

LEADER-FOLLOWING CONSENSUS OF MULTIPLE LINEAR SYSTEMS UNDER SWITCHING TOPOLOGIES: AN AVERAGING METHOD

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The leader-following consensus of multiple linear time invariant (LTI) systems under switching topology is considered. The leader-following consensus problem consists of designing for each agent a distributed protocol to make all agents track a leader vehicle, which has the same LTI dynamics as the agents. The interaction topology describing the information exchange of these agents is time-varying. An averaging method is proposed. Unlike the existing results in the literatures which assume the LTI agents to be neutrally stable, we relax this condition, only making assumption that the LTI agents are stabilizable and detectable. Observer-based leader-following consensus is also considered.

Keywords: consensus, multi-agent systems, averaging method

Classification: 93C15, 93C35

1. INTRODUCTION

Multi-agent system is a hot topic in a variety of research communities, such as robotics, sensor networks, artificial intelligence, automatic control and biology. Of particular interest in this field is the consensus problem, since it lays foundations for many consensus-related problems, including formation, flocking and swarming. We refer to survey papers [17, 19] and references therein for details.

Integrator and double integrator models are the simplest abstraction, upon which a large part of results on consensus of multi-agent systems have been based (see [4, 5, 8, 16, 17, 18]). To deal with more complex models, a number of recent papers are devoted to consensus of multiple LTI systems [9, 13, 14, 15, 21, 22, 25, 26, 27, 28, 29, 30]. These results keep most of the concepts provided by earlier developments, and provide new design and analysis technique, such as LQR approach, low gain approach, H_∞ approach, parametrization and geometric approach, output regulation approach, and homotopy based approach. However, most of these results [9, 13, 14, 15, 21, 25, 29, 30] mainly focus on fixed interaction topology, rather than time-varying topology. How the switches of the interaction topology and agent dynamics jointly affect the collective behavior of the multi-agent system? Attempts to understand this issue had been hampered by the lack of suitable analysis tools. The results of Scardovi et al. [22] and Ni et al. [15] are mentioned here, because of their contributions to dealing with switching topology in

the setup of high-order agent model. However, when dealing with switching topology, [22] and [15] assumed that the system of each agent is neutrally stable; thus it has no positive real parts eigenvalues. This assumption was widely assumed in the literatures when the interaction topology is fixed or switching. Unfortunately, when the agent is stabilizable and detectable rather than neutrally stable, and when the interaction topology is switching, there is no result reported in the literature to investigate the consensus of these agents.

To deal with switching graph topology and to remove the neutral stability condition, we provide a modified averaging approach, which is motivated by [1, 2, 10]. The averaging approach was initially proposed by Krylov and Bogoliubov in celestial mechanics [12], and was further developed in the work of [3, 11]; for more details refer to the recent book [20]. Closely related to the averaging theory is the stability of fast time-varying nonautonomous systems [1, 2, 10], and more specifically the fast switching systems [23, 24]. The modified approach in this paper is motivated by the work of Stilwell et al. [23], and also the work of [1, 2, 10]. Although this work borrows the idea from [23], the main difference of this work from [23] is as follows. The synchronization in [23] is achieved under fast switching condition; that is, synchronization is realized under two time scales: a time scale t for the agent dynamics and a time scale for the switching signal parameterized by t/ε with ε small enough. In our paper, we further establish that the two time scales can be made to be the same and thus the consensus in our paper is not limited to fast switching case. Furthermore, We present an extended averaging approach for consensus: a sequence of averaged systems (rather a single averaged systems) are the indicators of consensus of the the multi-agent systems. This allows to obtain more relax conditions for consensus. At last, We give further investigation on how to render these sequence of averaged systems to achieve consensus, and thus ensure the consensus of the original multi-agent systems. This was not investigated in [22]. The result in our paper shows that if there exists an infinite sequence of uniformly bounded and contiguous time intervals such that during each such interval the interaction graph is jointly connected and if the dwell time for each subgraph is appropriately small, then consensus can be achieved.

In summary, the contributions of this paper are as follows:

- Averaging method is applied to leader-following consensus of multiple LTI systems.
- Results are obtained for a wider class of agent dynamics which is stabilizable and detectable than the existing class of neutrally stable agent dynamics.
- The agent dynamics and the switching signal considered in this paper have the same time scale, rather than having different time scales considered in [23]. Thus the results in our paper are not limited to fast time switching case.

The rest of this paper is organized as follows. Section 2 contains the problem formulation and some preliminary results. Section 3 provides the main result of leader-following consensus, and extensions are made in Section 4 which devotes to observer-based protocols design and analysis. Two illustrated examples are presented in Section 5. Section 6 is a brief conclusion.

2. PROBLEM FORMULATION AND PRELIMINARIES

This section presents the multi-agent system model, with each agent being a stabilizable LTI system, which includes integrator or double integrator as its special cases. The leader-following consensus problem is formulated by use of the graph theory. Some supporting lemmas are also included here.

Consider N agents with the same dynamics

$$\dot{x}_i = Ax_i + Bu_i, \quad i = 1, 2, \dots, N, \quad (1)$$

where $x_i \in \mathbb{R}^n$ is the agent i 's state, and $u_i \in \mathbb{R}^m$ is agent i 's input through which the interactions or coupling between agent i and other agents are realized. The matrix B is of full column rank. The state information is transmitted among these agents, and the agents together with the transmission channels form a network. We use a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to describe the topology of this network, where $\mathcal{V} = \{1, 2, \dots, N\}$ is the set of nodes representing N agents and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of ordered edges (i, j) , meaning that agents i can send information to agent j .

The leader, labeled as $i = 0$, has linear dynamics as

$$\dot{x}_0 = Ax_0, \quad (2)$$

where $x_0 \in \mathbb{R}^n$ is the state. Referring agent $i \in \{1, \dots, N\}$ as follower agent, the leader's dynamics is obviously independent of follower agents. More specifically, the leader just sends information to some follower agents, without receiving information from them. The interaction structure of the whole agents $\{0, 1, \dots, N\}$ is described by an extended directed graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$, which consists of graph \mathcal{G} , node 0 and directed edges from node 0 to its information-sending follower nodes.

Definition 2.1. (Definitions related to graph) Consider a graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ with $\bar{\mathcal{V}} = \{0, 1, \dots, N\}$.

