ON A FUNCTIONAL EQUATION CONNECTED TO THE DISTRIBUTIVITY OF FUZZY IMPLICATIONS OVER TRIANGULAR NORMS AND CONORMS

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Distributivity of fuzzy implications over different fuzzy logic connectives have a very important role to play in efficient inferencing in approximate reasoning, especially in fuzzy control systems (see [9, 15] and [4]). Recently in some considerations connected with these distributivity laws, the following functional equation appeared (see [5])

$$f(\min(x+y,a)) = \min(f(x)+f(y),b),$$

where $a,b>0$ and $f\colon [0,a]\to[0,b]$ is an unknown function. In this paper we consider in detail a generalized version of this equation, namely the equation

$$f(m_1(x+y)) = m_2(f(x)+f(y)),$$

where m_1, m_2 are functions defined on some intervals of $\mathbb{R}$ satisfying additional assumptions. We analyze the cases when m_2 is injective and when m_2 is not injective.

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1. INTRODUCTION

Distributivity of fuzzy implication functions over different fuzzy logic connectives has been thoroughly investigated in recent past by many authors (see [1, 2, 3, 5, 6, 8, 20, 21, 22, 23, 24]). In general we can consider four such distributivity equations:

$$I(x, C_1(y,z)) = C_2(I(x,y), I(x,z)), \tag{D1}$$

$$I(x, D_1(y,z)) = D_2(I(x,y), I(x,z)), \tag{D2}$$

$$I(C(x,y), z) = D(I(x,z), I(y,z)), \tag{D3}$$

$$I(D(x,y), z) = C(I(x,z), I(y,z)), \tag{D4}$$

satisfied for all $x,y,z\in[0,1]$, where I is some generalization of classical implication, C, C_1, C_2 are some generalizations of classical conjunction and D, D_1, D_2 are some generalizations of classical disjunction.

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The importance of such equations in fuzzy control and fuzzy systems has been firstly emphasized by Combs and Andrews [9], wherein they exploit the following classical tautology

$$(p \wedge q) \to r \equiv (p \to r) \vee (q \to r),$$

in their inference mechanism towards reduction in the complexity of fuzzy "IF-THEN" rules. Subsequently, there were many discussions [10, 11, 13, 19], most of them pointing out the need for a theoretical investigation required for employing such equations. Later, a similar method but for similarity based reasoning was demonstrated by Jayaram [15]. For the details and concrete examples see also [4, Section 8.5].

Let us have a closer look at the situation, when C, C_1 and C_2 are continuous Archimedean triangular norms, while D, D_1 and D_2 are continuous Archimedean triangular conorms. It is well known that every continuous Archimedean triangular norm T is of the form

$$T(x,y) = t^{-1}(\min(t(x)+t(y),t(0))), \qquad x,y \in [0,1],$$

where $t\colon [0,1] \to [0,\infty]$ is a continuous, strictly decreasing function with $t(1)=0$, while every continuous Archimedean triangular conorm S is of the form

$$S(x,y) = s^{-1}(\min(s(x)+s(y),s(1))), \qquad x,y \in [0,1],$$

where $s\colon [0,1] \to [0,\infty]$ is a continuous, strictly increasing function with $s(0)=0$ (see Ling [18] and Klement et. al [16]). If we use these representations in the above distributivity laws (D1) – (D4), then we obtain the following four equations

$$\begin{aligned}
f_x(\min(t_1(y)+t_1(z),t_1(0))) &= \min(f_x(t_1(y))+f_x(t_1(z)),t_2(0)),\\
g_x(\min(s_1(y)+s_1(z),s_1(1))) &= \min(g_x(s_1(y))+g_x(s_1(z)),s_2(1)),\\
h^z(\min(t(x)+t(y),t(0))) &= \min(h^z(s(x))+h^z(s(y)),s(1)),\\
k^z(\min(s(x)+s(y),s(1))) &= \min(k^z(t(x))+k^z(t(y)),t(0)),
\end{aligned}$$

where

- t_1, t_2, t are functions occurring in the representations of T_1, T_2, T, respectively,
- s_1, s_2, s are functions occurring in the representations of S_1, S_2, S, respectively,
- $f_x(\,\cdot\,) = t_2 \circ I(x, t_1^{-1}(\,\cdot\,))$, for a fixed $x \in [0,1]$,
- $g_x(\,\cdot\,) = s_2 \circ I(x, s_1^{-1}(\,\cdot\,))$, for a fixed $x \in [0,1]$,
- $h^z(\,\cdot\,) = s \circ I(t^{-1}(\,\cdot\,), z)$, for a fixed $z \in [0,1]$,
- $k^z(\,\cdot\,) = t \circ I(s^{-1}(\,\cdot\,), z)$, for a fixed $z \in [0,1]$.

