KYBERNETIKA — VOLUME *48* (2012), NUMBER 5, PAGES 988–1006

ON EXTREMAL DEPENDENCE OF BLOCK VECTORS

Helena Ferreira and Marta Ferreira

Due to globalization and relaxed market regulation, we have assisted to an increasing of extremal dependence in international markets. As a consequence, several measures of tail dependence have been stated in literature in recent years, based on multivariate extreme-value theory. In this paper we present a tail dependence function and an extremal coefficient of dependence between two random vectors that extend existing ones. We shall see that in weakening the usual required dependence allows to assess the amount of dependence in d-variate random vectors based on bidimensional techniques. Simple estimators will be stated and can be applied to the well-known *stable tail dependence function*. Asymptotic normality and strong consistency will be derived too. An application to financial markets will be presented at the end.

Keywords: multivariate extreme value theory, tail dependence, extremal coefficients

Classification: 60G70

1. INTRODUCTION

Dependence between extremal events have increased in recent time periods in financial markets, especially during bear markets and market crashes. The globalization and the lack of supervision are well-known contributions for this phenomena. Therefore, modern risk management is highly interested in assessing the amount of extremal dependence. The concept of tail dependence is the current tool used to this end, although it was first introduced far back in the sixties (Sibuya [28], Tiago de Oliveira [29]). Tail dependence coefficients measure the probability of occurring extreme values for one random variable (r.v.) given that another assumes an extreme value too. These coefficients can be defined via copulas of random vectors which refers to their dependence structure concerning extreme events independently of their marginal distributions. The upper tail dependence coefficient,

$$\lambda = \lim_{t \downarrow 0} P(F_X(X) > 1 - t | F_Y(Y) > 1 - t), \tag{1}$$

where F_X and F_Y are the distribution functions (d.f.'s) of X and Y, respectively, is perhaps the most referred in literature and characterizes the dependence in the tail of a random pair (X, Y), i. e., $\lambda > 0$ corresponds to tail dependence and $\lambda = 0$ means tail independence. Further references on this topic are Ledford and Tawn [14, 15], Joe [12], Coles et al. [2], Embrechts et al. [5], among others.

Multivariate formulations for tail dependence coefficients can be used to describe the amount of dependence in the orthant tail of a multivariate distribution (Marshall-Olkin [20], Wolff [30], Nelsen [21], Frahm [7], Schmid and Schmidt [24], Li [16, 17, 18], among others). These have been increasingly used in the most recent and higher demanding times. Most of the multivariate measures consider that extremal events must occur to all the components of the random vector, and obviously they are more complicated to deal with and to understand than in the bivariate case. Not surprisingly, applications hardly go any further than the three-dimensional case.

But maybe this is a too demanding condition and the occurrence of at least one extremal event in sub-vectors (blocks) of a random vector can be enough to assess dependence. For instance, how the occurrence of at least one market crash in Europe can influence the occurrence of a crash of at least one USA market too?

Based on this, we define a new tail dependence function for a random vector as a measure of the probability of occurring extreme values for the maximum of one block given that the maximum of another block assumes an extreme value too. At the unit point, this function gives rise to the here called *extremal coefficient of dependence* since it relates to the *extremal coefficient* (Tiago de Oliveira [29], Smith [26]). Connections with other tail dependence concepts known from literature will also be stated. In deriving moments we find simple estimators that can be also applied to the *stable tail dependence function*. Asymptotic normality and strong consistency are proved.

This paper is organized as follows. In Section 2 we define our new *upper-tail dependence function* and the *extremal coefficient of dependence*. We present some properties and examples. We also analyze the case of asymptotic independence. In Section 3 we present estimators and derive the respective properties of asymptotic normality and strong consistency. An application to financial data will illustrate our approach.

2. EXTREMAL DEPENDENCE BETWEEN TWO RANDOM VECTORS

Let $\mathbf{X} = (X_1, \ldots, X_d)$ be a random vector with d.f. F and continuous marginal d.f.'s F_i. For $I \subset \{1, \ldots, d\}$, define $M(I) = \bigvee_{i \in I} F_i(X_i)$ and $\mathbf{X}_I$ the sub-vector of $\mathbf{X}$ having r.v.'s with indexes in I. Consider C_F the copula function of F, i. e.,

$$F(x_1, \ldots, x_d) = C_F(F_1(x_1), \ldots, F_d(x_d)), \;\; (x_1, \ldots, x_d) \in \mathbb{R}^d. \tag{2}$$

We are going to study the dependence between extremal events concerning two sub-vectors (blocks), $\mathbf{X}_{I_1}$ and $\mathbf{X}_{I_2}$, where I_1 and I_2 are disjoint subsets of $\{1, \ldots, d\}$.