- The set of neighbors of node i relative to subgraph \mathcal{G} of $\bar{\mathcal{G}}$ is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}, j \neq i\}$.
- A directed path is a sequence of edges $(i_1, i_2), (i_2, i_3), (i_3, i_4), \dots$ in that graph.
- Node i is reachable to node j if there is a directed path from i to j .
- The graph $\bar{\mathcal{G}}$ is called connected if node 0 is reachable to any other node.

Definition 2.2. (Structure matrices of graph)

- For a directed graph \mathcal{G} on nodes $\{1, \dots, N\}$, its structure is described by its adjacency matrix $\mathcal{A} \in \mathbb{R}^{N \times N}$, whose ij th entry is 1 if (j, i) is an edge of \mathcal{G} and 0 if it is not; or by its Laplacian matrix $\mathcal{L} = -\mathcal{A} + \Lambda$, where $\Lambda \in \mathbb{R}^{N \times N}$ is the in-degree matrix of \mathcal{G} which is diagonal with i th diagonal element be $|\mathcal{N}_i|$, the cardinal of \mathcal{N}_i , which equals $\sum_{j \neq i} a_{ij}$.

- For a directed graph $\bar{\mathcal{G}}$ on the node set $\{0, 1, \dots, N\}$, one uses a matrix $\mathcal{H} = \mathcal{L} + \mathcal{D}$ to describe its structure, where \mathcal{L} is the Laplacian matrix of its subgraph \mathcal{G} and $\mathcal{D} = \text{diag}(d_1, \dots, d_N)$ with $d_i = 1$ if node $(0, i)$ is an edge of graph $\bar{\mathcal{G}}$, and with $d_i = 0$ otherwise. Obviously, the structure of the graph $\bar{\mathcal{G}}$ can also be described by its Laplacian $\bar{\mathcal{L}}$.

It is noted that the graph describing the interaction topology of nodes $\{0, 1, \dots, N\}$ can vary with time. To account this we need to consider all possible graphs $\{\bar{\mathcal{G}}_p : p \in \mathcal{P}\}$, where \mathcal{P} is an index set for all graphs defined on nodes $\{0, 1, \dots, N\}$. Obviously, \mathcal{P} is a finite set. We use $\{\mathcal{G}_p : p \in \mathcal{P}\}$ to denote subgraphs defined on vertices $\{1, \dots, N\}$. The dependence of the graphs upon time can be characterized by a switching law $\sigma : [0, \infty) \rightarrow \mathcal{P}$ which is a piece-wise constant and right continuous map; that is, at each time t , the underlying graph is $\bar{\mathcal{G}}_{\sigma(t)}$.

For each agent $i \in \{1, \dots, N\}$, if agent j is a neighbor of agent i , the relative information $x_j - x_i$ is feedback to agent i with a gain matrix K to be design later. The leader-following consensus problem consists of designing for each agent $i \in \{1, \dots, N\}$ a distributed protocol which is a linear feedback, or a dynamical feedback of

$$z_i = \sum_{j \in \mathcal{N}_i(t)} (x_j - x_i) + d_i(t)(x_0 - x_i) \quad (3)$$

such that the closed-loop systems (1)–(2) achieve the following collected behaviors:

$$\lim_{t \rightarrow \infty} \|x_i(t) - x_0(t)\| = 0, \quad i = 1, \dots, N. \quad (4)$$

To solve the leader-following consensus problem, the following assumption is proposed throughout this paper.

Assumption 2.3. The pair (A, B) is stabilizable.

The following result presents an averaging method for stability of fast time-varying linear systems. The general result for nonlinear systems can be found in [1, 2, 10]. For convenient of our later use, we rewrite the result in the following form.

Lemma 2.4. Consider a linear time-varying systems $\dot{x}(t) = A(t)x(t)$ with $A(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$, if there exists an increasing sequence of times $t_k, k \in \mathbb{Z}$, with $t_k \rightarrow +\infty$ as $k \rightarrow +\infty$, $t_k \rightarrow -\infty$ as $k \rightarrow -\infty$, and $t_{k+1} - t_k \leq T$ for some $T > 0$, such that $\forall t_k$, the following average system

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t), \quad \bar{A} = \frac{\int_{t_k}^{t_{k+1}} A(t) dt}{t_{k+1} - t_k} \quad (5)$$

is asymptotically stable, then there exists $\alpha^* > 0$ such that the following fast time-varying system

$$\dot{x}(t) = A(\alpha t)x(t) \quad (6)$$

is asymptotically stable for all $\alpha > \alpha^*$.

Remark 2.5. It has been shown in [1, Remark 4] that the value α^* can be estimated from T by solving the equation

$$e^{\frac{KT}{\alpha}} \frac{T}{\alpha} = \frac{1}{K} \left(-1 + \sqrt{1 + \frac{v}{K_v KT}} \right) \tag{7}$$

for α , where $T > 0$ is defined above and $K_v > 0, K > 0, v > 0$ are parameters which can be determined from the system matrix; furthermore, this equation has for every $T > 0, K_v > 0, K > 0, v > 0$ exactly one positive solution α . Now fixing $K_v > 0, K > 0, v > 0$, we show that as $T \rightarrow 0$ the corresponding solution $\alpha = \alpha(T) \rightarrow 0$; indeed, $T \rightarrow 0$ rends the right hand side of (7) go to infinity, thus requiring $\frac{T}{\alpha}$ on the left hand side of (7) to go to infinity, being thus resulting in $\alpha \rightarrow 0$. Therefore, appropriately choosing a small $T > 0$ gives a solution $\alpha = \alpha^* < 1$.

The following rank property of Kronecker product will be used. The proof is straightforward, being thus omitted.

Lemma 2.6. For any matrices P, Q_1, \dots, Q_n of appropriate dimensions, the following property holds:

$$\text{rank} \left(P \otimes \begin{pmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{pmatrix} \right) = \text{rank} \left(\begin{pmatrix} P \otimes Q_1 \\ P \otimes Q_1 \\ \vdots \\ P \otimes Q_n \end{pmatrix} \right).$$

The following result will also be used later.

Lemma 2.7. Consider an n -order differential system $\dot{x}(t) = A_1x(t) + A_2y(t)$ with $A_1 \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times m}$, and $y(t) \in \mathbb{R}^m$. If A_1 is Hurwitz and $\lim_{t \rightarrow \infty} y(t) = 0$, then $\lim_{t \rightarrow \infty} x(t) = 0$.

