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CALCULATIONS OF GRADED ILL-KNOWN SETS

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To represent a set whose members are known partially, the graded ill-known set is proposed. In this paper, we investigate calculations of function values of graded ill-known sets. Because a graded ill-known set is characterized by a possibility distribution in the power set, the calculations of function values of graded ill-known sets are based on the extension principle but generally complex. To reduce the complexity, lower and upper approximations of a given graded ill-known set are used at the expense of precision. We give a necessary and sufficient condition that lower and upper approximations of function values of graded ill-known sets are obtained as function values of lower and upper approximations of graded ill-known sets.

Keywords: ill-known set, lower approximation, upper approximation

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1. INTRODUCTION

Various models have been proposed to represent uncertainty: probability theory [8], fuzzy sets [10], belief functions [7], possibility theory [12], random sets [5], rough sets [6], and so on. Most of those models treat the uncertainty of a single-valued variable while others treat the uncertainty of a set-valued variable. Person's height, weight and age, the stock price at expiration and the cost of a cab ride between certain places, etc., are considered single-valued variables because the true values of those are unique. On the other hand, someone's favorite food, person's belongings, the day when a person stays in Osaka, candidate for the research topic, and so on are considered set-valued variables because the true values are not always unique. The former is called a disjunctive variable while the latter is called a conjunctive variable (see [3, 9]).

While a disjunctive variable takes a value of the universe, a conjunctive variable takes a subset of the universe. Then the set of possible realizations of conjunctive variables becomes a collection of subsets and the number of possible realizations is exponentially many. Therefore, unlike that of a disjunctive variable, the treatment of the uncertainty of a conjunctive variable becomes complex. Moreover, a conjunctive variable might be considered less encountered in the real world than a disjunctive variable. Because of those possible reasons, conjunctive variables have been studied much less than disjunctive variables. Nevertheless, several models such as belief functions [1], ill-known sets [2]

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and graded ill-known sets [4] to represent uncertainty of conjunctive variables have been proposed.

A variation range of a single-valued (disjunctive) variable can be seen as a conjunctive variable (see [4]). For example, to predict the variation range of a stock price, we should treat it as a conjunctive variable. Moreover, the predicted values of stock price by many experts can also be treated by a conjunctive variable. In addition, from its original definition, a conjunctive variable is useful to represent a set-valued variable, e. g., a satisfactory range, feasible range, members of a group, and so on. From these points of view, conjunctive variables can be encountered as often as disjunctive variables.

In this paper, we concentrate on the graded ill-known set as a model of conjunctive variable and investigate the calculations of graded ill-known sets. An ill-known set is a subset whose members are not known exactly. They can be represented by a family of subsets that can be true. A graded ill-known set is an ill-known set represented by a family of subsets with possible degrees. In other word, a graded ill-known set can be represented by a possibility distribution on the power set. The treatments of graded ill-known sets are primitively very complex because their manipulations are defined in the power set in principle. Namely, the number of elements in the power set is exponential, and thus the processing of graded ill-known sets usually requires an exponential order of computations.

Lower and upper approximations of graded ill-known sets are proposed for its simplified model. Generally speaking, lower approximation is composed of sure members while upper approximation is composed of possible members. In some real world problems, we may know only lower and upper approximations and, in this case, it is shown that possibility and necessity measures of graded ill-known sets are calculated by its lower and upper approximations [2, 4]. Because lower and upper approximations are defined in the universe, the treatments of those approximations are much computationally less than the treatments of graded ill-known sets. Therefore the results about possibility and necessity measures of graded ill-known sets defined by their approximations are very computationally advantageous.

In this paper, we show a similar result about function calculations of graded ill-known sets. We introduce the extension principle to graded ill-known sets to calculate function values of graded ill-known sets. The calculations of function values with graded ill-known sets are performed on the power set. Therefore, as described earlier, it would require a lot of computational efforts. In some real world applications, it can be sufficient to know the lower and upper approximations of the function value of graded ill-known sets. From this point of view, we consider the lower and upper approximations of function value of graded ill-known sets. We investigate the necessary and sufficient condition for the lower and upper approximations of function value of graded ill-known sets to be calculated by the lower and upper approximations of given graded ill-known sets. Moreover, we give some simpler sufficient conditions useful for the applications of graded ill-known sets to various fields.

This paper is organized as follows. In Section 2, we briefly review graded ill-known sets. The main results on calculations of graded ill-known sets are given in Section 3. In Section 4, a simple example is given. Concluding remarks are given in Section 5.

2. GRADED ILL-KNOWN SETS

Let X be a universe. Let A be a crisp set whose members are not known exactly. For example, consider student participants of a conference held 10 years ago in a laboratory under Prof. X. Prof. X knows that there were six students at that time, say a, b, c, d, e and f. However, his memory is not certain. He is sure that three students attended the conference and f was absent at the conference. Moreover, he remembers that a and b attended the conference. From this memory, we know that the set of the student participants in Prof. X's laboratory was $\{a,b,c\}$, $\{a,b,d\}$ or $\{a,b,e\}$. Such a crisp set with imprecise members is called an ill-known set.

To represent an ill-known set, collecting possible realizations of A, we obtain the following family:

$$\mathcal{A} = \{A_1, A_2, \ldots, A_n\}, \tag{1}$$

where A_i is a crisp set such that $A = A_i$ is consistent with the partial knowledge about A.

Given $\mathcal{A}$, we obtain a set of elements which certainly belong to A, say A^- and a set of elements which possibly belong to A, say A^+ are defined as

$$A^- = \bigcap \mathcal{A} = \bigcap_{i=1,\ldots,n} A_i, \quad A^+ = \bigcup \mathcal{A} = \bigcup_{i=1,\ldots,n} A_i. \tag{2}$$

We call A^- and A^+ "the lower approximation" of $\boldsymbol{A}$ and "the upper approximation" of $\boldsymbol{A}$, respectively.

In the previous example about the student participants of a conference in Prof. X's laboratory, we may define $X = \{a,b,c,d,e,f\}$, $A_1 = \{a,b,c\}$, $A_2 = \{a,b,d\}$ and $A_3 = \{a,b,e\}$. Then we have $A^- = \{a,b\}$ and $A^+ = \{a,b,c,d,e\}$. A^- coincides with the sure participants in Prof. X's memory and $X - A^+ = \{f\}$ coincides with the sure non-participants in Prof. X's memory.

In the real world, we sometimes may know sure members and sure non-members of A only. In other words, we know the lower approximation A^- as a set of sure members and the upper approximation A^+ as a complementary set of sure non-members. Given A^- and A^+ (or equivalently, the complement of A^+), we obtain a family $\hat{\mathcal{A}}$ of possible realizations as

$$\hat{\mathcal{A}} = \{A_i \mid A^- \subseteq A_i \subseteq A^+\}. \tag{3}$$

We note that A^- and A^+ are recovered by applying (2) to the family $\hat{\mathcal{A}}$ induced from A^- and A^+ by (3). On the other hand, a given family $\mathcal{A}$ of (1) cannot be always recovered by applying (3) to A^- and A^+ defined by (2). For example, in the example of the student participants of a conference in Prof. X's laboratory, $\mathcal{A}$ is not recovered.

If all A_i's of (1) are not regarded as equally possible, we may assign a possibility degree $\pi_{\mathcal{A}}(A)$ to each $A \subseteq X$ so that

$$\exists A \subseteq X, \quad \pi(A) = 1. \tag{4}$$

A possibility distribution $\pi_{\mathcal{A}} : 2^X \to [0,1]$ can be seen as a membership function of a fuzzy set $\mathcal{A}$ in 2^X. Thus, we may identify $\boldsymbol{A}$ with $\mathcal{A}$. The ill-known set having such a possibility distribution is called "a graded ill-known set".

For example, assume Prof. X feels $\{a,b,c\}$ is most conceivable and $\{a,b,d\}$ is more conceivable than $\{a,b,e\}$ in the setting of the previous example. His feeling may be expressed by a possibility distribution $\pi_{\mathcal{A}}(\{a,b,c\})=1$, $\pi_{\mathcal{A}}(\{a,b,d\})=0.6$, $\pi_{\mathcal{A}}(\{a,b,e\})=0.3$ and $\pi_{\mathcal{A}}(A)=0$ for any other subset $A\subseteq X=\{a,b,c,d,e,f\}$.