We start by extending in Definition 2.1 the concept of *upper tail dependence function* (see Schmidt and Stadtmüller (2006) and references therein) and from this we define a new tail dependence coefficient between two block vectors.

Definition 2.1. Let I_1 and I_2 be two non-empty subsets of $\{1, \ldots, d\}$. The upper-tail dependence function of $\mathbf{X}_{I_1}$ given $\mathbf{X}_{I_2}$ is defined as, for $(x, y) \in (0, \infty)^2$,

$$\Lambda_U^{(I_1|I_2)}(x, y) = \lim_{t \to \infty} P\Big(M(I_1) > 1 - \frac{x}{t} \Big| M(I_2) > 1 - \frac{y}{t}\Big),$$

provided the limit exists.

By taking $x = y = 1$, we have

$$\Lambda_U^{(I_1|I_2)}(1,1) = \lim_{t\to\infty} P\Big(M(I_1) > 1 - \frac{1}{t}\Big|M(I_2) > 1 - \frac{1}{t}\Big),$$

which is a tail dependence coefficient greater than the one considered in Li and Sun [19],

$$\gamma = \lim_{t\to\infty} P\Big(\bigcap_{i\in I_1}\Big\{F_i(X_i) > 1 - \frac{1}{t}\Big\}\Big|\bigcup_{i\in I_2}\Big\{F_i(X_i) > 1 - \frac{1}{t}\Big\}\Big),$$

which in turn is greater than the coefficient of Li [18] for $I_1 = \{1,\ldots,d\} - I_2$,

$$\tau = \lim_{t\to\infty} P\Big(\bigcap_{i\in I_1}\Big\{F_i(X_i) > 1 - \frac{1}{t}\Big\}\Big|\bigcap_{i\in I_2}\Big\{F_i(X_i) > 1 - \frac{1}{t}\Big\}\Big).$$

The tail dependence coefficient $\Lambda_U^{(I_1|I_2)}(1,1)$ give us information about the probability of occurring some extreme value in block $\{F_i(X_i), i \in I_1\}$ given that some extreme value occurs in block $\{F_i(X_i), i \in I_2\}$. Observe also that if $I_1 = \{1\}$ and $I_2 = \{2\}$, $\Lambda_U^{(I_1|I_2)}(1,1)$ is the upper tail dependence coefficient λ in (1).

Before presenting the properties of function $\Lambda_U^{(I_1|I_2)}(x,y)$ that will be the basis for the definition of our coefficient, consider the following notation:
for $(x,y) \in (0,\infty)^2$, $\emptyset \neq I_1, I_2 \subseteq \{1,\ldots,d\}$ and $i \in \{1,\ldots,d\}$, let

$$\alpha_i^{(I_1,I_2)}(u,v) = u\mathbf{1}_{I_1}(i) + v\mathbf{1}_{I_2}(i) + \mathbf{1}_{\overline{I_1\cup I_2}}(i)$$

where $\mathbf{1}(\cdot)$ is the indicator function, and for G a multivariate extreme value distribution (MEV) and C_G the respective copula function, let

$$l^{(I_1,I_2)}(x^{-1},y^{-1}) = -\log C_G(\alpha_1^{(I_1,I_2)}\big(\exp(-x),\exp(-y)\big),\ldots,\alpha_d^{(I_1,I_2)}\big(\exp(-x),\exp(-y)\big)).$$

The extremal coefficient of $\mathbf{X}_{I_1\cup I_2}$ denoted $\epsilon_{I_1\cup I_2}$, is defined as

$$C_G(\alpha_1^{(I_1,I_2)}\big(\exp(-x),\exp(-x)\big),\ldots,\alpha_d^{(I_1,I_2)}\big(\exp(-x),\exp(-x)\big)) = \exp(-x)^{\epsilon_{I_1\cup I_2}}$$