Proof. Let $x(t, x_0, y(t))$ denote the solution of $\dot{x}(t) = A_1x(t) + A_2y(t)$ with initial state x_0 at $t = 0$. Since A_1 is Hurwitz, there exist positive number α, γ_1 and γ_2 such that

$$\|x(t, x_0, y(t))\| \leq \gamma_1 \|x_0\| e^{-\alpha t} + \gamma_2 \|y(t)\|_\infty,$$

where $\|y(t)\|_\infty = \text{ess sup}_{t \geq 0} \|y(t)\|$. Since $\lim_{t \rightarrow \infty} y(t) = 0$, then for any $\varepsilon > 0$, there exists a $T > 0$ such that $\gamma_2 \|y(t)\| < \varepsilon/2$. Similarly, $\gamma_1 \|x_0\| e^{-\alpha t} < \varepsilon/2$. Therefore, $\|x(t, x_0, y(t))\| < \varepsilon$. This completes the proof. \square

3. LEADER-FOLLOWING CONSENSUS OF MULTIPLE LTI SYSTEMS

This section presents the leader-following consensus of multiple stablizable LTI systems under switching topology. Unlike most results in the literature, we do not impose assumption that A is neutrally stable. For completeness, we first review a result from [15] when the graph is fixed and undirected.

Theorem 3.1. For the multi-agent system (1)–(2) associated with connected graph $\bar{\mathcal{G}}$ under Assumption 2.3, let $P > 0$ be a solution to the Riccati inequality

$$PA + A^T P - 2\delta P B B^T P + I_n < 0, \quad (8)$$

where δ is the smallest eigenvalue of the structure matrix \mathcal{H} of graph $\bar{\mathcal{G}}$ (which is shown to be positive therein), then under the control law $u_i = K z_i$ with $K = B^T P$ all the agents follow the leader from any initial conditions.

We now treat the leader-following consensus problem under switching topologies and directed graph case. Denoting the state error between the agent i and the leader as $\varepsilon_i = x_i - x_0$, then the dynamics of ε_i is

$$\begin{aligned} \dot{\varepsilon}_i &= A\varepsilon_i + Bu_i \\ &= A\varepsilon_i + BK \sum_{j \in \mathcal{N}_i(t)} (\varepsilon_j - \varepsilon_i) - BK d_i(t) \varepsilon_i, \quad i = 1, \dots, N. \end{aligned}$$

By introducing $\varepsilon = (\varepsilon_1^T, \varepsilon_2^T, \dots, \varepsilon_N^T)^T$, one has

$$\begin{aligned} \dot{\varepsilon} &= (I_N \otimes A) \varepsilon - (I_N \otimes B)(\mathcal{L}_{\sigma(t)} \otimes I_m)(I_N \otimes K) \varepsilon - (I_N \otimes B)(\mathcal{D}_{\sigma(t)} \otimes I_m)(I_N \otimes K) \varepsilon \\ &= [I_N \otimes A - (\mathcal{L}_{\sigma(t)} + \mathcal{D}_{\sigma(t)}) \otimes (BK)] \varepsilon \\ &= [I_N \otimes A - \mathcal{H}_{\sigma(t)} \otimes (BK)] \varepsilon. \end{aligned} \quad (9)$$

The remaining issue is finding conditions on the switching topologies (i.e., conditions on the switching law σ) under which one can synthesize a feedback gain matrix K such that the zero solution of systems (9) is asymptotically stable.

As treated in [5], consider an infinite sequence of nonempty, bounded and contiguous time intervals $[t_k, t_{k+1})$, $k = 0, 1, \dots$, with $t_0 = 0$, $t_{k+1} - t_k \leq T$ for some constant $T > 0$. Suppose that in each interval $[t_k, t_{k+1})$ there is a sequence of m_k nonoverlapping subintervals

$$[t_k^1, t_k^2), \dots, [t_k^j, t_k^{j+1}), \dots, [t_k^{m_k}, t_k^{m_k+1}), \quad t_k = t_k^1, \quad t_{k+1} = t_k^{m_k+1},$$

satisfying $t_k^{j+1} - t_k^j \geq \tau$, $1 \leq j \leq m_k$ for a given constant $\tau > 0$, such that during each of such subintervals, the interconnection topology does not change. That is, during each time interval $[t_k^j, t_k^{j+1})$, the graph $\bar{\mathcal{G}}_{\sigma(t)}$ is fixed and we denote it by $\bar{\mathcal{G}}_{k_j}$. The number τ is usually call the minimal dwell time of the graphs. The $\tau > 0$ can be arbitrarily small and the existence of such an number ensures that Zero phenomena dose not happen. During each time interval $[t_k, t_{k+1})$, some or all of $\bar{\mathcal{G}}_{k_j}$, $j = 1, \dots, m_k$ are permitted to be disconnected. We only require the graph to be jointly connected, which is defined as follows:

Definition 3.2. (Joint Connectivity)

- The union of a collection of graphs is a graph whose vertex and edge sets are the unions of the vertex and edge sets of the graphs in the collection.
- The graphs are said to be jointly connected across the time interval $[t, t+T]$, $T > 0$ if the union of graphs $\{\bar{\mathcal{G}}_{\sigma(s)} : s \in [t, t+T]\}$ is connected.

Assumption 3.3. The graphs $\bar{\mathcal{G}}_{\sigma(t)}$ are jointly connected across each interval $[t_k, t_{k+1})$, $k = 0, 1, \dots$, with their length being uniformly up-bounded by a positive number T and lower-bounded by a positive number τ .

The following lemma gives a property of jointly connected graphs. When the graph is undirected, this result has been reported in [5, 6]. We show this result is still valid when the graph is directed; its proof is put in the appendix.

Lemma 3.4. Let matrices $\mathcal{H}_1, \dots, \mathcal{H}_m$ be structure matrices associated with the graphs $\bar{\mathcal{G}}_1, \dots, \bar{\mathcal{G}}_m$ respectively. If these graphs are jointly connected, then

- (1) all the eigenvalues of $\sum_{i=1}^m \mathcal{H}_i$ have positive real parts.
- (2) all the eigenvalues of $\sum_{i=1}^m \tau_i \mathcal{H}_i$ have positive real parts, where $\tau_i > 0$ and $\sum_{i=1}^m \tau_i = 1$.