The first equation may be written in the following form

$$f_x(\min(u+v,t_1(0))) = \min(f_x(u)+f_x(v),t_2(0)),$$

where $u, v \in [0, t_1(0)]$, and f_x is an unknown function. The second equation may be written in the form

$$g_x(\min(u+v, s_1(1))) = \min(g_x(u) + g_x(v), s_2(1)),$$

here $u, v \in [0, s_1(1)]$, and g_x is an unknown function. The other equations can be written in a similar way. Thus, in the paper [5], authors have found the general form of $f\colon [0, r_1] \to [0, r_2]$ (for fixed $r_1, r_2 \in (0, \infty)$) satisfying the functional equation

$$f(\min(x+y, r_1)) = \min(f(x) + f(y), r_2). \tag{1}$$

This article extends significantly the results obtained before in the conference article [7], where we have considered the generalized version of this equation i. e., we have replaced functions $\min(\cdot, r_1)$, $\min(\cdot, r_2)$ occurring directly in this equation, by functions m_1, m_2 satisfying some assumptions. This means that we study here the following equation

$$f(m_1(x+y)) = m_2(f(x) + f(y)). \tag{2}$$

In particular, in this paper we present the full proofs of Lemma 3.1 and Theorem 3.2. Moreover, we shall not only find the general form of a function f, but we shall also prove that functions m_1 and m_2 must satisfy some properties, if we want the equation (2) to have some nontrivial solutions f. We believe that the results obtained in this article are not only theoretical, but they can be used in the future also in fuzzy control and approximate reasoning or in other theories like fuzzy mathematical morphology (see [12] or [14]), where solutions of functional equations play an important role.

2. SOLUTIONS OF (2) WHEN m_2 IS INJECTIVE

First we consider the situation when m_2 is injective (in particular it is a bijection).

Lemma 2.1. Let $r_1, r_2 \in (0, \infty)$ be some real numbers and let $m_1\colon [0, 2r_1] \to [0, r_1]$, $m_2\colon [0, 2r_2] \to [0, r_2]$ be given functions. If m_2 is injective and a function $f\colon [0, r_1] \to [0, r_2]$ satisfies the functional equation (2), then f satisfies the Jensen equation, i. e.,

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y), \qquad x, y \in [0, r_1]. \tag{3}$$

Proof. From (2) we obtain

$$m_2^{-1}\left(f(m_1(x+y))\right) = f(x) + f(y), \qquad x, y \in [0, r_1],$$

and putting $F(t) := m_2^{-1}(f(m_1(t)))$, for $t \in [0, 2r_1]$, we get

$$F(x+y) = f(x) + f(y), \qquad x, y \in [0, r_1]. \tag{4}$$

Now, if we take any $x, y \in [0, r_1]$, then from (4) we have

$$f(x) + f(y) = F(x+y) = F\left(\left(\frac{x+y}{2}\right) + \left(\frac{x+y}{2}\right)\right) = 2f\left(\frac{x+y}{2}\right),$$

and therefore f satisfies (3). □

Theorem 2.2. Let $r_1, r_2 \in (0, \infty)$ be some numbers and let $m_1 \colon [0, 2r_1] \to [0, r_1]$, $m_2 \colon [0, 2r_2] \to [0, r_2]$, $f \colon [0, r_1] \to [0, r_2]$ be given functions. Further, let m_2 be injective. Then the following sentences are equivalent:

(i) The triple of functions m_1, m_2, f satisfies the equation (2).

(ii) Either $f = b$ for some $b \in [0, r_2]$ and $m_2(2b) = b$, or $f(x) = ax + b$ for some $a, b \in \mathbb{R}$, $a \neq 0$ such that

$$ax + b \in [0, r_2], \qquad \text{for all } x \in [0, r_1] \tag{5}$$

and

$$m_1(x) = \frac{m_2(ax + 2b) - b}{a}. \tag{6}$$

Proof. *(ii)* $\Longrightarrow$ *(i)* It is easy to check that these functions satisfy (2). Indeed, in the case $f(x) = b$ our equation is satisfied provided that $m_2(2b) = b$. In the second case, for all $x, y \in [0, r_1]$, we have

$$\begin{aligned} f(m_1(x+y)) = am_1(x+y) + b &= a\frac{m_2(a(x+y) + 2b) - b}{a} + b \\ &= m_2(ax + b + ay + b) = m_2(f(x) + f(y)). \end{aligned}$$

(i) $\Longrightarrow$ *(ii)* From Lemma 2.1 we obtain that f satisfies the Jensen equation (3). However, since f is bounded, there exist $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ (see [17, Theorem III.2.2]). If we consider the case $a = 0$, then $f(x) = b$ for all $x \in [0, r_1]$ and from (2) we obtain that $m_2(2b) = b$. If we assume that $a \neq 0$, then using the form of f in (2) we have

$$am_1(x + y) + b = m_2(ax + b + ay + b)$$

and, taking here $y = 0$, we obtain

$$am_1(x) + b = m_2(ax + 2b)$$

which yields the equality (6). Clearly, the condition (5) must be satisfied, since f is defined on $[0, r_1]$ and takes values in $[0, r_2]$. □

3. SOME SOLUTIONS OF (2) WHEN m_2 IS NOT INJECTIVE

In the case when m_2 is not injective we will have some additional assumptions on functions m_1 and m_2. We start our discussion with the following result.