In this case, the lower approximation A^- and the upper approximation A^+ are defined as fuzzy sets with the following membership functions (see Dubois and Prade [2], Inuiguchi [4]):

$$\mu_{A^-}(x)=\inf_{\substack{A\subseteq X\\ x\notin A}}(1-\pi_{\mathcal{A}}(A)),\quad \mu_{A^+}(x)=\sup_{\substack{A\subseteq X\\ x\in A}}\pi_{\mathcal{A}}(A). \tag{5}$$

We have the following property:

$$\forall x\in X,\ \mu_{A^-}(x)>0 \text{ implies } \mu_{A^+}(x)=1. \tag{6}$$

In the example of possibility distribution for student participants of a conference in Prof. X's laboratory, we obtain $\mu_{A^-}(a)=\mu_{A^-}(b)=1$, $\mu_{A^-}(c)=0.4$ and $\mu_{A^-}(x)=0$ for $x\in\{d,e,f\}$. On the other hand, we obtain $\mu_{A^+}(a)=\mu_{A^+}(b)=\mu_{A^+}(c)=1$, $\mu_{A^+}(d)=0.6$, $\mu_{A^+}(e)=0.3$ and $\mu_{A^+}(f)=0$.

Because the specification of possibility distribution $\pi_{\mathcal{A}}$ may need a lot of information, as is in the usual ill-known sets, we may know only the lower approximation A^- and the upper approximation A^+ as fuzzy sets satisfying (6). The consistent possibility distribution $\pi_{\mathcal{A}}$ for any A^- and A^+ is not unique.

However, the following possibility distribution $\pi^*_{\mathcal{A}}(A_i)$ is the maximal possibility distribution among the consistent possibility distributions

$$\pi^*_{\mathcal{A}}(A)=\min\left(\inf_{x\notin A}(1-\mu_{A^-}(x)),\ \inf_{x\in A}\mu_{A^+}(x)\right), \tag{7}$$

where we define $\inf\emptyset=1$. We identify the maximal possibility distribution $\pi^*_{\mathcal{A}}(A)$ with the given fuzzy sets A^- and A^+ unless the other information is available.

For example, when $\pi_{\mathcal{A}}$ of $X=\{a,b,c,d,e,f\}$ are given by $\mu_{A^-}(a)=\mu_{A^-}(b)=1$, $\mu_{A^-}(c)=0.4$ and $\mu_{A^-}(x)=0$ for $x\in\{d,e,f\}$, and $\mu_{A^+}(a)=\mu_{A^+}(b)=\mu_{A^+}(c)=1$, $\mu_{A^+}(d)=0.6$, $\mu_{A^+}(e)=0.3$ and $\mu_{A^+}(f)=0$, we obtain $\pi^*_{\mathcal{A}}(\{a,b\})=0.6$, $\pi^*_{\mathcal{A}}(\{a,b,c\})=1$, $\pi^*_{\mathcal{A}}(\{a,b,d\})=0.6$, $\pi^*_{\mathcal{A}}(\{a,b,e\})=0.3$, $\pi^*_{\mathcal{A}}(\{a,b,c,d\})=0.6$, $\pi^*_{\mathcal{A}}(\{a,b,c,e\})=0.3$, $\pi^*_{\mathcal{A}}(\{a,b,d,e\})=0.3$, $\pi^*_{\mathcal{A}}(\{a,b,c,d,e\})=0.3$ and $\pi^*_{\mathcal{A}}(A)=0$ for any other $A\subseteq X$.

When A^- and A^+ are obtained from a possibility distribution $\pi_{\mathcal{A}}$, $\pi^*_{\mathcal{A}}$ obtained from A^- and A^+ through (7) is not always same as the original $\pi_{\mathcal{A}}$. We only have $\pi_{\mathcal{A}}(A)\le\pi^*_{\mathcal{A}}(A)$, $A\subseteq X$. Indeed, this fact can be observed in the examples above. Namely, the possibility distribution $\pi_{\mathcal{A}}$ defined for the student participants of a conference in Prof. X's laboratory has lower and upper approximations A^- and A^+ which are used for the calculation of $\pi^*_{\mathcal{A}}$ in the example above. We observe $\pi_{\mathcal{A}}(A)\le\pi^*_{\mathcal{A}}(A)$, $A\subseteq X$.

3. EXTENSION PRINCIPLE FOR GRADED ILL-KNOWN SETS

In this paper, we consider graded ill-known sets in real line $\mathbf{R}$ and investigate the calculations of graded ill-known sets in $\mathbf{R}$. Graded ill-known sets in real line $\mathbf{R}$ are called "graded ill-known sets of quantities". The set of graded ill-known sets of quantities is denoted by $\mathcal{IQ}$.

Because graded ill-known sets are characterized by possibility distributions on the power set which can be seen as a membership function of a fuzzy set in the power set, the function values of graded ill-known sets of quantities can be defined by the extension principle [11] in fuzzy set theory.

When a function $\psi : (2^{\mathbf{R}})^m \to 2^{\mathbf{R}}$ is given, we extend this function to a function from $\mathcal{IQ}^m$ to $\mathcal{IQ}$ in the following definition.

Definition 3.1. Let $\mathcal{A}_i$, $i = 1, 2, \ldots, m$ be graded ill-known sets of quantities. Given a function $\psi : (2^{\mathbf{R}})^m \to 2^{\mathbf{R}}$, the image $\psi(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ is defined by a graded ill-known set of quantities associated with the following possibility distribution:

$$\pi_{\psi(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)}(Y) = \begin{cases} \displaystyle\sup_{\substack{Q_1,Q_2,\ldots,Q_m \subseteq \mathbf{R} \\ Y=\psi(Q_1,\ldots,Q_m)}} \min\left(\pi_{\mathcal{A}_1}(Q_1), \pi_{\mathcal{A}_2}(Q_2), \ldots, \pi_{\mathcal{A}_m}(Q_m)\right), & \text{if } \psi^{-1}(Y) \neq \emptyset, \\ 0, & \text{if } \psi^{-1}(Y) = \emptyset, \end{cases} \tag{8}$$

where $\pi_{\mathcal{A}_i}$ is a possibility distribution associated with graded ill-known set of quantities $\mathcal{A}_i$ and ψ^{-1} is the inverse image of ψ.

Note that, function $f : \mathbf{R}^m \to \mathbf{R}$ can be extended to a function $f : (2^{\mathbf{R}})^m \to 2^{\mathbf{R}}$ by $f(A_1, A_2, \ldots, A_m) = \{f(x_1, x_2, \ldots, x_m) \mid x_i \in A_i,\ i = 1, 2, \ldots, m\}$. The extended function $f : (2^{\mathbf{R}})^m \to 2^{\mathbf{R}}$ can be further extended to a function $f : \mathcal{IQ}^m \to \mathcal{IQ}$ by Definition 3.1.

The calculation of $\psi(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ is very complex because we should consider all elementary sets of power set $2^{\mathbf{R}}$. This implies that at least an exponential order of calculations are requested. In this paper, we investigate the necessary and sufficient condition for the lower and upper approximations of $\psi(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ to be calculated in smaller order of complexity when ψ is the extension of $f : \mathbf{R}^m \to \mathbf{R}$. The lower and upper approximations provides the approximated values and, in some special cases, the exact values (see [2, 4]), Therefore it is very useful to know those approximations.

We obtain the following theorem about the upper approximation.

Theorem 3.2. The upper approximation $f^+(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ of $f(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$ can be calculated by upper approximations of $\mathcal{A}_i$, $i = 1, 2, \ldots, m$. More concretely, we obtain

$$\begin{aligned} \mu_{f^+(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)}(y) &= \sup_{y\in Y} \pi_{f(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)}(Y) \\ &= \sup_{\substack{x_1,x_2,\ldots,x_m \in \mathbf{R} \\ y=f(x_1,x_2,\ldots,x_m)}} \min(\mu_{A_1^+}(x_1), \mu_{A_2^+}(x_2), \ldots, \mu_{A_m^+}(x_m))) = \mu_{f(A_1^+,A_2^+,\ldots,A_m^+)}(y), \end{aligned} \tag{9}$$

where $\mu_{f^+(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)}$ is the membership function of $f^+(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)$ and $\mu_{A_i^+}$ is the membership function of the upper approximation A_i^+ of $\mathcal{A}_i$. Similarly, $\mu_{f(A_1^+,A_2^+,\ldots,A_m^+)}$ is the membership function of the image $f(A_1^+,A_2^+,\ldots,A_m^+)$.

Proof. It can be proved straightforwardly from the definitions. □

For the lower approximation, we only have an inequality as shown in the following theorem.