With this, an averaging method by using Lemma 2.4 is applied to study the stability of system (9), whose average system during each time interval $[t_k, t_{k+1})$, $k = 0, 1, \dots$, is

$$\dot{\bar{x}} = \bar{A}\bar{x} \quad (10)$$

with

$$\begin{aligned} \bar{A} &= \frac{\int_{t_k}^{t_{k+1}} [I_N \otimes A - \mathcal{H}_{\sigma(t)} \otimes (BK)] dt}{t_{k+1} - t_k} \\ &= I_N \otimes A - \bar{\mathcal{H}}_{[t_k, t_{k+1}]} \otimes (BK), \end{aligned}$$

where $\bar{\mathcal{H}}_{[t_k, t_{k+1}]} = \sum_{t \in [t_k, t_{k+1})} \tau_{\sigma(t)} \mathcal{H}_{\sigma(t)}$, $\tau_j = (t_k^{j+1} - t_k^j) / (t_{k+1} - t_k)$, $j = k_1, \dots, k_{m_k}$. Define by $Re\lambda_{\min}(\cdot)$ the least real part of the eigenvalues of a matrix. Define

$$\bar{\delta} = \min \left\{ \inf_{(\tau_{k_1}, \dots, \tau_{k_{m_k}}) \in \Gamma_k} Re\lambda_{\min}(\bar{\mathcal{H}}_{[t_k, t_{k+1}]}) \mid k = 0, 1, \dots \right\}, \quad (11)$$

where

$$\Gamma_k = \left\{ (\tau_{k_1}, \dots, \tau_{k_{m_k}}) \mid \sum_{j=1}^{m_k} \tau_{k_j} = 1, \tau_j \leq \tau_j < 1, j = 1, \dots, m_k \right\}.$$

Noting that $Re\lambda_{\min}(\bar{\mathcal{H}}_{[t_k, t_{k+1}]})$ depends continuously on $\tau_{k_1}, \dots, \tau_{k_{m_k}}$ and the set Γ_k is compact, also by referring to Lemma 3.4, one has

$$\inf_{(\tau_{k_1}, \dots, \tau_{k_{m_k}}) \in \Gamma_k} Re\lambda_{\min}(\bar{\mathcal{H}}_{[t_k, t_{k+1}]}) = Re\lambda_{\min}(\tau_{k_1}^* \mathcal{H}_{k_1} + \dots + \tau_{k_{m_k}}^* \mathcal{H}_{k_{m_k}}) > 0,$$

which, together with the fact that the set in (11) is finite due to finiteness of all graphs, implies that $\bar{\delta}$ in (11) is a positive number. Then the leader-following consensus control can be achieved through the following theorem.

Theorem 3.5. For the multi-agent system (1)-(2) under Assumption 2.3, associated with switched graphs $\bar{\mathcal{G}}_{\sigma(t)}$ under Assumption 3.3 with T small enough, let $P > 0$ be a solution to the Riccati inequality

$$PA + A^T P - 2\bar{\delta} P B B^T P + I_n < 0, \quad (12)$$

then under the control law $u_i = K z_i$ with $K = B^T P$ all the agents follow the leader from any initial conditions.

Proof. We first prove that for each $k = 0, 1, \dots$, the average system (10) is asymptotically stable. To this end, let $T_k \in \mathbb{C}^{N \times N}$ be a unitary matrix such that $T_k \bar{\mathcal{H}}_{[t_k, t_{k+1}]} T_k^* = \bar{\Lambda}_k$ be an upper triangular matrix with the diagonal elements $\bar{\lambda}_1^k, \dots, \bar{\lambda}_N^k$ be the eigenvalues of matrix $\bar{\mathcal{H}}_{[t_k, t_{k+1}]}$, where T_k^* denote the Hermitian adjoint of matrix T_k . Setting $\tilde{x} = (T_k \otimes I_n) \bar{x}$, (10) becomes

$$\dot{\tilde{x}} = (I_N \otimes A - \bar{\Lambda}_k \otimes BK) \tilde{x}. \quad (13)$$

The stability of (13) is equivalent to stability of its diagonal system

$$\dot{\tilde{x}} = [I_N \otimes A - \text{diag}(\bar{\lambda}_1^k, \dots, \bar{\lambda}_N^k) \otimes BK] \tilde{x}, \quad (14)$$

or equivalent to the stability of the following N systems

$$\dot{\tilde{x}}_i = (A - \bar{\lambda}_i^k BB^T P) \tilde{x}_i, \quad i = 1, \dots, N. \quad (15)$$

Denoting $\bar{\lambda}_i^k = \bar{\mu}_i^k + j\bar{\nu}_i^k$, where $j^2 = -1$, then

$$\begin{aligned} & P(A - \bar{\lambda}_i^k BB^T) + (A - \bar{\lambda}_i^k BB^T)^* P \\ &= P[A - (\bar{\mu}_i^k + j\bar{\nu}_i^k) BB^T] + [A - (\bar{\mu}_i^k + j\bar{\nu}_i^k) BB^T]^* P \\ &= PA + A^T P - 2\bar{\mu}_i^k PBB^T P \\ &\leq PA + A^T P - 2\bar{\delta} PBB^T P \\ &\leq -I < 0. \end{aligned}$$

Therefore system (10) is globally asymptotically stable for each $k = 0, 1, \dots$

Using Lemma 2.4, we conclude that there exists a positive α^* dependent of T , such that $\forall \alpha > \alpha^*$, the switching system

$$\dot{\varepsilon}(t) = [I_N \otimes A - \mathcal{H}_{\sigma(\alpha t)} \otimes (BK)] \varepsilon(t) \quad (16)$$

is asymptotically stable. According to Remark 2.5, α^* can be made smaller than one if we choose T small enough. Since $\alpha > \alpha^*$ is arbitrary, just pick $\alpha = 1$. That is, system (9) is asymptotically stable, which implies that leader-following consensus is achieved. \square

Although the exact value of $\bar{\delta}$ is hard to obtain, this difficulty can be removed as follows. Noting that for two positive parameters $\bar{\delta}^* < \bar{\delta}$, if $P > 0$ is a solution of (12) for parameter $\bar{\delta}^*$, then this P is also a solution of (12) for parameter $\bar{\delta}$. Thus we can compute a positive definite matrix P with a small enough parameter $\bar{\delta}^*$ which is obviously independent the global information. This treatment has an extra advantage that it make consensus control law really distributed since the feedback gain $K = B^T P$ does not include global information.

