underline{\underline{W}} < \underline{W}, \\ \hat{\beta} & \text{if} \quad \underline{\underline{W}} \geq \underline{W} \end{cases} \tag{31}$$

guarantees that $P(W_T \geq \underline{W}) \geq 1 - \alpha$.

The proof is clear from the derivation of the strategy.

### 4.6. Portfolio insurance with spreads in the constrained models

We investigate the problem

$$\begin{aligned} & \max_{\theta} E\left[\frac{W_T^{1-\gamma}}{1-\gamma}\right] \\ s.t. \quad & P(W_T \geq \underline{W}) \geq 1-\alpha, \\ & \mathcal{C} = \left\{\theta^i \geq 0,\ i = 1,2,\dots,d; \sum \theta^i \leq 1\right\}. \end{aligned} \tag{32}$$

We provide an admissible solution for the problem (32) in the following theorem.

**Theorem 4.2.** Let $\hat{\beta}$ be the optimal portfolio strategy with convex constraints, computed by

$$\hat{\beta} = \arg\max_{\beta \in \mathcal{C}} \ \beta^\top(\mu - r\mathbf{1}) - \frac{1}{2}\gamma\beta^\top c^R \beta, \tag{33}$$

where $\mathcal{C}$ is defined as in problem (32). Then the portfolio strategy

$$\theta_t = \begin{cases} \frac{\left[1+\varphi_t(\underline{W})-\varphi_t(\underline{\underline{W}})\right]\hat{\beta}X_t}{W_t} & \text{if} \quad \underline{\underline{W}} < \underline{W}, \\ \hat{\beta} & \text{if} \quad \underline{\underline{W}} \geq \underline{W} \end{cases}$$

is admissible for the problem (32).

P r o o f. Since $\hat{\beta}$ is calculated by (33), it holds that $\hat{\beta}^i \geq 0$ for $i = 1,\dots,d$ and $\sum_{i=1}^d \hat{\beta}^i \leq 1$. In case when $\underline{\underline{W}} < \underline{W}$, the hedging ratio is

$$\varphi(\underline{W}) - \varphi(\underline{\underline{W}}) = \Phi(d_1(\underline{W})) - \Phi(d_1(\underline{\underline{W}})).$$

It can easily be shown that

$$-1 < \Phi(d_1(\underline{W})) - \Phi(d_1(\underline{\underline{W}})) < 0 \quad \text{and} \quad 0 < \frac{X_t}{W_t} < 1.$$

Therefore the vector $\theta_t$ satisfies

$$\theta_t = \frac{\left[1+\varphi_t(\underline{W})-\varphi_t(\underline{\underline{W}})\right]\hat{\beta}X_t}{W_t} \geq \mathbf{0}.$$

Now, we show that $\sum_{i=1}^{d} \theta_t^i \leq 1$:

$$\begin{aligned}\sum_{i=1}^{d} \theta_t^i &= \sum_{i=1}^{d} \left[1 + \varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})\right] \hat{\beta}^i \frac{X_t}{W_t} \\ &\leq \sum_{i=1}^{d} \left[1 + \varphi_t(\underline{W}) - \varphi_t(\underline{\underline{W}})\right] \hat{\beta}^i \\ &\leq \sum_{i=1}^{d} \hat{\beta}^i \leq 1.\end{aligned}$$

In case when $\underline{\underline{W}} \geq \underline{W}$, it holds that $\theta_t = \hat{\beta}$ is admissible for the problem (32). □

### 4.7. Example

Let us compare the portfolio insurance with spreads in the constrained model with the VaR-RM. Since the latter is used only in the unconstrained models, the comparison is not precisely adequate. Despite this fact, it still provides a fair illustration of how the VaR-RM performs better compared to the portfolio insurance with spreads in the constrained model.

We use the same settings as the default parameters in Example 3.4. In addition we set the probability level $\alpha = 0.05$.

Table 2 compares the certainty equivalents of the VaR-RM and those of the portfolio insurance with spreads in the constrained model for different levels of the floor, using $T = 1$ and $T = 3$.

$T = 1$

| $\underline{W}$ | $C^{VaR}$ | $C$ |
|---|---|---|
| 0.98 | 1.089074 | 1.075588 |
| 0.99 | 1.088262 | 1.071753 |
| 1 | 1.087253 | 1.066867 |
| 1.01 | 1.086015 | 1.060392 |
| 1.015 | 1.085303 | 1.056223 |

$T = 3$

| $\underline{W}$ | $C^{VaR}$ | $C$ |
|---|---|---|
| 0.98 | 1.299863 | 1.288040 |
| 0.99 | 1.299829 | 1.285884 |
| 1 | 1.299765 | 1.282696 |
| 1.01 | 1.299665 | 1.278792 |
| 1.015 | 1.299584 | 1.276543 |

**Tab. 2.** Certainty equivalents of the VaR-RM and those of the portfolio insurance with spreads in the constrained model.