Theorem 3.3. The membership function of lower approximation $f^-(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)$ of $f(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)$ is not smaller than that of $f(A_1^-,A_2^-,\ldots,A_m^-)$, i. e.,

$$\begin{aligned}\mu_{f^-(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)}(y) &= \inf_{y\notin Y}(1-\pi_{f(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)}(Y))\\ \geq \sup_{\substack{x_1,x_2,\ldots,x_m\in\mathbf{R}\\ y=f(x_1,x_2,\ldots,x_m)}} &\min(\mu_{A_1^-}(x_1),\mu_{A_2^-}(x_2),\ldots,\mu_{A_m^-}(x_m)) = \mu_{f(A_1^-,A_2^-,\ldots,A_m^-)}(y),\end{aligned} \tag{10}$$

where $\mu_{A_i^-}$ is the membership function of lower approximation A_i^- of $\mathcal{A}_i$. $\mu_{f(A_1^-,A_2^-,\ldots,A_m^-)}$ is the membership function of the image $f(A_1^-,A_2^-,\ldots,A_m^-)$ of fuzzy sets $A_1^-,A_2^-,\ldots,A_m^-$.

Proof. For the sake of simplicity, we prove when $m=2$. In cases where $m\neq 2$, it can be proved in the same way. From the definition, we have

$$\begin{aligned}\mu_{f^-(\mathcal{A}_1,\mathcal{A}_2)}(y) &= \inf_{y\notin Y}\left(1-\sup_{\substack{A_1,A_2\\ Y=f(A_1,A_2)}}\min\left(\pi_{\mathcal{A}_1}(A_1),\pi_{\mathcal{A}_2}(A_2)\right)\right)\\ &= \inf_{\substack{A_1,A_2\\ y\notin f(A_1,A_2)}}\max\left(\left(1-\pi_{\mathcal{A}_1}(A_1)\right),n\left(\pi_{\mathcal{A}_2}(A_2)\right)\right).\end{aligned}$$

Assume $\mu_{f(A_1^-,A_2^-)}(y)\geq\alpha$. By definition of $f(A_1^-,A_2^-)$, there exist x_1 and x_2 such that $y=f(x_1,x_2)$, $(\forall A_1\not\ni x_1,\ n(\pi_{\mathcal{A}_1}(A_1))\geq\alpha)$ and $(\forall A_2\not\ni x_2,\ n(\pi_{\mathcal{A}_2}(A_2))\geq\alpha)$. Under this assumption, we prove $\mu_{f^-(\mathcal{A}_1,\mathcal{A}_2)}(y)\geq\alpha$. For all A_1 and A_2 such that $y\notin f(A_1,A_2)$, from the assumption, we have $x_1\notin A_1$ or $x_2\notin A_2$. Moreover, from the assumption, $x_i\notin A_i$ implies $n(\pi_{\mathcal{A}_i}(A_i))\geq\alpha$ $(i=1,2)$. Hence, applying those to the equation above, we obtain $\mu_{f^-(\mathcal{A}_1,\mathcal{A}_2)}(y)\geq\alpha$. □

The equality of (10) does not hold generally but in special cases. In the following section, we investigate the necessary and sufficient condition for the equality of (10).

4. THE MAIN RESULT AND ITS IMPLICATIONS

We obtain the following theorem.

Theorem 4.1. We have $f^-(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m) = f(A_1^-, A_2^-, \ldots, A_m^-)$, i.e.,

$$\begin{aligned}\mu_{f^-(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)}(y) &= \inf_{y\notin Y}(1-\pi_{f(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)}(Y)) \\ &= \sup_{\substack{x_1,x_2,\ldots,x_m\in\mathbf{R} \\ y=f(x_1,x_2,\ldots,x_m)}} \min(\mu_{A_1^-}(x_1), \mu_{A_2^-}(x_2), \ldots, \mu_{A_m^-}(x_m)) = \mu_{f(A_1^-,A_2^-,\ldots,A_m^-)}(y),\end{aligned} \tag{11}$$

if and only if

$$\begin{aligned}\forall\alpha\in[0,1),\qquad &\bigcap\{f(Q_1,Q_2\ldots,Q_m) \mid Q_1\in(\mathcal{A}_1)_\alpha, (\mathcal{A}_2)_\alpha, \ldots, Q_m\in(\mathcal{A}_m)_\alpha\} \\ &= f\left(\bigcap(\mathcal{A}_1)_\alpha, (\mathcal{A}_2)_\alpha, \ldots, \bigcap(\mathcal{A}_m)_\alpha\right),\end{aligned} \tag{12}$$

where $(\mathcal{A}_i)_\alpha = \{Q \mid \pi_{\mathcal{A}_i}(Q) > \alpha\}$.

Proof. For the sake of simplicity, we prove when $m = 2$. In cases where $m \neq 2$, it can be proved in the same way. From Theorem 2, we consider the necessary and sufficient condition of

$$\mu_{f^-(\mathcal{A}_1,\mathcal{A}_2)}(y) \le \mu_{f(A_1^-,A_2^-)}(y).$$

This is equivalent to

$$\forall\alpha\in(0,1],\ \mu_{f^-(\mathcal{A}_1,\mathcal{A}_2)}(y)\ge\alpha \text{ implies } \mu_{f(A_1^-,A_2^-)}(y)\ge\alpha. \tag{$*$}$$

Then we consider the equivalent condition of (a) $\mu_{f^-(\mathcal{A}_1,\mathcal{A}_2)}(y) \ge \alpha$ and that of (b) $\mu_{f(A_1^-,A_2^-)}(y) \ge \alpha$.

First let us investigate the equivalent condition of (a). By definition, we have

$$\begin{aligned}\mu_{f^-(\mathcal{A}_1,\mathcal{A}_2)}(y)\ge\alpha &\Leftrightarrow \inf_{Y\not\ni y}(1-\pi_{f(\mathcal{A}_1,\mathcal{A}_2)}(Y))\ge\alpha \\ &\Leftrightarrow\quad y\notin Y \text{ implies } \pi_{f(\mathcal{A}_1,\mathcal{A}_2)}(Y)\ge 1-\alpha \\ &\Leftrightarrow\quad \pi_{f(\mathcal{A}_1,\mathcal{A}_2)}(Y) > 1-\alpha \text{ implies } y\in Y \\ &\Leftrightarrow\quad \sup_{Q_1,Q_2:Y=f(Q_1,Q_2)}\min(\pi_{\mathcal{A}_1}(Q_1),\pi_{\mathcal{A}_2}(Q_2)) > 1-\alpha \text{ implies } y\in Y \\ &\Leftrightarrow\quad y\in\bigcap\{f(Q_1,Q_2)\mid Q_1\in(\mathcal{A}_1)_{1-\alpha},\ Q_2\in(\mathcal{A}_2)_{1-\alpha}\}.\end{aligned}$$

Now let us investigate the equivalent condition of (b). By definition, we obtain

$$\begin{aligned}
\mu_{f(A_1^-,A_2^-)}(y) \geq \alpha \Leftrightarrow & \sup_{x_1,x_2:y=f(x_1,x_2)} \min\left(\mu_{A_1^-}(x_1),\mu_{A_2^-}(x_2)\right) \geq \alpha \\
\Leftrightarrow\ & \forall\varepsilon>0,\ \exists x_1,\ x_2,\ y=f(x_1,x_2),\ \mu_{A_1^-}(x_1)>\alpha-\varepsilon,\ \mu_{A_2^-}(x_2)>\alpha-\varepsilon \\
\Leftrightarrow\ & \forall\varepsilon>0,\ \exists x_1,\ x_2,\ y=f(x_1,x_2),\ \inf_{Q_i\not\ni x_i}(1-\pi_{\mathcal{A}_i}(Q_i))>\alpha-\varepsilon,\ i=1,2 \\
\Leftrightarrow\ & \forall\varepsilon>0,\ \exists x_1,\ x_2,\ y=f(x_1,x_2), \\
& (\forall Q_1\not\ni x_1,\ \pi_{\mathcal{A}_1}(Q_1)<1-\alpha+\varepsilon),\ (\forall Q_2\not\ni x_2,\ \pi_{\mathcal{A}_2}(Q_2)<1-\alpha+\varepsilon) \\
\Leftrightarrow\ & \forall\varepsilon>0,\ \exists x_1,\ x_2,\ y=f(x_1,x_2), \\
& x_i\in\bigcap\left\{Q_i \mid \pi_{\mathcal{A}_i}(Q_i)\geq 1-\alpha+\varepsilon\right\},\ i=1,2 \\
\Leftrightarrow\ & \exists x_1,\ x_2,\ y=f(x_1,x_2),\ x_i\in\bigcap\left\{Q_i \mid \pi_{\mathcal{A}_i}(Q_i)>1-\alpha\right\},\ i=1,2 \\
\Leftrightarrow\ & y\in f\left(\bigcap(\mathcal{A}_1)_{1-\alpha},\bigcap(\mathcal{A}_2)_{1-\alpha}\right).
\end{aligned}$$