Remark 3.6. During each interval $[t_k, t_{k+1})$, the total dwell times of the m_k graphs is upper bounded by a positive number T , which is required to be appropriately small to make $\alpha^* < 1$. This means that the dwell time of each graph can not exceed a certain bound. However, in [15] the dwell time for each graph can be arbitrary since there T is not constrained and can be chosen arbitrarily large.

Remark 3.7. Note that in (16) the switching signal $\sigma(\alpha t)$ and state $\varepsilon(t)$ have different time scales, while our result is obtained for system (9) with $\sigma(t)$ and $\varepsilon(t)$ have the same time scale, and thus the result in our paper is not limited to fast time switching case. This distinguishes this work from [23].

Remark 3.8. It can be seen that if $P > 0$ is a solution to (24), then $\kappa P, \kappa \geq 1$, is also a solution to (24). Indeed, $\kappa PA + \kappa A^T P - 2\kappa^2 \bar{\delta} P B B^T P + \bar{\delta} I_n = \kappa(PA + A^T P - 2\kappa \bar{\delta} P B B^T P + 1/\kappa I_n) \leq \kappa(PA + A^T P - 2\bar{\delta} P B B^T P + I_n) < 0$. Therefore, κK is also a stabilizing feedback matrix and the κ can be understood as the coupling strength.

4. OBSERVER-BASED LEADER-FOLLOWING CONSENSUS

This section extends the result in last section to observer-based leader-following consensus. Consider a multi-agent system consisting of N agents and a leader. The leader agent, labeled as $i = 0$, has linear dynamics as

$$\begin{aligned}\dot{x}_0 &= Ax_0, \\ y_0 &= Cx_0,\end{aligned}\tag{17}$$

where $y_0 \in \mathbb{R}^p$ is the output of the leader. The dynamics of each follower agent, labeled as $i \in \{1, \dots, N\}$, is

$$\begin{aligned}\dot{x}_i &= Ax_i + Bu_i, \\ y_i &= Cx_i,\end{aligned}\tag{18}$$

where $y_i \in \mathbb{R}^p$ is the agent i 's observed output information, and $u_i \in \mathbb{R}^m$ is agent i 's input through which the interaction or coupling between other agents is realized. More specifically, u_i is a dynamical feedback of z_i . In this section, we assume

Assumption 4.1. The pair (A, B) is stabilizable, and the pair (A, C) is detectable.

The observer-based feedback controller is represented as

$$\begin{aligned}\dot{\hat{\varepsilon}}_i &= A\hat{\varepsilon}_i + K_o(\hat{z}_i - z_i) + Bu_i, \\ u_i &= F\hat{\varepsilon}_i,\end{aligned}\tag{19}$$

where

$$\hat{z}_i = \sum_{j \in \mathcal{N}_i} (C\hat{\varepsilon}_j - C\hat{\varepsilon}_i) - d_i C\hat{\varepsilon}_i,\tag{20}$$

and the matrices K_o and F are to be designed later.

Remark 4.2. The term z_i in (19) indicates that the observer receives the output variable information from this agent's neighbors as input, and the term \hat{z}_i indicates that this observer exchanges its state with its neighboring observers. That is, each observer is implemented according to its local sensing resources. Since z_i and \hat{z}_i are local, the observer is essentially distributed, thus then feeding the state of each observer back to the corresponding agent is again a distributed control scheme.

By further introducing the following stacked vector $\hat{\varepsilon} = (\hat{\varepsilon}_1^T, \dots, \hat{\varepsilon}_N^T)^T$, $\hat{z} = (\hat{z}_1^T, \dots, \hat{z}_N^T)^T$, and by using the structure matrices of graph $\bar{\mathcal{G}}_{\sigma(t)}$, one has

$$\begin{aligned} \dot{\hat{\varepsilon}} &= (I_N \otimes A)\hat{\varepsilon} - [\mathcal{L}_{\sigma(t)} \otimes (K_o C)]\hat{\varepsilon} - [\mathcal{D}_{\sigma(t)} \otimes (K_o C)]\hat{\varepsilon} + \\ &\quad [\mathcal{L}_{\sigma(t)} \otimes (K_o C)]\varepsilon + [\mathcal{D}_{\sigma(t)} \otimes (K_o C)]\varepsilon + (I_N \otimes B)u \\ &= [I_N \otimes (A + BF) - \mathcal{H}_{\sigma(t)} \otimes (K_o C)]\hat{\varepsilon} + [\mathcal{H}_{\sigma(t)} \otimes (K_o C)]\varepsilon. \end{aligned} \quad (21)$$

Then

$$\begin{aligned} \dot{\varepsilon} &= (I_N \otimes A)\varepsilon + [I_N \otimes (BF)]\hat{\varepsilon} \\ \dot{\hat{\varepsilon}} &= [\mathcal{H}_{\sigma(t)} \otimes (K_o C)]\varepsilon + [I_N \otimes A + I_N \otimes (BF) - \mathcal{H}_{\sigma(t)} \otimes (K_o C)]\hat{\varepsilon}. \end{aligned} \quad (22)$$

Let $e = \hat{\varepsilon} - \varepsilon$, that is

$$\begin{pmatrix} \varepsilon \\ e \end{pmatrix} = \begin{pmatrix} I_{nN} & 0 \\ -I_{nN} & I_{nN} \end{pmatrix} \begin{pmatrix} \varepsilon \\ \hat{\varepsilon} \end{pmatrix}.$$

Under this coordinate transformation, system (22) becomes

$$\begin{pmatrix} \dot{\varepsilon} \\ \dot{e} \end{pmatrix} = \begin{pmatrix} I_N \otimes (A + BF) & I_N \otimes BF \\ 0 & I_N \otimes A - \mathcal{H}_{\sigma(t)} \otimes (K_o C) \end{pmatrix} \begin{pmatrix} \varepsilon \\ e \end{pmatrix}. \quad (23)$$

Therefore, observer-based leader-following consensus consists in, under jointly connected graph condition, designing matrices K_o and F such that system (23) is asymptotically stable. By separate principle and by referring to Lemma 2.7, system (23) can be made asymptotically stable through carrying out the following two-procedure design:

- Design matrix K_o such the switched system $\dot{e} = [I_N \otimes A - \mathcal{H}_{\sigma(t)} \otimes (K_o C)]e$ is asymptotically stable;
- Design matrix F such the $\dot{\varepsilon} = [I_N \otimes (A + BF)]\varepsilon$ is asymptotically stable.