(Tiago de Oliveira 1962-63, Smith 1990), and hence we can write

$$l^{(I_1,I_2)}(x^{-1},x^{-1}) = x\epsilon_{I_1\cup I_2}.$$

Proposition 2.2. If F is in the domain of attraction of an MEV distribution G, then function $\Lambda_U^{(I_1|I_2)}(x,y)$ is defined and verifies

$$\Lambda_U^{(I_1|I_2)}(x,y) = 1 + \frac{x\epsilon_{I_1}}{y\epsilon_{I_2}} - \frac{l^{(I_1,I_2)}(x^{-1},y^{-1})}{y\epsilon_{I_2}}.$$

Proof. We have

$$\Lambda_U^{(I_1|I_2)}(x,y) = \lim_{t\to\infty}\left(1 + \frac{1 - P\big(M(I_1) \le 1 - \frac{x}{t}\big)}{1 - P\big(M(I_2) \le 1 - \frac{y}{t}\big)} - \frac{1 - P\big(M(I_1) \le 1 - \frac{x}{t}, M(I_2) \le 1 - \frac{y}{t}\big)}{1 - P\big(M(I_2) \le 1 - \frac{y}{t}\big)}\right). \quad (3)$$

On the other hand, it holds

$$
\begin{aligned}
&\lim_{t\to\infty} -t\log P\big(M(I_1)\le 1-\tfrac{x}{t}, M(I_2)\le 1-\tfrac{y}{t}\big)\\
=\ &\lim_{t\to\infty} -t\log C_F\Big(\alpha_1^{(I_1,I_2)}\big(1-\tfrac{x}{t},1-\tfrac{y}{t}\big),\ldots,\alpha_d^{(I_1,I_2)}\big(1-\tfrac{x}{t},1-\tfrac{y}{t}\big)\Big)\\
=\ &\lim_{t\to\infty} -\log C_F\Big(\alpha_1^{(I_1,I_2)}\big((1-\tfrac{x}{t})^t,(1-\tfrac{y}{t})^t\big),\ldots,\alpha_d^{(I_1,I_2)}\big((1-\tfrac{x}{t})^t,(1-\tfrac{y}{t})^t\big)\Big)\\
=\ &-\log C_G\Big(\alpha_1^{(I_1,I_2)}\big(\exp(-x),\exp(-y)\big),\ldots,\alpha_d^{(I_1,I_2)}\big(\exp(-x),\exp(-y)\big)\Big)\\
=\ &l^{(I_1,I_2)}\big(x^{-1},y^{-1}\big).
\end{aligned}
$$

Therefore, dividing the numerator and denominator of the fractions in (3) by t, we obtain

$$
\begin{aligned}
\Lambda_U^{(I_1|I_2)}(x,y) =\ & 1+\frac{l^{(I_1,\emptyset)}(x^{-1},x^{-1})}{l^{(\emptyset,I_2)}(y^{-1},y^{-1})}-\frac{l^{(I_1,I_2)}(x^{-1},y^{-1})}{l^{(\emptyset,I_2)}(y^{-1},y^{-1})}\\
=\ & 1+\frac{-\log(\exp(-x))^{\epsilon_{I_1}}}{-\log(\exp(-y))^{\epsilon_{I_2}}}-\frac{l^{(I_1,I_2)}(x^{-1},y^{-1})}{-\log(\exp(-y))^{\epsilon_{I_2}}}.
\end{aligned}
$$

□

Therefore, under the conditions of Proposition 2.2, we have

$$
y\epsilon_{I_2}\Lambda_U^{(I_1|I_2)}(x,y) = x\epsilon_{I_1}\Lambda_U^{(I_2|I_1)}(y,x) = x\epsilon_{I_1}+y\epsilon_{I_2}-l^{(I_1,I_2)}(x^{-1},y^{-1})
$$

and we will denote this common value as $\Lambda_U^{(I_1,I_2)}(x,y)$, corresponding to the probability of occurring simultaneously some extreme value in block $\{F_i(X_i), i\in I_1\}$ and in block $\{F_i(X_i), i\in I_2\}$.