From those equivalent conditions of (a) and (b), the necessary and sufficient condition of $(*)$ is obtained as

$$\forall\alpha\in[0,1),\ \bigcap\left\{f(Q_1,Q_2)\mid Q_1\in(\mathcal{A}_1)_\alpha,\ Q_2\in(\mathcal{A}_2)_\alpha\right\}=f\left(\bigcap(\mathcal{A}_1)_\alpha,\bigcap(\mathcal{A}_2)_\alpha\right).$$

□

The necessary and sufficient condition for $f^-(\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_m)=f(A_1^-,A_2^-\dots,A_m^-)$ obtained in Theorem 4.1 is not easily confirmed. Then we will give a sufficient conditions which are easily confirmed. To this end, we define a class $\mathcal{IQ}_{\text{int}}\subseteq\mathcal{IQ}$ of graded ill-known sets of quantities $\mathcal{A}$ satisfying the following properties:

$$\begin{aligned}
&\forall\alpha\in[0,1),\ A(\alpha)=\bigcap(\mathcal{A})_\alpha \text{ is nonempty and convex, and} \\
&\qquad \text{there exists a family of convex sets } \{Q_j\}_{j\in J} \\
&\qquad \text{such that } Q_j\in(\mathcal{A})_\alpha, j\in J \text{ and } A(\alpha)=\textstyle\bigcap_{j\in J}Q_j.
\end{aligned}\tag{13}$$

A graded ill-known set of quantities $\mathcal{A}$ satisfying (13) can be seen as an extension of an interval in $\mathbf{R}$. Then $\mathcal{IQ}_{\text{int}}$ is considered the set of ill-known intervals.

Then we obtain the following theorem.

Theorem 4.2. Let $f:\mathbf{R}^m\to\mathbf{R}$ be continuous and monotone (monotonically increasing or monotonically decreasing with respect to each argument). Let $\mathcal{A}_i\in\mathcal{IQ}_{\text{int}}$, $i=1,2,\dots,m$. Then we have (11), i. e., $f^-(\mathcal{A}_1,\mathcal{A}_2,\dots,\mathcal{A}_m)=f(A_1^-,A_2^-\dots,A_m^-)$.

Proof. By the same reason as Theorem 4.1, we prove when $m=2$. Without loss of generality, we assume f is monotonically increasing with respect to all arguments.

From $A_i(\alpha)=\bigcap(\mathcal{A}_i)_\alpha\subseteq Q_i$ for $Q_i\in(\mathcal{A}_i)_\alpha$, $f(A_1(\alpha),A_2(\alpha))\subseteq\bigcap\{f(Q_1,Q_2)\mid Q_1\in(\mathcal{A}_1)_\alpha,\ Q_2\in(\mathcal{A}_2)_\alpha\}$. Then we prove

$$y\notin f(A_1(\alpha),A_2(\alpha)) \text{ implies } y\notin\bigcap\{f(Q_1,Q_2)\mid Q_1\in(\mathcal{A}_1)_\alpha,\ Q_2\in(\mathcal{A}_2)_\alpha\}.\tag{$*$}$$

Because f is continuous and $A_i(\alpha)$, $i = 1, 2$ are nonempty and convex, $f(A_1(\alpha), A_2(\alpha))$ becomes an interval (a convex set in the real line). Then we prove $(*)$ dividing into two cases: (a) $y \le \inf f(A_1(\alpha), A_2(\alpha))$ and $y \notin f(A_1(\alpha), A_2(\alpha))$ and (b) $y \ge \sup f(A_1(\alpha), A_2(\alpha))$ and $y \notin f(A_1(\alpha), A_2(\alpha))$.

Because $\mathcal{A}_i \in \mathcal{IQ}_{\rm int}$, there exists a family $\mathcal{Q}_i$ of convex sets $\{Q_{ij}\}_{j\in J_i}$ such that $Q_{ij} \in (\mathcal{A}_i)_\alpha$ and $A_i(\alpha) = \bigcap_{j\in J_i} Q_{ij}$ for $i = 1, 2$. From the convexity of Q_{ij}, $j \in J_i$, $i = 1, 2$, there exist subfamilies $\underline{\mathcal{Q}}_i = \{\underline{Q}_{ij}\}_{j\in \underline{J}_i} \subseteq \mathcal{Q}_i$ and $\overline{\mathcal{Q}}_i = \{\overline{Q}_{ij}\}_{j\in \overline{J}_i} \subseteq \mathcal{Q}_i$ such that $\sup_{j\in\underline{J}_i} \inf \underline{Q}_{ij} = \inf A_i(\alpha)$ and $\inf_{j\in\overline{J}_i} \sup \overline{Q}_{ij} = \sup A_i(\alpha)$.

From the monotonicity, we obtain

$$\begin{aligned}
&\forall r_1 \in A_1(\alpha),\ \forall r_2 \in A_2(\alpha),\ y < f(r_1, r_2) \text{ implies}\\
&\qquad\qquad \exists k_1 \in \underline{J}_1,\ \exists k_2 \in \underline{J}_2,\ \forall q_1 \in \underline{Q}_{1k_1},\ \forall q_2 \in \underline{Q}_{2k_2},\ y < f(q_1, q_2),\\
&\forall r_1 \in A_1(\alpha),\ \forall r_2 \in A_2(\alpha),\ y > f(r_1, r_2) \text{ implies}\\
&\qquad\qquad \exists l_1 \in \overline{J}_1,\ \exists l_2 \in \overline{J}_2,\ \forall q_1 \in \overline{Q}_{1l_1},\ \forall q_2 \in \overline{Q}_{2l_2},\ y > f(q_1, q_2).
\end{aligned}$$

Therefore, in case (a) $y \le \inf f(A_1(\alpha), A_2(\alpha))$ and $y \notin f(A_1(\alpha), A_2(\alpha))$, we have $y \notin f(\underline{Q}_1, \underline{Q}_2)$. This implies that $y \notin \bigcap\{f(Q_1, Q_2) \mid Q_1 \in (\mathcal{A}_1)_\alpha,\ Q_2 \in (\mathcal{A}_2)_\alpha\}$. Similarly, in case (b) $y \ge \sup f(A_1(\alpha), A_2(\alpha))$ and $y \notin f(A_1(\alpha), A_2(\alpha))$, we have $y \notin f(\overline{Q}_1, \overline{Q}_2)$. This implies that $y \notin \bigcap\{f(Q_1, Q_2) \mid Q_1 \in (\mathcal{A}_1)_\alpha,\ Q_2 \in (\mathcal{A}_2)_\alpha\}$. Hence, $(*)$ is proved. □

If $\mathcal{A}_i(\alpha) = \bigcap(\mathcal{A}_i)_\alpha \in (\mathcal{A}_i)_\alpha$, $i = 1, 2, \ldots, m$ for any $\alpha \in [0, 1)$, we have (12). From Theorem 4.1, we have the following corollary.

Corollary 4.3. If $\mathcal{A}_i(\alpha) \in (\mathcal{A}_i)_\alpha$, $i = 1, 2, \ldots, m$ for any $\alpha \in [0, 1)$, then we have (11), i. e., $f^-(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m) = f(A_1^-, A_2^-, \ldots, A_m^-)$.

Proof. It suffices to prove that $Q_i^1 \subseteq Q_i^2$ implies $f(Q_1^1, Q_2^1, \ldots, Q_m^1) \subseteq f(Q_1^2, Q_2^2, \ldots, Q_m^2)$. This is obvious from definition, $f(Q_1, Q_2, \ldots, Q_m) = \{f(x_1, x_2, \ldots, x_m) \mid x_i \in Q_i,\ i = 1, 2, \ldots . m\}$. □

When $\mathcal{A}_i(\alpha) \in (\mathcal{A}_i)_\alpha$, $i = 1, 2, \ldots, m$ for any $\alpha \in [0, 1)$, we have (11) without any condition on f. The strong condition $\mathcal{A}_i(\alpha) \in (\mathcal{A}_i)_\alpha$, $i = 1, 2, \ldots, m$ for any $\alpha \in [0, 1)$ is satisfied by a graded ill-known set of quantities defined by lower and upper approximations. This can be understood directly from the following proposition.