The first step can be realized by referring Theorem 3.5, replacing the pair (A, B) with (A^T, C^T) . The second step is a standard state feedback control problem. We summarize above analysis in the following theorem. The rest of its proof is essentially similar to that of Theorem 3.5, and is thus omitted for save of space.

Theorem 4.3. Consider the multi-agent systems (17-18) associated with switching graphs $\bar{\mathcal{G}}_{\sigma(t)}$ under the Assumptions 3.3, 4.1 with T small enough. Let $P > 0$ be a solution to the Riccati inequality

$$PA^T + AP - 2\bar{\delta}PC^T CP + I_n < 0, \quad (24)$$

then under the control law (19) with $K_o = PC^T$ and F being such that $A + BF$ is Hurwitz, all the agents follow the leader from any initial conditions.

5. SIMULATION RESULTS

In this section, we give two examples to illustrate the validity of the results. Consider a multi-agent system consisting of a leader and four agents. Assume the system matrices are

$$A = \begin{pmatrix} 0.5548 & -0.5397 & -0.0757 \\ 0.3279 & -0.0678 & -0.4495 \\ -0.0956 & -0.6640 & 0.0130 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 5 \\ 3 & -2 \\ -8 & -8 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 & 2 \\ -4 & 2 & -3 \end{pmatrix}.$$

We suppose that possible interaction graphs are $\{\bar{G}_1, \bar{G}_2, \bar{G}_3, \bar{G}_4, \bar{G}_5, \bar{G}_6\}$ which are shown in Figure 1, and the interaction graphs are switching as $\bar{G}_1 \rightarrow \bar{G}_2 \rightarrow \bar{G}_3 \rightarrow \bar{G}_4 \rightarrow \bar{G}_5 \rightarrow \bar{G}_6 \rightarrow \bar{G}_1 \rightarrow \dots$, and each graph is active for $1/2$ second. Since the graphs $\bar{G}_1 \cup \bar{G}_2 \cup \bar{G}_3$ and $\bar{G}_4 \cup \bar{G}_5 \cup \bar{G}_6$ are connected, we can choose $t_k = k, t_{k+1} = k + 3/2$ and $t_k^0 = k, t_k^1 = k + 1/2, t_k^2 = k + 2/2, t_k^3 = k + 3/2$ with $k = 0, 1 \dots$. We choose a small parameter $\bar{\delta} = 1/3 \min(0.3820, 0.1732) = 0.0057$. The matrices K in Theorem 3.5 and K_O, F Theorem 4.3 are calculated as

$$K = \begin{pmatrix} 0.7520 & 5.9852 & -2.7041 \\ 12.6966 & -3.8441 & 1.6419 \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0.6338 & -0.5087 & 0.3731 \\ -0.9077 & 0.4509 & -0.1938 \end{pmatrix}, \quad K_O = \begin{pmatrix} -6.7092 & -9.1532 \\ -9.6111 & 4.1353 \\ 7.7514 & -1.1756 \end{pmatrix},$$

respectively. With the same initial condition, the simulation results of Theorem 2 and Theorem 4 are shown in Figure 2 and Figure 3, respectively.

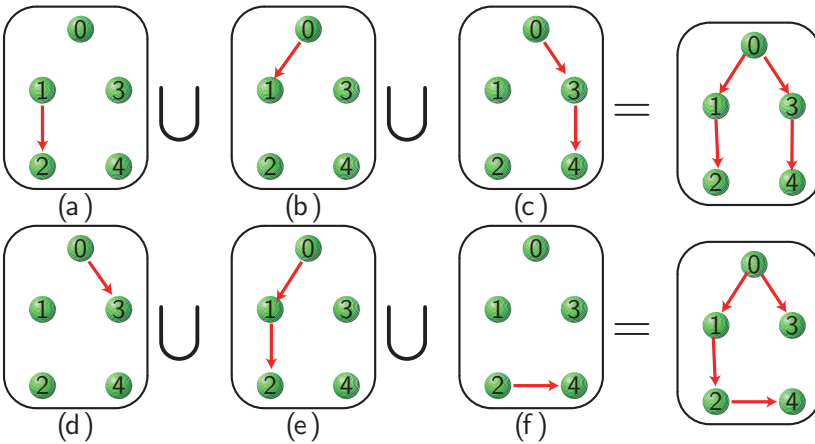


Fig. 1. Six possible interaction topologies between the leader and the agents.