Lemma 3.1. Let $r_1, r_2 \in (0, \infty)$ be some numbers and let functions $m_1 \colon [0, 2r_1] \to [0, r_1]$, $m_2 \colon [0, 2r_2] \to [0, r_2]$ be continuous (on their whole domains) and strictly increasing on some intervals $[0, x_1]$, $[0, x_2]$, respectively, and then be equal respectively to r_1, r_2 on intervals $[x_1, 2r_1]$, $[x_2, 2r_2]$, where $x_1 \leq r_1$ and $x_2 \leq r_2$. Further, let m_1, m_2 satisfy

$$m_1(0) = 0, \qquad 2m_1(x) > x, \quad x \in (0, 2r_1) \tag{7}$$

and

$$m_2(0) = 0, \qquad 2m_2(x) > x, \quad x \in (0, 2r_2). \tag{8}$$

If a function $f\colon [0, r_1] \to [0, r_2]$ satisfies (2), then one of the following conditions is satisfied:

(i) $f = r_2$;

(ii) $f = 0$;

(iii) $f(0) = 0,\ f(x) = r_2$ for $x > 0$;

(iv) there exists $x_0 \in (0, x_1]$ such that $f(x) = \dfrac{x_2}{x_0}x$ for $x < x_0$ and $f(x) \geq x_2$ for $x \geq x_0$ (in particular $f(r_1) = r_2$).

Proof. Putting $y = 0$ in (2), we obtain

$$f(m_1(x)) = m_2(f(x) + f(0)), \tag{9}$$

for $x \in [0, r_1]$ and, using this equality in (2), we arrive at

$$m_2(f(x + y) + f(0)) = m_2(f(x) + f(y)), \tag{10}$$

for $x, y, x + y \in [0, r_1]$. If we take $x = 0$ in (9), then, by (7), we get $f(0) = m_2(2f(0))$ which means, by (8), that either $f(0) = 0$ or $f(0) = r_2$.

Firstly we consider the case $f(0) = r_2$. From (9) we get $f(m_1(x)) = m_2(f(x) + r_2)$, hence $f(m_1(x)) = r_2$, since $f(x) + r_2 \geq r_2 \geq x_2$. This simply means that $f(x) = r_2$ for $x \in m_1([0, r_1])$. However, since m_1 is continuous and $x_1 \leq r_1$, we have $m_1([0, r_1]) = [0, r_1]$ and, consequently, $f(x) = r_2$ for all $x \in [0, r_1]$.

Now let us consider the case $f(0) = 0$. Then from (10) we have

$$m_2(f(x + y)) = m_2(f(x) + f(y)), \tag{11}$$

for all $x, y, x + y \in [0, r_1]$. But from (2) we get

$$f(r_1) = f(m_1(r_1 + r_1)) = m_2(2f(r_1))$$

and, in view of (8), this means that $f(r_1) \in \{0, r_2\}$. If $f(r_1) = 0$, then from (11) we have, for $x \in (0, r_1)$,

$$0 = m_2(0) = m_2(f(r_1)) = m_2(f(x + (r_1 - x))) = m_2(f(x) + f(r_1 - x))$$

and therefore $f(x) = 0$ for $x \in (0, r_1)$. Thus in this case we obtain $f = 0$.

Consequently, we may assume that $f(0) = 0$ and $f(r_1) = r_2$. Observe that if $x \in [0, r_1]$ is such that $f(x) \geq x_2$, then using (11) and the monotonicity of m_2, we obtain for every $y \in [0, r_1]$, $y \geq x$

$$m_2(f(y)) = m_2(f(x + (y - x))) = m_2(f(x) + f(y - x)) \geq m_2(f(x)) \geq m_2(x_2) = r_2,$$

which is equivalent to $f(y) \geq x_2$. Therefore we may take

$$x_0 := \inf\{x \in [0, r_1] \ : \ m_2(f(x)) = r_2\} = \inf\{x \in [0, r_1] \ : \ f(x) \geq x_2\},$$

and for all $x > x_0$ we have $f(x) \geq x_2$.

We will show that $x_0 \leq x_1$. Indeed, we have

$$m_2(f(x_1)) = f(m_1(x_1)) = f(r_1) = r_2,$$

which means that $f(x_1) \geq x_2$ and, in view of the definition of x_0, we obtain the desired inequality.

If $x_0 = 0$, then $m_2(f(x)) = r_2$ for $x > 0$. Since $f(0) = 0$, from (9) we have $f(m_1(x)) = r_2$ for $x > 0$, thus $f(z) = r_2$ for all $z > 0$ and we obtain next solution (iii) in this case. Now assume that $x_0 > 0$ and take $x, y \in [0, \frac{x_0}{2})$, then $f(x), f(y), f(x+y) < x_2$ and since m_2 is injective on the interval $(0, x_2)$ we have, from (11),

$$f(x+y) = f(x) + f(y).$$

This means that the Cauchy equation is satisfied for $x, y \in [0, \frac{x_0}{2})$ and from [17, Theorem XIII.3.3], we know that f can be uniquely extended on $\mathbb{R}$ to an additive function. Moreover, f is bounded and therefore

$$f(x) = kx, \qquad x \in [0, x_0),$$

for some $k \in \mathbb{R}$.

Now we shall show that $k \leq \frac{x_2}{x_0}$. Indeed, if we had $k > \frac{x_2}{x_0}$, then for some $x \in (0, x_0)$

$$f(x) > x_2,$$

i. e.,

$$m_2(f(x)) = r_2,$$

which contradicts the definition of x_0. To finish the proof it suffices to show that $k \geq \frac{x_2}{x_0}$. Assume for the indirect proof that $k < \frac{x_2}{x_0}$. Then we may take $x, y \in (0, x_0)$ such that $x + y > x_0$ and $k(x+y) < x_2$. Consequently, we have

$$r_2 = m_2(f(x+y)) = m_2(f(x) + f(y)) = m_2(kx + ky) < r_2,$$

a contradiction. □

It is also possible to obtain some sufficient conditions, as the following theorem will show.