Proposition 4.4. Let $\mathcal{A}$ be a graded ill-known set defined by lower and upper approximations A^- and A^+. Then we have

$$(\mathcal{A})_\alpha = \left\{ A \;\middle|\; [A^-]_{1-\alpha} \subseteq A \subseteq (A^+)_\alpha \right\}, \tag{14}$$

where $[A^-]_\beta$ is a weak β-level set of A^-, i. e., $[A^-]_\beta = \{x \mid \mu_{A^-}(x) \ge \beta\}$, $\beta \in (0, 1]$ while $(A^+)_\gamma$ is a strong γ-level set of A^+, i. e., $(A^+)_\gamma = \{x \mid \mu_{A^+}(x) > \gamma\}$, $\gamma \in [0, 1)$.

P r o o f. From (7), we obtain the following equivalences:

$$\begin{aligned} & A \in (\mathcal{A})_\alpha \\ & \quad \Leftrightarrow \inf_{x \notin A} (1 - \mu_{A^-}(x)) > \alpha \text{ and } \inf_{x \in A} \mu_{A^+}(x) > \alpha \\ & \quad \Leftrightarrow (x \in A \text{ implies } \mu_{A^+}(x) > \alpha) \text{ and } (\mu_{A^-}(x) \geq 1 - \alpha \text{ implies } x \in A) \\ & \quad \Leftrightarrow [A^-]_{1-\alpha} \subseteq A \subseteq (A^+)_\alpha. \end{aligned}$$

□

From Proposition 4.4, we know that $\mathcal{A}(\alpha) = [A^-]_{1-\alpha}$ if $\mathcal{A}$ is defined by lower and upper approximations A^- and A^+. Because A^+ is not related to $\mathcal{A}(\alpha)$, we may have a weaker sufficient condition for $\mathcal{A}_i(\alpha) \in (\mathcal{A}_i)_\alpha$. Namely, we obtain the following theorem.

Theorem 4.5. If the possibility distribution $\pi_{\mathcal{A}_i}$ of a graded ill-known set of quantities $\mathcal{A}_i$ satisfies

$$\pi_{\mathcal{A}_i}(A) = \inf_{x \notin A} (1 - \mu_{A_i^-}(x)), \ \forall A \text{ such that } \inf_{x \notin A} (1 - \mu_{A_i^-}(x)) \leq \inf_{x \in A} \mu_{A_i^+}(x), \quad i = 1, 2, \ldots, m, \tag{15}$$

we have (11), i. e., $f^-(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m) = f(A_1^-, A_2^-, \ldots, A_m^-)$, where A_i^- and A_i^+ are lower and upper approximations of $\mathcal{A}_i$, respectively, and $\mu_{A_i^-}$ and $\mu_{A_i^+}$ are their membership functions.

P r o o f. From Corollary 4.3, it suffices to prove $\mathcal{A}_i(\alpha) \in (\mathcal{A}_i)_\alpha$ under condition (15). Because

$$\pi_{\mathcal{A}_i}(A_i) \leq \pi^*_{\mathcal{A}_i}(A) = \min\left(\inf_{x \notin A} (1 - \mu_{A_i^-}(x)), \inf_{x \in A} \mu_{A_i^+}(x) \right) \leq \inf_{x \notin A} (1 - \mu_{A_i^-}(x)),$$

we obtain

$$\pi_{\mathcal{A}_i}(A_i) > \alpha \Rightarrow \inf_{x \notin A} (1 - \mu_{A_i^-}(x)) > \alpha \Leftrightarrow [A_i^-]_{1-\alpha} \subseteq A \Leftrightarrow [A_i^-]_{1-\alpha} \subseteq \mathcal{A}_i(\alpha).$$

Now we prove $[A_i^-]_{1-\alpha} = \mathcal{A}_i(\alpha) = \bigcap (\mathcal{A}_i)_\alpha$ using (15). From (6), for $\forall \varepsilon \in (0, 1-\alpha)$, we have

$$\inf_{x \notin [A_i^-]_{1-\alpha-\varepsilon}} (1 - \mu_{A_i^-}(x)) \leq \inf_{x \in [A_i^-]_{1-\alpha-\varepsilon}} \mu_{A_i^+}(x).$$

From (15), we obtain

$$\pi_{\mathcal{A}_i}([A_i^-]_{1-\alpha-\varepsilon}) = \inf_{x \notin [A_i^-]_{1-\alpha-\varepsilon}} (1 - \mu_{A_i^-}(x)) \geq \alpha + \varepsilon > \alpha.$$

Namely, we have $[A_i^-]_{1-\alpha-\varepsilon} \in (\mathcal{A}_i)_\alpha$ for any $\varepsilon \in (0, 1-\alpha)$. From the property of weak level set, we have $[A_i^-]_{1-\alpha-\varepsilon} \supseteq [A_i^-]_{1-\alpha}$ and $\bigcap_{\varepsilon \in (0,1-\alpha)} [A_i^-]_{1-\alpha-\varepsilon} = [A_i^-]_{1-\alpha}$. Hence, we obtain $[A_i^-]_{1-\alpha} = \mathcal{A}_i(\alpha)$ □

5. CASES OF $F(\langle A_1^-, A_1^+\rangle, \ldots, \langle A_M^-, A_M^+\rangle) = \langle F(A_1^-, \ldots, A_M^-), F(A_1^+, \ldots, A_M^+)\rangle$

In this section, we investigate cases where

$$f(\langle A_1^-, A_1^+\rangle, \langle A_2^-, A_2^+\rangle, \ldots, \langle A_m^-, A_m^+\rangle) = \langle f(A_1^-, A_2^-, \ldots, A_m^-), f(A_1^+, A_2^+ \ldots, A_m^+)\rangle. \tag{16}$$

Contrary to our expectation, (16) does not always hold. Counter examples are given as follows.

Example 5.1. Let us consider a function $f_1 : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$f_1(x_1, x_2) = \begin{cases} x_1 + x_2, & \text{if } x_1 + x_2 \le 6, \\ 0, & \text{if } x_1 + x_2 \in (6, 10], \\ x_1 + x_2 - 4, & \text{if } x_1 + x_2 > 10. \end{cases}$$

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be ill-known sets defined by lower approximations $A_1^- = [2,3]$ and $A_2^- = [2,3]$ and upper approximations $A_1^+ = [1,7]$ and $A_2^+ = [1,8]$, respectively. Then we have

$$[4,8] \notin f_1(\mathcal{A}_1, \mathcal{A}_2) = f_1(\langle A_1^-, A_1^+\rangle, \langle A_2^-, A_2^+\rangle),$$

but

$$\{0\} \cup [4,8] \in f_1(\mathcal{A}_1, \mathcal{A}_2) = f_1(\langle A_1^-, A_1^+\rangle, \langle A_2^-, A_2^+\rangle).$$

On the other hand, we obtain $f_1(A_1^-, A_2^-) = [4,6]$ and $f_1(A_1^+, A_2^+) = \{0\} \cup [2,11]$. Then we have

$$[4,8] \in \langle f_1(A_1^-, A_2^-), f_1(A_1^+, A_2^+)\rangle.$$

Therefore, we have

$$f_1(\langle A_1^-, A_1^+\rangle, \langle A_2^-, A_2^+\rangle) \neq \langle f_1(A_1^-, A_2^-), f_1(A_1^+, A_2^+)\rangle.$$

Even when function is continuous and monotone, we have a similar result. Consider the following example.

Example 5.2. Consider a function $f_2 : \mathbf{R}^2 \to \mathbf{R}$ defined by $f_2(x_1, x_2) = x_1 + x_2$. Let A_i^- and A_i^+ $(i = 1, 2)$ be the same as above, i.e., $A_1^- = [2,3]$, $A_2^- = [2,3]$, $A_1^+ = [1,7]$ and $A_2^+ = [1,8]$. We have $f_2(A_1^-, A_2^-) = [4,6]$ and $f_2(A_1^+, A_2^+) = [2,15]$. Then $[4,6] \cup [11,12] \in \langle f_2(A_1^-, A_2^-), f_2(A_1^+, A_2^+)\rangle$. On the contrary, $[4,6] \cup [11,12] \notin f_2(\langle A_1^-, A_1^+\rangle, \langle A_2^-, A_2^+\rangle)$. This is because there is no $Q_1 \subseteq \mathbf{R}$ and $Q_2 \subseteq \mathbf{R}$ such that $f_2(Q_1, Q_2) = [4,6] \cup [11,12]$.