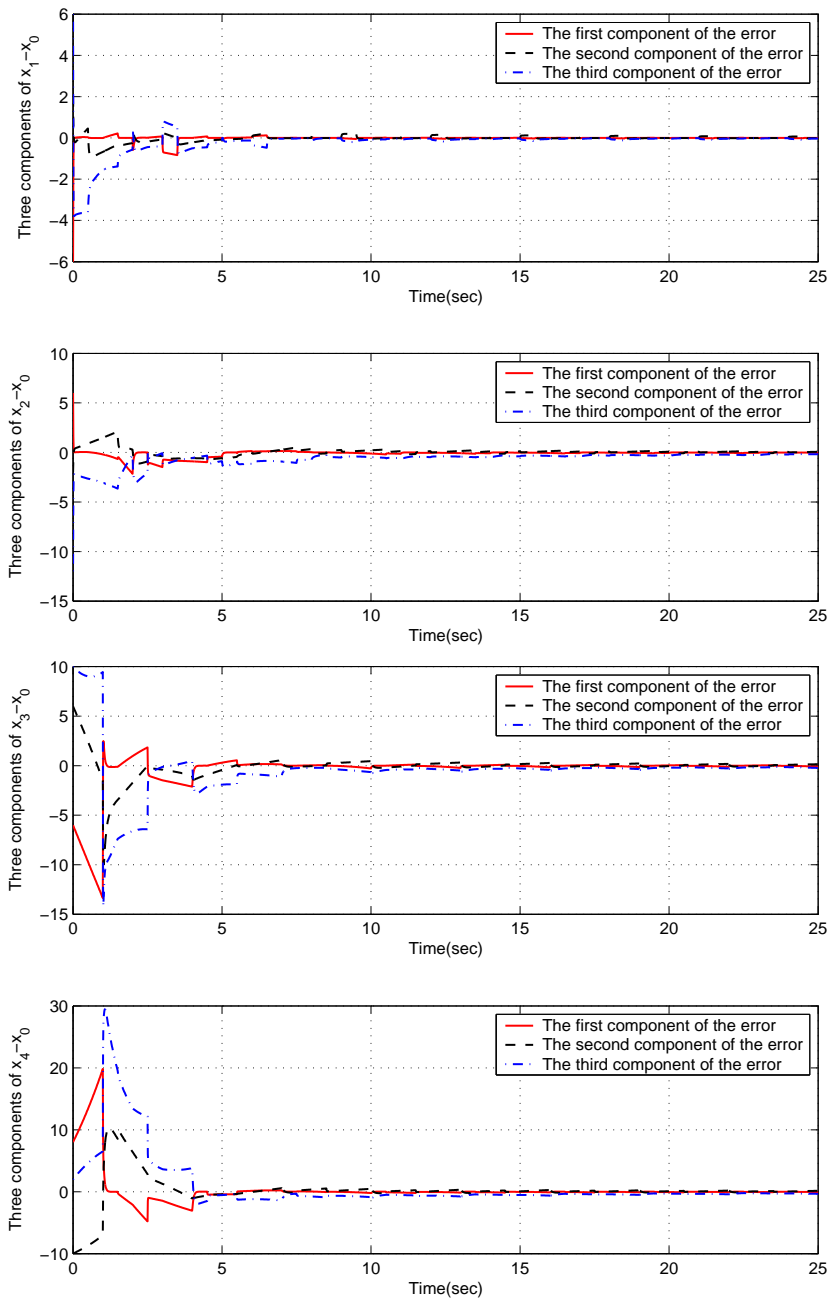


Fig. 2. Simulation for Theorem 3.5: The error trajectories between the leader and each agent.

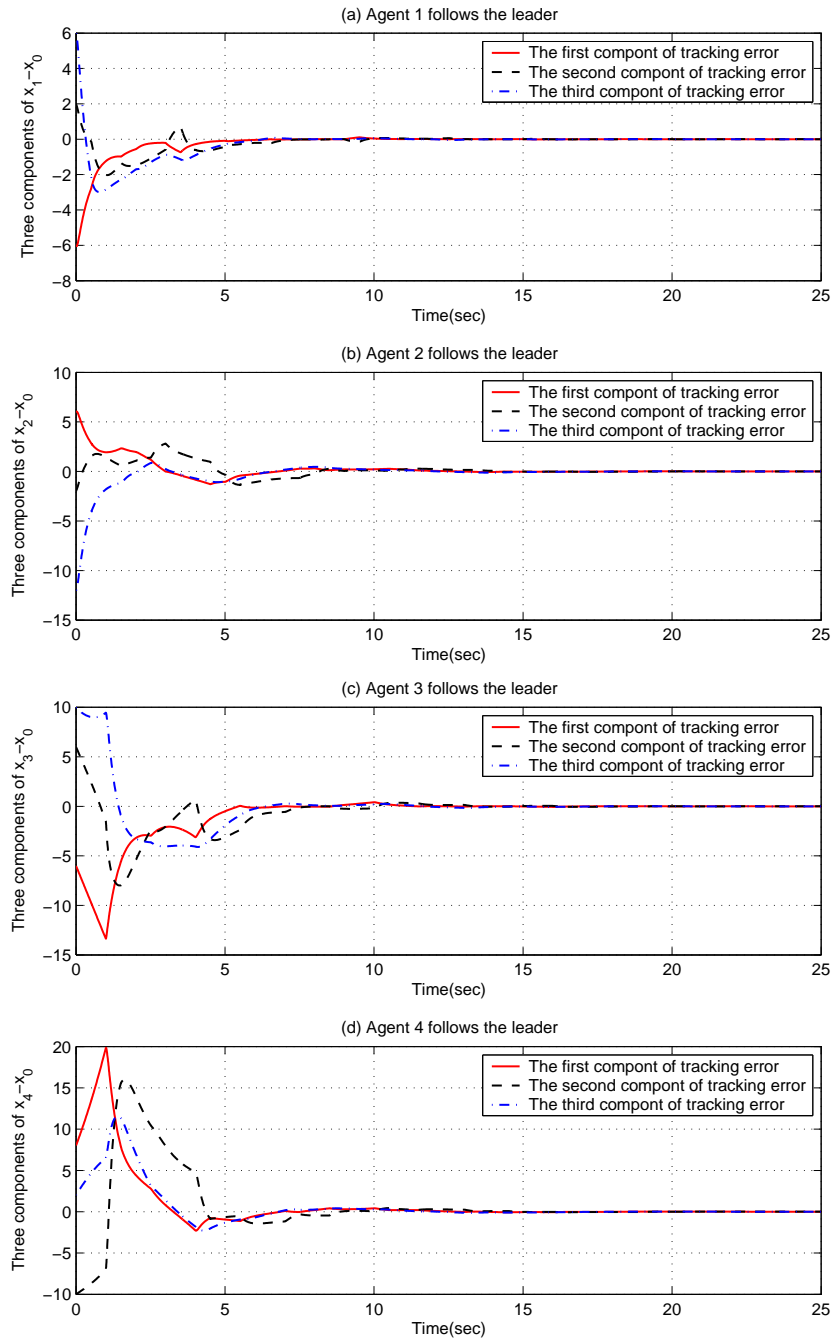


Fig. 3. Simulation for Theorem 4.3: The error trajectories between the leader and each agent.

6. CONCLUSIONS

This paper presents an averaging approach to leader-following consensus problem of multi-agent with linear dynamics, without imposing neutrally stable condition on agent dynamics. The interaction topology is switching. The proposed protocols force the follower agents to follow the independent leader trajectory. The result is extended to observer-based protocols design. Such design can be separated as a two-step procedure: design asymptotically stable distributed observers and asymptotically stable observer-state-feedback protocols.