Theorem 3.2. Let f, m_1, m_2 satisfy the assumptions of Lemma 3.1. If that triple of functions satisfies the equation (2), then one of the following possibilities is satisfied:

(i) $f = r_2$;

(ii) $f = 0$;

(iii) $f(0) = 0, f(x) \geq x_2$ for $x > 0$ and $f(r_1) = r_2$;

(iv) there exists $x_0 \in (0, x_1]$ such that $f(x) \geq x_2$ for $x \geq x_0$, $f(x) = r_2$ for $x \in [m_1(x_0), r_1]$ and $f(x) = \frac{x_2}{x_0}x$ for $x < x_0$. Moreover in this case there exists exactly one $y_0 \leq x_1$ such that $m_1(y_0) = x_0$ and

$$m_1(x) = \frac{x_0 m_2(\frac{x_2}{x_0}x)}{x_2},$$

for $x < y_0$.

Conversely, if we add to (iv) the assumption that $y_0 = x_0$ or $f(m_1(x)) = m_2(f(x))$ for $x \in [y_0, x_0)$, then each of the triples of functions described above satisfies the equation (2).

Proof. In view of Lemma 3.1 we only have to show that if (i), (ii) and (iii) are not satisfied, then $f(x) = r_2$ for $x \geq m_1(x_0)$ and that $m_1(x) = \frac{m_2(kx)}{k}$ for $x < y_0$ (where $k := \frac{x_2}{x_0}$). To end this let us take $x \geq x_0$. This implies $f(x) \geq x_2$ and then from (2) we have

$$r_2 = m_2(f(x)) = m_2(f(x) + f(0)) = f(m_1(x + 0)) = f(m_1(x)).$$

Function m_1 is increasing and continuous, thus $f([m_1(x_0), r_1]) = \{r_2\}$.

Now let us notice that $y_0 \leq x_0$. This is true, because for all $x \geq x_0$ we have $f(m_1(x)) = r_2$. From Lemma 3.1 we have

$$f(x) \geq x_2 \Leftrightarrow x \geq x_0.$$

Thus $m_1(x) \geq x_0$ for all $x \geq x_0$ and we get $m_1(x_1) \geq m_1(x_0) \geq x_0 = m_1(y_0)$. Since m_1 is strictly increasing on the interval $[0, x_1]$ we get $y_0 \leq x_0 \leq x_1$. Now it is easy to check, that

$$m_1(x) = \frac{x_0 m_2(\frac{x_2}{x_0}x)}{x_2},$$

if we put $x < y_0 (\leq x_0)$ into the equation (2).

Finally we prove the second part of Theorem 3.2 – that the obtained functions, with additional assumptions in the case (iv), satisfy (2).

Cases (i), (ii) and (iii) are obvious, we consider only the case (iv). Take $x, y \in [0, r_1]$ and consider four cases:

1. $x, y, x + y < y_0$. Then $m_1(x), m_1(y), m_1(x + y) < x_0$ and with $k = \frac{x_2}{x_0}$ we have

$$\begin{aligned} f(m_1(x + y)) &= k m_1(x + y) = k\frac{m_2(k(x + y))}{k} \\ &= m_2(kx + ky) = m_2(f(x) + f(y)). \end{aligned}$$

2. $x \geq x_0$. Then we have

$$m_2(f(x) + f(y)) \geq m_2(f(x)) = r_2$$

and $f(m_1(x + y)) = r_2$ since $x + y > x_0$ and therefore $m_1(x + y) \geq m_1(x_0)$.

3. $x, y < x_0, x + y \geq x_0$. In this case we have $k(x+y) = \frac{x_2}{x_0}(x+y) \geq x_2$, thus
$$m_2(f(x) + f(y)) = m_2(kx + ky) = m_2(k(x+y)) = r_2$$
and
$$f(m_1(x+y)) = r_2.$$
4. $x, y < x_0$ and $x + y \in [y_0, x_0]$. This case we split into two subcases, according to an additional assumption in the converse to Theorem 3.2:
   (a) If $f(m_1(z)) = m_2(f(z))$ for $z \in [y_0, x_0)$, then if we put $z = x + y$, we obtain:
$$f(m_1(x+y)) = m_2(f(x+y)) = m_2(f(x) + f(y)).$$
   The last equation results from (11).
   (b) If $x_0 = y_0$, then $x + y = x_0$, thus
$$m_2(f(x) + f(y)) = m_2(kx + ky) = m_2(k(x+y)) = m_2(kx_0) = m_2(x_2) = r_2$$
   and
$$f(m_1(x+y)) = f(m_1(x_0)) = r_2.$$

□

Remark 3.3. We will show that the additional assumption in the converse to Theorem 3.2 (i. e., $y_0 = x_0$ or $f(m_1(x)) = m_2(f(x))$ for $x \in [y_0, x_0)$) is necessary that is, we will point out a triple of functions m_1, m_2, f such that they have all the properties enumerated in (iv) of the last theorem, but the functional equation (2) does not hold. Let $r_1 = r_2 = 1$ and $m_1(x) = \min(\sqrt{x}, 1)$ for $x \in [0, 2]$, and
$$m_2(x) = \begin{cases} \sqrt{x}, & x \leq \frac{1}{16}, \\ 4x, & \frac{1}{16} < x \leq \frac{1}{4}, \\ 1, & \frac{1}{4} < x \leq 2. \end{cases}$$
Let us consider
$$f(x) = \begin{cases} x, & x \leq \frac{1}{4}, \\ 3x - \frac{1}{2}, & \frac{1}{4} < x \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < x \leq 1. \end{cases}$$
The plots of these three functions are presented in Figure 1.