From the examples above, we know that we may have

$$\pi_{f(\langle A_1^-, A_1^+\rangle, \langle A_2^-, A_2^+\rangle, \ldots, \langle A_m^-, A_m^+\rangle)}(Y) = \pi_{\langle f(A_1^-, A_2^-, \ldots, A_m^-), f(A_1^+, A_2^+, \ldots, A_m^+)\rangle}(Y), \tag{17}$$

only for $Y \in f(2^{\mathbf{R}}, \ldots, 2^{\mathbf{R}})$.

The following theorem shows that (17) holds for a convex set $Y \subseteq \mathbf{R}$ and a monotone continuous function.

Theorem 5.3. Let $f : \mathbf{R}^m \to \mathbf{R}$ be continuous and monotone. Let A_i^- and A_i^+ be fuzzy sets showing lower and upper approximations of a graded ill-known set $\mathcal{A}_i$, $i = 1, 2, \ldots, m$. Then (17) holds for a convex set $Y \subseteq \mathbf{R}$.

Proof. We prove (17) when $m = 2$. (17) can be proved in the same way even when $m > 2$. Let $f^{-1}(Y) = \{Q_1 \times Q_2 \subseteq \mathbf{R}^2 \mid f(Q_1, Q_2) = Y\}$ for $Y \subseteq \mathbf{R}$ and $f^{-1}(y) = \{(x_1, x_2) \mid f(x_1, x_2) = y\}$ for $y \in \mathbf{R}$. For the sake of simplicity, we define graded ill-known sets $\mathcal{A}_i = \langle A_i^-, A_i^+\rangle$, $i = 1, 2$ and $\mathcal{F} = \langle f(A_1^-, A_2^-), f(A_1^+, A_2^+)\rangle$. The proof is given in two complementary cases: (i) $f^{-1}(Y) = \emptyset$ and (ii) $f^{-1}(Y) \neq \emptyset$.

First we consider (i) $f^{-1}(Y) = \emptyset$. Suppose $\forall y \in Y$, $f^{-1}(y) \neq \emptyset$. Then $\forall y \in Y$, $(Q_1 \times Q_2) \cap f^{-1}(y) \neq \emptyset$ implies $f(Q_1, Q_2) \supset Y$. It is obvious that there exists $\hat{Q}_1$ and $\hat{Q}_2$ such that $(\hat{Q}_1 \times \hat{Q}_2) \cap f^{-1}(y) \neq \emptyset$. Let $q_i^{\mathrm{L}} = \inf \hat{Q}_i$ and $q_i^{\mathrm{R}} = \sup \hat{Q}_i$, $i = 1, 2$. Because of the monotonicity, we have $f(q_1^{\mathrm{L}}, q_2^{\mathrm{L}}) \leq \inf Y$ and $f(q_1^{\mathrm{R}}, q_2^{\mathrm{R}}) \geq \sup Y$. Thus, we may find $0 \leq \lambda^{\mathrm{L}} \leq 1$ and $0 \leq \lambda^{\mathrm{R}} \leq 1$ such that

$$\begin{aligned}&f((1-\lambda^{\mathrm{L}})q_1^{\mathrm{L}} + \lambda^{\mathrm{L}} q_1^{\mathrm{R}}, (1-\lambda^{\mathrm{L}})q_2^{\mathrm{L}} + \lambda^{\mathrm{L}} q_2^{\mathrm{R}}) = \inf Y,\\ &\text{and } f((1-\lambda^{\mathrm{R}})q_1^{\mathrm{R}} + \lambda^{\mathrm{R}} q_1^{\mathrm{R}}, (1-\lambda^{\mathrm{R}})q_2^{\mathrm{R}} + \lambda^{\mathrm{R}} q_2^{\mathrm{R}}) = \inf Y,\end{aligned}$$

because of the continuity and monotonicity of f. Then we find convex sets $\bar{Q}_1 \subseteq \mathbf{R}$ and $Q_2 \subseteq \mathbf{R}$ such that

$$\begin{aligned}\inf \bar{Q}_1 &= (1-\lambda^{\mathrm{L}})q_1^{\mathrm{L}} + \lambda^{\mathrm{L}} q_1^{\mathrm{R}}, \quad \inf \bar{Q}_2 = (1-\lambda^{\mathrm{L}})q_2^{\mathrm{L}} + \lambda^{\mathrm{L}} q_2^{\mathrm{R}},\\ \sup \bar{Q}_1 &= (1-\lambda^{\mathrm{R}})q_1^{\mathrm{R}} + \lambda^{\mathrm{R}} q_1^{\mathrm{R}}, \quad \sup \bar{Q}_2 = (1-\lambda^{\mathrm{R}})q_2^{\mathrm{R}} + \lambda^{\mathrm{R}} q_2^{\mathrm{R}},\end{aligned}$$

where $\bar{Q}_1$ and $\bar{Q}_2$ include their infimums if Y includes its infimum, and $\bar{Q}_1$ and $\bar{Q}_2$ include their supremums if Y includes its supremum. For $\bar{Q}_1$ and $\bar{Q}_2$, because of the continuity of f, we have $f(\bar{Q}_1, \bar{Q}_2) = Y$. This contradicts $f^{-1}(Y) = \emptyset$. Therefore we know $f^{-1}(Y) = \emptyset$ implies $\exists y \in Y$, $f^{-1}(y) = \emptyset$.

By Definition 3.1, we have $\pi_{f(\mathcal{A}_1, \mathcal{A}_2)}(Y) = 0$ from $f^{-1}(Y) = \emptyset$. Moreover, $\inf_{y \in Y} \mu_{f(A_1^+, A_2^+)}(y) = 0$ because $\mu_{f(A_1^+, A_2^+)}(y) = 0$ for $f^{-1}(y) = \emptyset$ from the extension principle in fuzzy sets. This implies $\pi_{\mathcal{F}}(Y) = 0$. Hence, we have (17) when $f^{-1}(Y) = \emptyset$.

We now consider a case where $f^{-1}(Y) \neq \emptyset$. From the assumption, we have $f^-(\mathcal{A}_1, \mathcal{A}_2) = f(A_1^-, A_2^-)$ and $f^+(\mathcal{A}_1, \mathcal{A}_2) = f(A_1^+, A_2^+)$. Because $\pi_{\mathcal{F}}$ is the maximal possibility distribution of graded ill-known sets having lower and upper approximations $f(A_1^-, A_2^-)$ and $f(A_1^+, A_2^+)$, $\pi_{f(\mathcal{A}_1, \mathcal{A}_2)}(Y) \leq \pi_{\mathcal{F}}(Y)$. Therefore, we prove

$$\pi_{f(\mathcal{A}_1, \mathcal{A}_2)}(Y) \geq \pi_{\mathcal{F}}(Y). \tag{$*$}$$

Moreover, because $\mathcal{A}_i = \langle A_i^-, A_i^+\rangle$, $i = 1, 2$, we have

$$\begin{aligned}\pi_{f(\mathcal{A}_1, \mathcal{A}_2)}(Y) &= \sup_{\substack{Q_1, Q_2 \subseteq \mathbf{R}\\ Y = f(Q_1, Q_2)}} \min\left(\pi_{\mathcal{A}_1}(Q_1), \pi_{\mathcal{A}_2}(Q_2)\right)\\ &= \sup_{\substack{Q_1, Q_2 \subseteq \mathbf{R}\\ Y = f(Q_1, Q_2)}} \min\left(\min\left(\inf_{x \notin Q_1}(1 - \mu_{A_1^-}(x)), \inf_{x \in Q_1} \mu_{A_1^+}(x)\right),\right.\\ &\qquad\qquad \left.\min\left(\inf_{x \notin Q_2}(1 - \mu_{A_2^-}(x)), \inf_{x \in Q_2} \mu_{A_2^+}(x)\right)\right).\end{aligned}$$

expense	Expert 1	Expert 2	Expert 3
L_k	$[10,13]$	$[8,11]$	$[11,12]$
U_k	$[7,17]$	$[8,18]$	$[6,15]$
income	Expert 4	Expert 5	Expert 6
L_k	$[20,23]$	$[22,25]$	$[19,24]$
U_k	$[19,25]$	$[18,26]$	$[17,27]$

Tab. 1. Expense and income estimations ($\times 1,000$ \$).