The certainty equivalents of the portfolio insurance with spreads are lower than those of the VaR-RM. The average difference between them is approximately 2.1% when $T = 1$ and 1.73% when $T = 3$.

## 5. CONCLUSIONS

The main objective of our work was to examine the portfolio insurance when short-selling of both risky and risk-free assets is prohibited. Our goal was to provide a dynamic portfolio strategy that satisfies such constraints and maximizes the expected utility from the partially guaranteed terminal wealth.

Assuming that the terminal wealth of the portfolio is not allowed to fall under the predefined level with probability one and that short-selling is prohibited, we provided two admissible strategies, the OBPI in the constrained model and the alternative method. Based on the results of the sensitivity analysis, we concluded that none of the methods dominates the other.

Under the assumption that the terminal wealth is partially allowed to fall under the predefined floor, the Value-at-Risk based risk management in the constrained model turned out not to be admissible, hence we provided an alternative to it, the portfolio insurance with spreads, which is an admissible solution. Even though comparing the VaR-RM and the portfolio insurance with spreads is not quite fair, the example illustrated that the certainty equivalents of the portfolio insurance with spreads are lower than the certainty equivalents of the VaR-RM.

## ACKNOWLEDGEMENT

This work was supported by grant Vega 1/2429/12.

(Received July 24, 2013)

## REFERENCES

[1] S. Basak and A. Shapiro: Value-at-Risk based risk management: optimal policies and asset prices. Rev. Financ. Stud. *14* (2001), 371–405.

[2] M. Baxter and A. Rennie: Financial Calculus. Cambridge University Press, Cambridge 1996.

[3] P. Bertrand and J.-L. Prigent: Portfolio insurance strategies: Obpi versus Cppi. University of CERGY Working Paper No. 2001-30; GREQAM Working Paper (December 2001), available at SSRN: `http://ssrn.com/abstract=299688`.

[4] N. H. Hakansson: Optimal investment and consumption strategies under risk for a class of utility functions. Econometrica *38* (1970), 5, 587–607.

[5] H. E. Leland and M. Rubinstein: The evolution of portfolio insurance. In: The Evolution of Portfolio Insurance (D. L. Lushin, ed.), Wiley Sons, New York 1976.

[6] H. E. Leland and M. Rubinstein: Replicating options with positions in stock and cash. Financ. Anal. J. *37* (1981), 4, 63–71.

[7] Cs. Krommerová: Expected utility maximization with risk management and strategy constraints. In: Zborník z prvého česko-slovenského workshopu mladých ekonómov (2012), electronic document, pp. 1–21.

[8] R. Mehra and E. Prescott: The equity premium: a puzzle. J. Monetary Economics *15* (1985), 145–161.

[9] R. C. Merton: Lifetime portfolio selection under uncertainty: the continuous-time case. Rev. Econom. Statist. *51* (1969), 3, 247–257.

[10] M. Nutz: Power utility maximization in constrained exponential Lévy models. Math. Finance *22* (2012), 4, 690–709.

[11] M. Nutz: The Bellman equation for power utility maximization with semimartingales. Ann. Appl. Probab. *22* (2012), 1, 363–406.

[12] A. Perold and W. F. Sharpe: Dynamic strategies for asset allocation. Financ. Anal. J. *44* (1988), 1, 16–27.

[13] J.-L. Prigent: Portfolio Optimization and Performance Analysis. Chapman and Hall/CRC Financial Mathematics Series, Boca Raton 2007.

[14] P. A. Samuelson: Lifetime porfolio selection by dynamic stochastic programming. Rev. Econom. Statist. *51* (1969), 3, 239–246.

*Csilla Krommerová, Comenius University, Faculty of Mathematics, Physics and Informatics, Mlynská dolina, 842 48 Bratislava. Slovak Republic.*
*e-mail: csilla.krommerova@gmail.com*

*Igor Melicherčík, Comenius University, Faculty of Mathematics, Physics and Informatics, Mlynská dolina, 842 48 Bratislava. Slovak Republic.*
*e-mail: igor.melichercik@fmph.uniba.sk*