Thus $x_1 = 1$ and $x_2 = \frac{1}{4}$. It is easy to check that m_1 and m_2 satisfy assumptions of Lemma 3.1. Next, it is easy to see that also $x_0 = \frac{1}{4}$. Now we check that the above functions satisfy the conditions given in (iv) in Theorem 3.2. Of course $f(x) \geq \frac{1}{4}$ for $x \geq \frac{1}{4}$. We see that $f(x) = 1$ for $x \in [\frac{1}{2}, 1]$ and since $m_1\left(\frac{1}{4}\right) = \frac{1}{2}$ we get $f(x) = r_2$ for $x \in [m_1(x_0), r_1]$. Also $f(x) = x = \frac{x_2}{x_0}x$ for $x < x_0 = \frac{1}{4}$. Finally, $m_1\left(\frac{1}{16}\right) = \frac{1}{4}$ and
$$\frac{x_0 m_2(\frac{x_2}{x_0}x)}{x_2} = \frac{\frac{1}{4}m_2(x)}{\frac{1}{4}} = m_1(x) = \sqrt{x},$$

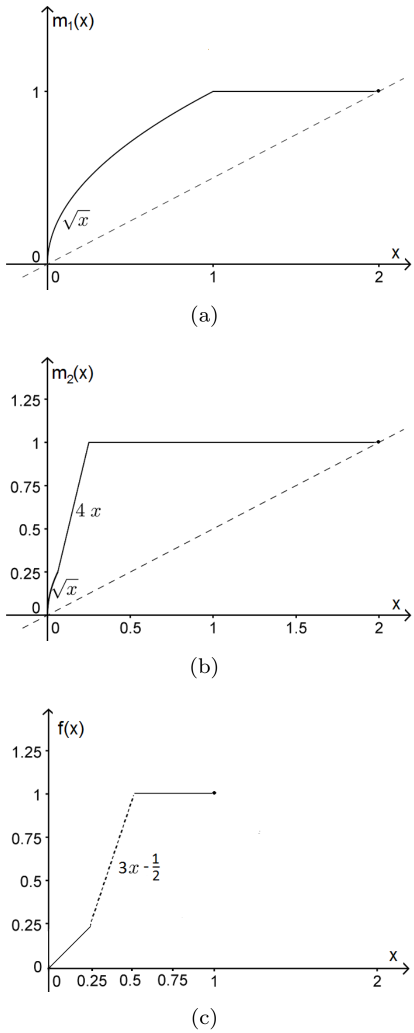

Fig. 1. Functions (a) m_1, (b) m_2 and (c) f from Remark 3.3.

for all $x < \frac{1}{16}$. However, the equation (2)

$$f(m_1(x+y)) = m_2(f(x)+f(y))$$

does not hold for all x, y. Indeed, for example

$$f\left(m_1\left(\frac{1}{16}+\frac{1}{16}\right)\right) = f\left(m_1\left(\frac{1}{8}\right)\right) = f\left(\frac{\sqrt{2}}{4}\right) = \frac{3\sqrt{2}}{4} - \frac{1}{2},$$

while

$$m_2\left(f\left(\frac{1}{16}\right)+f\left(\frac{1}{16}\right)\right)=m_2\left(\frac{1}{16}+\frac{1}{16}\right)=m_2\left(\frac{1}{8}\right)=\frac{1}{2}.$$

We conclude that in order to obtain the equivalence in Theorem 3.2, we have to add an artificial condition to the case (iv) that $x_0 = y_0$ or simply that (2) is satisfied on the interval $[y_0, x_0)$. The question of a complete characterization of the solutions of the equation (2) remains open.

Remark 3.4. In the case *(iv)* of Theorem 3.2, we know additionally that function f must be continuous and increasing on its whole domain $[0, r_1]$ (more precisely, for $x \in [0, m_1(x_0))$ the function f is strictly increasing and for $x \in [m_1(x_0), r_1]$ the function f is constant).