From Definition 3.1 and Proposition 4.4, we obtain

$$\begin{aligned}&\pi_{f(\mathcal{A}_1,\mathcal{A}_2)}(Y) > \alpha\\ &\Leftrightarrow\ \exists(Q_1,Q_2) \text{ such that } Y = f(Q_1,Q_2),\\ &\qquad [A_1^-]_{1-\alpha} \subseteq Q_1 \subseteq (A_1^+)_\alpha \text{ and } [A_2^-]_{1-\alpha} \subseteq Q_2 \subseteq (A_2^+)_\alpha. \qquad (\#)\end{aligned}$$

On the other hand, from the extension principle in fuzzy sets, we obtain

$$\begin{aligned}\pi_{\mathcal{F}}(Y) \quad = \quad \min\Bigg(&\inf_{y\notin Y}\Bigg(1 - \sup_{\substack{x_1,x_2\in\mathbf{R}\\ y=f(x_1,x_2)}} \min\left(\mu_{A_1^-}(x_1), \mu_{A_2^-}(x_1)\right)\Bigg),\\ &\inf_{y\in Y}\Bigg(\sup_{\substack{x_1,x_2\in\mathbf{R}\\ y=f(x_1,x_2)}} \min\left(\mu_{A_1^+}(x_1), \mu_{A_2^+}(x_1)\right)\Bigg)\Bigg).\end{aligned}$$

In the same way of the proof of Proposition 4.4, we obtain

$$\pi_{\mathcal{F}}(Y) > \alpha \Leftrightarrow \bigcap_{\varepsilon>0} f([A_1^-]_{1-\alpha-\varepsilon}, [A_1^-]_{1-\alpha-\varepsilon}) \subseteq Y \subseteq f((A_1^+)_\alpha, (A_2^+)_\alpha). \qquad (**)$$

Now we prove $(*)$ by showing

$$\pi_{\mathcal{F}}(Y) > \alpha \text{ implies } \pi_{f(\mathcal{A}_1,\mathcal{A}_2)}(Y) > \alpha.$$

For lower and upper approximations A_i^- and A_i^+, we have $\mu_{A_i^-}(x) > 0$ implies $\mu_{A_i^+}(x) = 1$. Then we obtain $[A_i^-]_{1-\alpha} \subseteq [A_i^-]_{1-\alpha-\varepsilon} \subseteq (A_i^+)_\alpha$ for any $\varepsilon > 0$ and for $i = 1, 2$.

Assume $\pi_{\mathcal{F}}(Y) > \alpha$, from $(**)$, we obtain $f([A_1^-]_{1-\alpha}, [A_1^-]_{1-\alpha}) \subseteq \bigcap_{\varepsilon>0} f([A_1^-]_{1-\alpha-\varepsilon}, [A_1^-]_{1-\alpha-\varepsilon}) \subseteq Y$. In the same way that we prove $f^{-1}(Y) =$ implies $\exists y \in Y$, $f^{-1}(y) = \emptyset$, from the continuity and monotonicity of f, the convexity of Y and $f([A_1^-]_{1-\alpha}, [A_1^-]_{1-\alpha}) \subseteq Y \subseteq f((A_1^+)_\alpha, (A_2^+)_\alpha)$, we find $[A_i^-]_{1-\alpha} \subseteq Q_i \subseteq (A_i^+)_\alpha$, $i = 1, 2$ such that $Y = f(Q_1, Q_2)$ (see Figure 1). Hence, from (#), we obtain $\pi_{f(\mathcal{A}_1,\mathcal{A}_2)}(Y) > \alpha$. □

6. A SIMPLE EXAMPLE

In order to give an image of graded ill-known sets in the real world as well as to demonstrate the efficiency in computation owing to (9) and (11), we consider a virtual profit

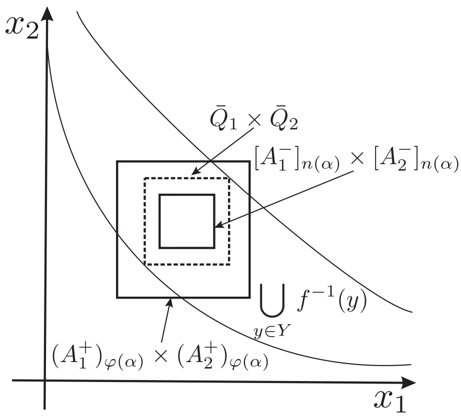

Fig. 1. $[A_i^-]_{1-\alpha} \subseteq \bar{Q}_i \subseteq (A_i^+)_\alpha$.

estimation problem. There is a small project requiring some expenses but producing incomes in future. To estimate the expected profit of the project, we asked six experts. Three of them are good at the estimation of expenses while the other three are good at the estimation of incomes. Although they are experts, due to the uncertain environment, they cannot estimate them in univocal values. Their estimations are twofold: highly possible intervals L_k and somehow possible intervals U_k such that $L_k \subseteq U_k$. As shown in Table 1, we assume that the estimations of expenses are L_k and U_k, $k = 1, 2, 3$ while the estimations of incomes are L_k and U_k, $k = 4, 5, 6$

Using L_k and U_k such that $L_k \subseteq U_k$, $k = 1, 2, \ldots, 6$, the possibility distributions $\pi_{\mathcal{A}_i}$, $i = 1, 2$ about expenses $\mathcal{A}_1$ and incomes $\mathcal{A}_2$ are defined by

$$\pi_{\mathcal{A}_i}(A) = \frac{|\{k \mid L_k \subseteq A \subseteq U_k,\ k \in [3i-2, 3i]\}|}{3}, \quad i = 1, 2, \tag{18}$$

where $|B|$ is the cardinality of set B. For the normality of $\pi_{\mathcal{A}_i}$, we assume $\bigcup_{k=3i-2,3i-1,3i} L_k \subseteq \bigcap_{k=3i-2,3i-1,3i} U_k$, $i = 1, 2$. Moreover, to satisfy (13), we assume $\bigcap_{k=3i-2,3i-1,3i} L_k \neq \emptyset$, $i = 1, 2$, otherwise $\hat{A}_i(\frac{1}{3}) = \bigcap(\mathcal{A})_{\frac{1}{3}} = \emptyset$, $i = 1, 2$.

Here, we note that the information of experts can be modeled by a basic probability assignment $Bpa_i : 2^{2^{\mathbf{R}}} \to [0, 1]$, $i = 1, 2$ such that

$$Bpa_i(\mathcal{C}) = \begin{cases} \dfrac{1}{3}, & \text{if } \exists k \in [3i-2, 3i],\ \mathcal{C} = \{A \mid L_k \subseteq A \subseteq U_k\}, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Then $\pi_{\mathcal{A}_i}$, $i = 1, 2$ of (18) can be seen as contour functions of Bpa_i, $i = 1, 2$, i.e., we have

$$\pi_{\mathcal{A}_i}(A) = \sum_{A \in \mathcal{C} \subseteq 2^{\mathbf{R}}} Bpa_i(\mathcal{C}), \quad i = 1, 2. \tag{20}$$

For $i = 1, 2$, let us define

$$\mathcal{S}_i = \left\{ A \subseteq \mathbf{R} \,\middle|\, \bigcup_{k=3i-2,3i-1,3i} L_k \subseteq A \subseteq \bigcap_{k=3i-2,3i-1,3i} U_k \right\}, \tag{21}$$

$$\mathcal{M}_i = \left\{ A \subseteq \mathbf{R} \,\middle|\, \bigcup_{k=3i-2,3i-1} L_k \subseteq A \subseteq \bigcap_{k=3i-2,3i-1} U_k, \right. \\ \left. \bigcup_{k=3i-1,3i} L_k \subseteq A \subseteq \bigcap_{k=3i-1,3i} U_k \text{ or } \bigcup_{k=3i-2,3i} L_k \subseteq A \subseteq \bigcap_{k=3i-2,3i} U_k \right\}, \tag{22}$$

$$\mathcal{W}_i = \{ A \subseteq \mathbf{R} \mid L_{3i-2} \subseteq A \subseteq U_{3i-2},\ L_{3i-1} \subseteq A \subseteq U_{3i-1} \text{ or } L_{3i} \subseteq A \subseteq U_{3i} \}. \tag{23}$$

Then $\pi_{\mathcal{A}_i}$, $i = 1, 2$ are obtained by

$$\pi_{\mathcal{A}_i}(A) = \begin{cases} 1, & \text{if } A \in \mathcal{S}_i, \\ \frac{2}{3}, & \text{if } A \notin \mathcal{S}_i \text{ and } A \in \mathcal{M}_i, \\ \frac{1}{3}, & \text{if } A \notin \mathcal{M}_i \text{ and } A \in \mathcal{W}_i, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

From (21) to (24), we easily confirm that $\mathcal{A}_i$, $i = 1, 2$ satisfy (13), i.e., $\mathcal{A}_i \in \mathcal{IQ}_{\text{int}}$, $i = 1, 2$. Then we can apply Theorem 4.2.