APPENDIX

The appendix is devoted to the proof of Lemma 3.4. To this end, we first cite the following result.

Lemma 6.1. (Geršgorin [7]) For any matrix $G = [g_{ij}] \in \mathbb{R}^{N \times N}$, all the eigenvalues of G are located in the union of N Geršgorin discs

$$\text{Ger}(G) := \cup_{i=1}^N \left\{ z \in \mathbb{C} : |z - g_{ii}| \leq \sum_{j \neq i} |g_{ij}| \right\}.$$

The definition of weighted graph will also be used in the proof of what follows. If we assign each edge (i, j) of graph $\bar{\mathcal{G}}$ a weight w_{ij} , we obtain a weighted graph $\bar{\mathcal{G}}_{\mathcal{W}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{W}})$, where $\bar{\mathcal{W}} = [w_{ij}]$. For an graph $\bar{\mathcal{G}}$ and any positive number $k > 0$, the graph $k\bar{\mathcal{G}}$ is defined to be a weighted graph obtained from $\bar{\mathcal{G}}$ by assigning a weight k to each existing edge of $\bar{\mathcal{G}}$. For two graphs $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$, their union is a weighted graph and the weight for edge (i, j) is the sum of weights for the two edges (i, j) in the graphs $\bar{\mathcal{G}}_1$ and $\bar{\mathcal{G}}_2$ respectively. The weighted Laplacian of graph $\bar{\mathcal{G}}_{\mathcal{W}}$ is defined as $\bar{\mathcal{L}}_{\mathcal{W}} = -\bar{\mathcal{A}}_{\mathcal{W}} + \bar{\Lambda}_{\mathcal{W}}$, where $\bar{\mathcal{A}}_{\mathcal{W}} = [w_{ij}a_{ij}]$ and $\bar{\Lambda}_{\mathcal{W}}(i, i) = \sum_{j \neq i} w_{ij}a_{ij}$; the weighted structure matrix of graph $\bar{\mathcal{G}}_{\mathcal{W}}$ is defined as $\bar{\mathcal{H}}_{\mathcal{W}} = \bar{\mathcal{L}}_{\mathcal{W}} + \bar{\mathcal{D}}_{\mathcal{W}}$, where $\bar{\mathcal{L}}_{\mathcal{W}}$ is the weighted Laplacian of the subgraph $\bar{\mathcal{G}}_{\mathcal{W}}$ of $\bar{\mathcal{G}}_{\mathcal{W}}$, and $\bar{\mathcal{D}}_{\mathcal{W}} = \text{diag}(w_{01}d_1, \dots, w_{0N}d_N)$.

Proof of Lemma 3.4: (1) Denote the Laplacian matrix and the structure matrix of graph $\bar{\mathcal{G}}_i$ by $\bar{\mathcal{L}}_i$ and $\bar{\mathcal{H}}_i$ respectively, and denote the Laplacian matrix of graph \mathcal{G}_i by \mathcal{L}_i .

We first prove the case when $m = 1$. By definitions, it can be easily verified the following relationship

$$\bar{\mathcal{L}}_1 = \begin{pmatrix} 0 & 0 & \dots & 0 \\ -d_1 & & & \\ \vdots & & \mathcal{H}_1 & \\ -d_N & & & \end{pmatrix}.$$

Since the graph $\bar{\mathcal{G}}_1$ is connected, then $\text{rank}(\bar{\mathcal{L}}_1) = N$ [18]. Thus the sub-matrix $\bar{\mathcal{M}}_1$ formed by the last N rows of $\bar{\mathcal{L}}_1$ has rank N . Note that

$$\begin{pmatrix} -d_1 \\ \vdots \\ -d_N \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \mathcal{L}_1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \mathcal{D}_1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \mathcal{H}_1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

that is, the first column of matrix $\bar{\mathcal{M}}_1$ is a linear combination of its last N columns. Therefore, $\text{rank}(\mathcal{H}_1) = N$. Furthermore, we claim the eigenvalues of the matrix \mathcal{H}_1 are located in closed-right half plan; indeed, from Geršgorin theorem, all the eigenvalues of H are located in

$$\text{Ger}(H) = \cup_{i=1}^N \{z \in \mathbb{C} : |z - l_{ii} - d_i| \leq |N_i|\},$$

and therefore they are located in the closed-right half plan by noting that $l_{ii} = |N_i|$ and $d_i \geq 0$. We thus conclude that all the eigenvalues of \mathcal{H}_1 have positive real parts.

We proceed to prove the case when $m > 1$. Obviously, for a union $\bar{\mathcal{G}}_U$ of a group of weighted graphs $\{\bar{\mathcal{G}}_1, \dots, \bar{\mathcal{G}}_m\}$, its weighted Laplacian matrix $\bar{\mathcal{L}}_U$ is the sum of the Laplacian matrices $\{\bar{\mathcal{L}}_1, \dots, \bar{\mathcal{L}}_m\}$ of graphs $\{\bar{\mathcal{G}}_1, \dots, \bar{\mathcal{G}}_m\}$, and

$$\bar{\mathcal{L}}_U = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -d_1^U & & & \\ \vdots & \mathcal{H}_1 + & \cdots & +\mathcal{H}_m \\ -d_N^U & & & \end{pmatrix},$$

where

$$\begin{pmatrix} -d_1^U \\ \vdots \\ -d_N^U \end{pmatrix} = \begin{pmatrix} -d_1^1 \\ \vdots \\ -d_N^1 \end{pmatrix} + \cdots + \begin{pmatrix} -d_1^m \\ \vdots \\ -d_N^m \end{pmatrix}$$

with $(d_1^j, d_2^j, \dots, d_N^j)^T$ be the diagonal elements of matrix $\mathcal{D}_j, j = 1, \dots, m$. When the graphs are jointly connected, that is, when the $\bar{\mathcal{G}}_U$ is connected, the matrix $\bar{\mathcal{L}}_U$ has a simple zero eigenvalue. Argue in a manner similar to that of $m = 1$ case, it can be shown that the all the eigenvalues of the matrix $\mathcal{H}_1 + \cdots + \mathcal{H}_m$ have positive real parts.

(2) Similar discussion as given in (1) for the weighted graphs $\tau_1 \bar{\mathcal{G}}_1, \dots, \tau_N \bar{\mathcal{G}}_N$ yields the conclusion. \square

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