Proof. For $x \in [0, x_0)$ function $f(x) = kx$ is continuous and strictly increasing.
For $x \in [m_1(x_0), r_1]$ function $f(x) = r_2$ is constant.
Thus we only have to show that the function f is strictly increasing and continuous on the interval $[x_0, m_1(x_0))$. Let $y_1, y_2 \in [x_0, m_1(x_0))$, $y_1 < y_2$. The function m_1 is continuous and strictly increasing on $[0, x_0)$, so there exist $z_1, z_2 \in [0, x_0)$, such that $m_1(z_1) = y_1$, $m_1(z_2) = y_2$ and $z_1 < z_2$. In the case (iv) of Theorem 3.2 the following equation is satisfied

$$f(m_1(x)) = m_2(f(x)),$$

thus $f(y_1) = f(m_1(z_1)) = m_2(f(z_1)) = m_2(kz_1)$ and $f(y_2) = f(m_1(z_2)) = m_2(f(z_2)) = m_2(kz_2)$. Therefore we have

$$f(y_1) < f(y_2) \Leftrightarrow m_2(kz_1) < m_2(kz_2) \Leftrightarrow kz_1 < kz_2 \Leftrightarrow z_1 < z_2,$$

which ends the proof of f being strictly increasing.
Similarly one can show the continuity of f on the interval $[x_0, m_1(x_0)]$ using the continuity of functions m_1, m_2 on their domains and f on the interval $[0, x_0]$ and from the equation $f(m_1(x)) = m_2(f(x))$. □

4. EXAMPLES

In this section we will discuss three examples which show how our results can be used with respect to some particular functions m_1 and m_2.

Example 4.1. Let us fix arbitrarily $r_1, r_2 > 0$ and $\alpha \geq 1$. Let us consider the case $m_1(x) = \min(\alpha x, r_1)$ for $x \in [0, 2r_1]$ and $m_2 = \min(\alpha x, r_2)$ for $x \in [0, 2r_2]$. In this case we obtain the following equation

$$f(\min(\alpha(x+y), r_1)) = \min(\alpha(f(x) + f(y)), r_2).$$

We will show that from Theorem 3.2 we obtain the following solutions:

(i) $f = r_2$;

(ii) $f = 0$;

(iii) $f(x) = \begin{cases} 0, & x = 0 \\ r_2, & x > 0 \end{cases};$

(iv) $f(x) = \min(kx, r_2)$, where $k = \frac{r_2}{\alpha x_0}$.

We only need to prove that in the case *(iv)* of Theorem 3.2 the only solution is $f(x) = \min(kx, r_2)$. We have

$$x_i = \min\{x \in [0, r_i] : m_i(x) = r_i\} \qquad \text{for } i = 1, 2,$$

i. e.

$$x_1 = \frac{r_1}{\alpha}, x_2 = \frac{r_2}{\alpha} \text{ and } k = \frac{x_2}{x_0} = \frac{r_2}{\alpha x_0}.$$

In this case from $f(m_1(x)) = m_2(f(x))$ we obtain the following equation

$$f(\min(\alpha x, r_1)) = \min(\alpha f(x), r_2)$$

- Let $x < x_0$. Then $\min(\alpha f(x), r_2) = \min(\alpha kx, r_2) = \alpha kx$, because $\alpha kx = \alpha \frac{r_2}{\alpha x_0} x = \frac{x}{x_0} r_2 < r_2$
- Let $x < x_1$. Then $f(\min(\alpha x, r_1)) = f(\alpha x)$, because $\alpha x < \alpha x_1 = \alpha \frac{r_1}{\alpha} = r_1$.

Thus for $x < \min(x_0, x_1) = x_0$ we obtain $f(\alpha x) = \alpha kx$, which means that for $y < \alpha x_0$ we have $f(y) = ky$. We know from the Proposition 3.4, that function f is continuous and increasing, so $f(\alpha x_0) = k\alpha x_0 = \frac{r_2}{\alpha x_0} \alpha x_0 = r_2$ and $f(y) = r_2$ for $y > \alpha x_0$. Finally we obtain $f(x) = \min(kx, r_2)$.

The plots of functions m_1, m_2 and f with $r_1 = 1, r_2 = \frac{3}{2}$ and $\alpha = \frac{3}{2}$ are presented in Figure 2.

Example 4.2. Let us fix arbitrarily $r_1, r_2 > 0$ and let $m_1(x) = \min(\sqrt{r_1 x}, r_1), m_2(x) = \min(\sqrt{r_2 x}, r_2)$. In this case we obtain the following equation

$$f\left(\min(\sqrt{r_1(x+y)}, r_1)\right) = \min\left(\sqrt{r_2(f(x) + f(y))}, r_2\right) \tag{12}$$

and from Theorem 3.2 we get that only nontrivial continuous solution is

$$f(x) = \frac{r_2}{r_1} x.$$

We obtain $x_1 = r_1$ and $x_2 = r_2$ from the form of functions m_1 and m_2. The only one nontrivial solution appears in the case *(iv)* of Theorem 3.2. For $y = 0$ the equation (12) gives:

$$f(\min(\sqrt{r_1 x}, r_1)) = \min(\sqrt{r_2 f(x)}, r_2).$$

Using an analogous argument to the one from the previous example we obtain for $x < \min(x_0, x_1) = x_0$ an expression $f(\sqrt{r_1 x}) = \sqrt{r_2 kx}$. For sufficiently small x, precisely for x such that $\sqrt{r_1 x} < x_0$, we have $f(\sqrt{r_1 x}) = k\sqrt{r_1 x}$. Thus for those x we obtain $\sqrt{r_2 kx} = k\sqrt{r_1 x}$, therefore $k = \frac{r_2}{r_1}$. However $k = \frac{x_2}{x_0} = \frac{r_2}{x_0}$. Thus $x_0 = r_1$ and finally we have $f(x) = kx = \frac{r_2}{r_1} x$ for $x < x_0 = r_1$.

The plots of functions m_1, m_2 and f with $r_1 = 1, r_2 = \frac{3}{2}$ are presented in Figure 3.