Let us calculate the range of profit $\mathcal{A}_2 - \mathcal{A}_1$ and ensure Theorems 3.2 and 4.2.

First we apply Definition 3.1 to $\mathcal{A}_2 - \mathcal{A}_1$. For parameters shown in Table 1, we obtain

$$\pi_{\mathcal{A}_2-\mathcal{A}_1}(A) \\ = \begin{cases} 1, & \text{if } A \in \mathcal{S}_2 - \mathcal{S}_1, \\ \frac{2}{3}, & \text{if } A \notin \mathcal{S}_2 - \mathcal{S}_1 \text{ and } A \in \mathcal{M}_2 - \mathcal{M}_1, \\ \frac{1}{3}, & \text{if } A \notin \mathcal{M}_2 - \mathcal{M}_1 \text{ and } A \in \mathcal{W}_2 - \mathcal{W}_1, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} 1, & \text{if } A \in \mathcal{S}, \\ \frac{2}{3}, & \text{if } A \notin \mathcal{S} \text{ and } A \in \mathcal{M}, \\ \frac{1}{3}, & \text{if } A \notin \mathcal{M} \text{ and } A \in \mathcal{W}, \\ 0, & \text{otherwise,} \end{cases} \tag{25}$$

where we define

$$\begin{aligned}
\mathcal{S} &= \{A \subseteq \mathbf{R} \mid [6,17] \subseteq A \subseteq [4,17]\}, &(26)\\
\mathcal{M} &= \{A \subseteq \mathbf{R} \mid [7,17] \subseteq A \subseteq [2,17],\ [6,17] \subseteq A \subseteq [1,18],\\
&\quad [8,17] \subseteq A \subseteq [4,17],\ [7,17] \subseteq A \subseteq [3,18],\ [7,15] \subseteq A \subseteq [4,18],\\
&\quad [6,15] \subseteq A \subseteq [3,19] \text{ or } [6,14] \subseteq A \subseteq [4,18]\}, &(27)\\
\mathcal{W} &= \{A \subseteq \mathbf{R} \mid [9,15] \subseteq A \subseteq [1,19],\ [6,14] \subseteq A \subseteq [0,20],\\
&\quad [11,17] \subseteq [0,18],\ [8,16] \subseteq A \subseteq [-1,19],\ [8,12] \subseteq A \subseteq [4,19],\\
&\quad [10,14] \subseteq A \subseteq [3,20] \text{ or } [7,13] \subseteq A \subseteq [2,21]\}. &(28)
\end{aligned}$$

Applying (5), we obtain the following lower and upper approximations $(A_2 - A_1)^-$ and $(A_2 - A_1)^+$:

$$\mu_{(A_2-A_1)^-}(x) = \begin{cases} 1, & \text{if } x \in [11,12],\\ \frac{2}{3}, & \text{if } x \in [8,11) \cup (12,14],\\ \frac{1}{3}, & \text{if } x \in [6,8) \cup (14,17],\\ 0, & \text{otherwise,} \end{cases} \qquad \mu_{(A_2-A_1)^+}(x) = \begin{cases} 1, & \text{if } x \in [4,17],\\ \frac{2}{3}, & \text{if } x \in [1,4) \cup (17,19],\\ \frac{1}{3}, & \text{if } x \in [-1,1) \cup (19,21],\\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

Now let us calculate $A_2^- - A_1^-$ and $A_2^+ - A_1^+$. For $i = 1, 2$, we define

$$\begin{aligned}
S_i^- &= \bigcap_{k=3i-2,3i-1,3i} L_k, \qquad S_i^+ = \bigcap_{k=3i-2,3i-1,3i} U_k,\\
M_i^- &= \bigcap_{k=3i-2,3i-1} L_k \cup \bigcap_{k=3i-1,3i} L_k \cup \bigcap_{k=3i-2,3i} L_k,\\
M_i^+ &= \bigcap_{k=3i-2,3i-1} U_k \cup \bigcap_{k=3i-1,3i} U_k \cup \bigcap_{k=3i-2,3i} U_k,\\
W_i^- &= \bigcup_{k=3i-2,3i-1,3i} L_k \quad \text{and} \quad W_i^+ = \bigcup_{k=3i-2,3i-1,3i} U_k.
\end{aligned} \tag{30}$$

For parameters given in Table 1, we obtain $S_1^- = [11,11]$, $S_1^+ = [8,15]$, $M_1^- = [10,12]$, $M_1^+ = [7,17]$, $W_1^- = [8,13]$, $W_1^+ = [6,18]$, $S_2^- = [22,23]$, $S_2^+ = [19,25]$, $M_2^- = [20,24]$, $M_2^+ = [18,26]$, $W_2^- = [19,25]$ and $W_2^+ = [17,27]$, We note that those sets always become closed interval because of $\bigcap_{k=3i-2,3i-1,3i} L_k \neq \emptyset$, $i = 1, 2$.

Then the lower and upper approximations of $\mathcal{A}_i$, $i = 1, 2$ are obtained as

$$\mu_{A_i^\pm}(x) = \begin{cases} 1, & \text{if } x \in S_i^\pm,\\ \frac{2}{3}, & \text{if } x \notin S_i^\pm \text{ and } x \in M_i^\pm,\\ \frac{1}{3}, & \text{if } x \notin M_i^\pm \text{ and } x \in W_i^\pm,\\ 0, & \text{otherwise,} \end{cases} \quad \text{(double sign in same order).} \tag{31}$$

Applying the extension principle in fuzzy set theory, we obtain

$$\mu_{A_2^- - A_1^-}(x) = \begin{cases} 1, & \text{if } x \in [11,12], \\ \frac{2}{3}, & \text{if } x \in [8,11) \cup (12,14], \\ \frac{1}{3}, & \text{if } x \in [6,8) \cup (14,17], \\ 0, & \text{otherwise,} \end{cases} \qquad \mu_{A_2^+ - A_1^+}(x) = \begin{cases} 1, & \text{if } x \in [4,17], \\ \frac{2}{3}, & \text{if } x \in [1,4) \cup (17,19], \\ \frac{1}{3}, & \text{if } x \in [-1,1) \cup (19,21], \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$

We have $(A_2 - A_1)^- = A_2^- - A_1^-$ and $(A_2 - A_1)^+ = A_2^+ - A_1^+$. Then we could confirm Theorems 3.2 and 4.2.

Even in this simple case, the calculation of $\mathcal{A}_2 - \mathcal{A}_1$ is rather complex because we should consider all combinations of minimal and maximal elements of α-level sets. A part of the complexity can be observed in the definitions of $\mathcal{M}$ and $\mathcal{W}$. On the other hand, as demonstrated above, calculations of $(A_2^- - A_1^-)$ and $(A_2^+ - A_1^+)$ are much simpler.

7. CONCLUDING REMARKS

We investigated the calculations of graded ill-known sets. We showed that the lower and upper approximations of function values of graded ill-known sets are obtained rather easily in some cases while the exact calculations are complex. We revealed the necessary and sufficient condition that lower and upper approximations of function values of graded ill-known sets are obtained by function values of lower and upper approximations of graded ill-known sets. Using this condition, we gave the sufficient conditions. From one of them, we know that the lower and upper approximations of function values of graded ill-known sets defined by lower and upper approximations are always obtained by function values of the given lower and upper approximations.

Moreover, we gave counterexamples to show that function values of graded ill-known sets defined by lower and upper approximations do not always equal to graded ill-known sets defined by function values of the given lower and upper approximations while their lower and upper approximations are. However the possibility distributions corresponding to those function values may take same membership values at function images of sets. We showed a sufficient condition that those membership values are equal. The results obtained in this paper are valuable for applications of graded ill-known sets to systems optimization, decision making, data analysis and so on. Those applications would be future topics.

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