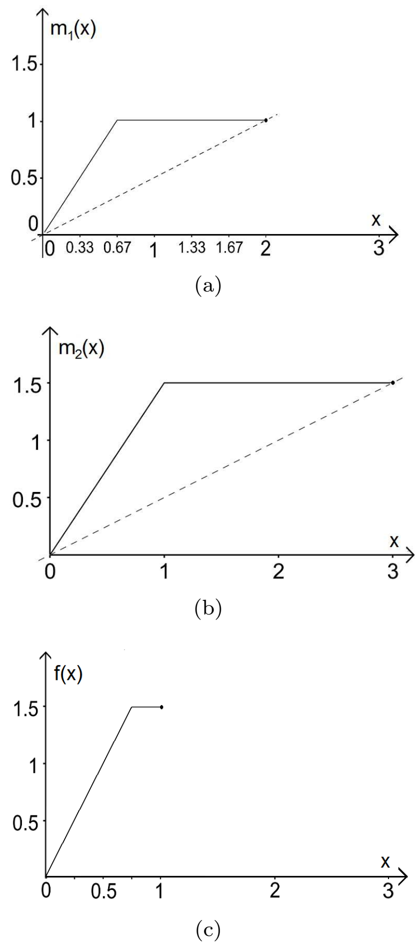

Fig. 2. Functions (a) m_1, (b) m_2 and (c) f from Example 4.1.

Example 4.3. Let us fix arbitrarily $r_1, r_2 > 0$ and $\alpha, \beta \geq 1$. Let us consider the case

$$m_1(x) = \begin{cases} r_1 \sin\left(\frac{\pi}{2}\frac{\alpha}{r_1}x\right), & x < \frac{r_1}{\alpha} \\ r_1, & x \geq \frac{r_1}{\alpha} \end{cases}, \qquad \text{for } x \in [0, 2r_1],$$

and

$$m_2(x) = \begin{cases} r_2 \sin\left(\frac{\pi}{2}\frac{\beta}{r_2}\, x\right), & x < \frac{r_2}{\beta} \\ r_2, & x \geq \frac{r_2}{\beta} \end{cases}, \qquad \text{for } x \in [0, 2r_2].$$

We will show that using Theorem 3.2 we obtain the following solutions of equation (2) (with just defined functions m_1 and m_2):

(i) $f = r_2$;

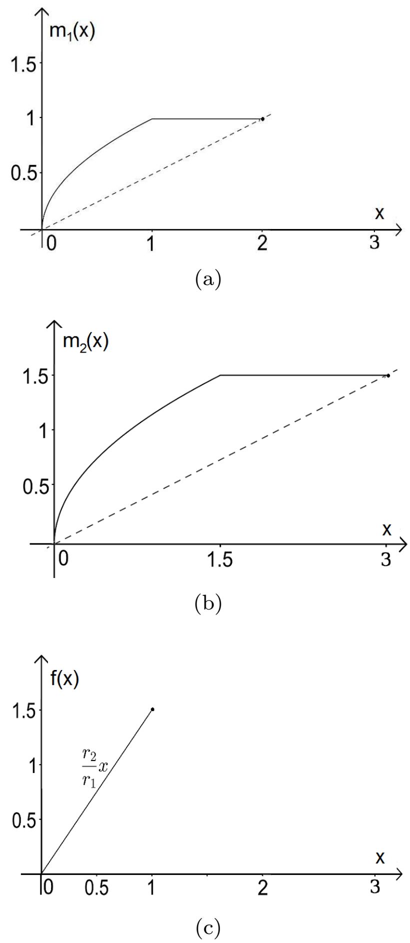

Fig. 3. Functions (a) m_1, (b) m_2 and (c) f from Example 4.2.

(ii) $f = 0$;

(iii) $f(x) = \begin{cases} 0, & x = 0 \\ r_2, & x > 0 \end{cases}$;

(iv) $f(x) = kx$, where $k = \frac{r_2}{r_1}$.

Moreover, we will show that the last solution can be obtained only when $\alpha = \beta$.

We just need to prove that in the case (iv) of Theorem 3.2 the only solution is $f(x) = \frac{r_2}{r_1}x$. From Theorem 3.2 we know that in this case $f(0) = 0$ and $f(x) = kx = \frac{x_2}{x_0}x$ for $x < x_0$.

We obtain $x_1 = \frac{r_1}{\alpha}$, $x_2 = \frac{r_2}{\beta}$ and $k = \frac{x_2}{x_0} = \frac{r_2}{\beta x_0}$ from the form of functions m_1 and m_2.

Because of $f(0) = 0$ the following equation is held for all $x \in [0, r_1]$

$$f(m_1(x)) = m_2(f(x)).$$

Using an analogous argument to the one from the first example we obtain for $x < \min(x_0, x_1) = x_0$ the following equation

$$L = R \Longleftrightarrow f\left(r_1 \sin\left(\frac{\pi}{2}\frac{\alpha}{r_1}x\right)\right) = r_2 \sin\left(\frac{\pi}{2}\frac{1}{x_0}x\right). \tag{13}$$

For sufficiently small x, precisely for x such that $r_1 \sin\left(\frac{\pi}{2}\frac{\alpha}{r_1}x\right) < x_0$, we have

$$f\left(r_1 \sin\left(\frac{\pi}{2}\frac{\alpha}{r_1}x\right)\right) = k r_1 \sin\left(\frac{\pi}{2}\frac{\alpha}{r_1}x\right).$$

Therefore, for $x < \min(x_0, \frac{2r_1}{\pi\alpha} \arcsin\left(\frac{x_0}{r_1}\right))$, we obtain

$$\frac{r_2}{\beta x_0} r_1 \sin\left(\frac{\pi}{2} \cdot \frac{\alpha}{r_1} \cdot x\right) = r_2 \sin\left(\frac{\pi}{2} \cdot \frac{1}{x_0} x\right).$$

Now, let us consider three cases:

- $x_0 \neq \frac{r_1}{\alpha}$. Then the last equation takes the following form:

  $$a \sin(bx) = \sin(cx),$$

  where a, b, c are some constants, $b \neq c$. Such equation can not be true for all x from any nonempty interval.

- $x_0 = \frac{r_1}{\alpha}$ and $\alpha \neq \beta$. Then the last equation takes the form

  $$a \sin(bx) = \sin(bx),$$

  where $a \neq 1$. This equation again can not be true for all x from some nonempty interval.

- $x_0 = \frac{r_1}{\alpha}$ and $\alpha = \beta$. Then the last equation takes the following form

  $$\sin\left(\frac{\pi}{2}\frac{1}{x_0}x\right) = \sin\left(\frac{\pi}{2}\frac{1}{x_0}x\right),$$

  which is obviously true.

Therefore, for $x_0 = \frac{r_1}{\alpha}$ and $\alpha = \beta$, the equation (13) takes the form

$$f\left(r_1 \sin\left(\frac{\pi}{2}\frac{1}{x_0}x\right)\right) = r_2 \sin\left(\frac{\pi}{2}\frac{1}{x_0}x\right) = \frac{r_2}{r_1} \cdot r_1 \sin\left(\frac{\pi}{2}\frac{1}{x_0}x\right), \quad x < x_0. \tag{14}$$

We know, from Remark 3.4, that function f is continuous and increasing. Moreover $\lim_{x \to x_0} r_1 \sin\left(\frac{\pi}{2}\frac{1}{x_0}x\right) = r_1 \sin\left(\frac{\pi}{2}\right) = r_1$, so finally we obtain $f(x) = kx = \frac{r_2}{r_1}x$, for all $x \in [0, r_1]$.

The plots of functions m_1, m_2 and f with $r_1 = 1, r_2 = \frac{3}{2}$ and $\alpha = \frac{3}{2}$ are presented in Figure 4.

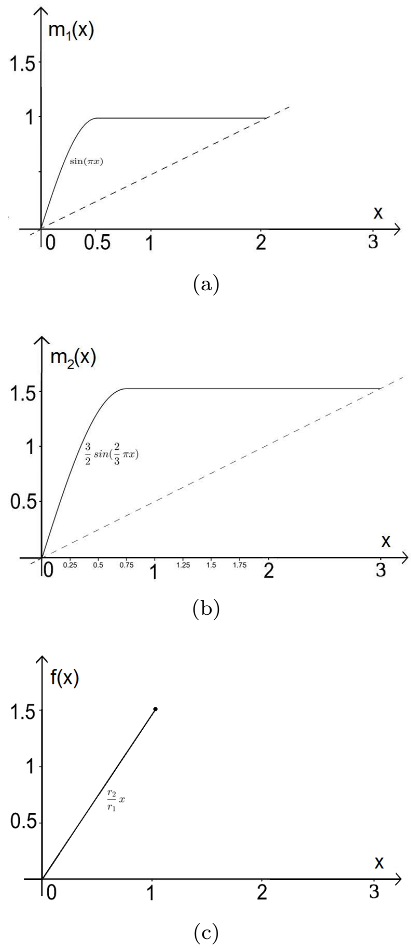

Fig. 4. Functions (a) m_1, (b) m_2 and (c) f from Example 4.3.

5. CONCLUSION

In this paper we presented some solutions of the following functional equation (2)

$$f(m_1(x+y)) = m_2(f(x)+f(y)),$$

where m_1, m_2 are given functions defined on some intervals of $\mathbb{R}$ and f is an unknown function. In fact the above equation generalizes the equation (1), which helps us in describing solutions of the distributivity equations of fuzzy implication functions over continuous Archimedean triangular norms and/or conorms.

Our investigations probably do not give more solutions for the original problem of distributivity of fuzzy implication functions over continuous Archimedean triangular norms and/or conorms. But, for example, using results from this article, it is possible to find some solutions of the following distributivity equation

$$I(x, M_1(y, z)) = M_2(I(x, y), I(x, z)), \tag{15}$$

where M_i, for $i = 1, 2$ are functions of the following form

$$M_i(x, y) = f_i^{-1}(m_i(f_i(x) + f_i(y))), \tag{16}$$

where functions f_i for $i = 1, 2$ are some continuous, monotonic generators (like for continuous Archimedean t-norms or t-conorms), while functions m_i, for $i = 1, 2$, should satisfy conditions from Section 2 or 3. Of course such defined functions M_i need not be t-norms or t-conorms. At this moment it is quite difficult for us to show possible practical applications (in fuzzy logic) of such equations as (2) with other functions than minimum, but it is the beginning of our work with such type of equations and functions.

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