Definition 2.3. The upper-tail dependence function for random vector $(\mathbf{X}_{I_1},\mathbf{X}_{I_2})$ with d.f. in the domain of attraction of an MEV is defined as

$$
\Lambda_U^{(I_1,I_2)}(x,y) = x\epsilon_{I_1}+y\epsilon_{I_2}-l^{(I_1,I_2)}(x^{-1},y^{-1}) \tag{4}
$$

and the extremal coefficient of dependence between $\mathbf{X}_{I_1}$ and $\mathbf{X}_{I_2}$ is given by $\Lambda_U^{(I_1,I_2)}(1,1)$, which we denote $\epsilon_{(I_1,I_2)}$ and hence

$$
\epsilon_{(I_1,I_2)} = \epsilon_{I_1}+\epsilon_{I_2}-\epsilon_{I_1\cup I_2}. \tag{5}
$$

The upper-tail dependence function (4) generalizes the relation of Huang [11] corresponding to $I_1=\{1\}$ and $I_2=\{2\}$,

$$
\Lambda_U(x,y) = x+y-l(x,y),
$$

where the *bivariate stable tail dependence function* in the right-side is given by

$$
l(x,y) = \lim_{t\to\infty} tP\Big(\big\{F_1(X_1)>1-\frac{x}{t}\big\}\text{ or }\big\{F_2(X_2)>1-\frac{y}{t}\big\}\Big). \tag{6}
$$

In the following, we present the expression of the tail-dependence function $\Lambda_U^{(I_1,I_2)}(x,y)$ and the value of the corresponding extremal coefficient $\epsilon_{(I_1,I_2)}$ for a d-variate random vector $\mathbf{X}$ with well-known distribution functions for its margins.

Example 2.4. Consider vector $\mathbf{X}$ with unit Fréchet margins and copula function $C_{\mathbf{X}}(u_1,\ldots,u_d)=\prod_{l=1}^{\infty}\prod_{k=-\infty}^{\infty}u_1^{\alpha_{lk1}}\wedge\ldots\wedge u_d^{\alpha_{lkd}}$, where $u_j\in[0,1]$, $j=1,\ldots,d$, and $\{\alpha_{lkj},-\infty<k<\infty,1\leq j\leq d,l\geq 1\}$ is a family of non negative constants such that $\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\alpha_{lkj}=1$, $j=1,\ldots,d$. The distribution of $\mathbf{X}$ is the MEV marginal distribution of *multivariate maxima of moving maxima* processes considered in Smith and Weissman [27]. We have

$$l^{(I_1,I_2)}(x^{-1},y^{-1})=\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\bigvee_{j=1}^{d}-\log\alpha_j^{(I_1,I_2)}\big(\exp(-x),\exp(-y)\big)\alpha_{lkj}.$$

Therefore,

$$\begin{aligned}\Lambda_U^{(I_1,I_2)}(x,y)= &\ l^{(I_1,\emptyset)}(x^{-1},x^{-1})+l^{(\emptyset,I_2)}(y^{-1},y^{-1})-l^{(I_1,I_2)}(x^{-1},y^{-1})\\ = &\ x\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\bigvee_{j\in I_1}\alpha_{lkj}+y\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\bigvee_{j\in I_2}\alpha_{lkj}\\ &-\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\left(\Big(x\bigvee_{j\in I_1}\alpha_{lkj}\Big)\vee\Big(y\bigvee_{j\in I_2}\alpha_{lkj}\Big)\right)\end{aligned}$$

and

$$\epsilon_{(I_1,I_2)}(x,y)=\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\bigvee_{j\in I_1}\alpha_{lkj}+\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\bigvee_{j\in I_2}\alpha_{lkj}-\sum_{l=1}^{\infty}\sum_{k=-\infty}^{\infty}\bigvee_{j\in I_1\cup I_2}\alpha_{lkj}.$$

Illustrating with

$$\begin{aligned}&C_{\mathbf{X}}(u_1,u_2,u_3,u_4)\\ =&\ (\textstyle\bigwedge_{i=1}^{4}u_i^{1/8}).(u_1^{5/8}\wedge u_2^{4/8}\wedge u_3^{7/8}\wedge u_4^{1/8}).(u_1^{1/8}\wedge u_2^{2/8}).(u_1^{1/8}\wedge u_2^{1/8}\wedge u_4^{6/8}),\end{aligned}$$

$I_1=\{1,2\}$ and $I_2=\{3,4\}$, we obtain

$$\begin{aligned}&\Lambda_U^{(I_1,I_2)}(x,y)\\ =&\ \Big(\tfrac{1}{8}+\tfrac{5}{8}+\tfrac{2}{8}+\tfrac{1}{8}\Big)x+\Big(\tfrac{1}{8}+\tfrac{7}{8}+\tfrac{6}{8}\Big)y-\Big(\big(x\tfrac{1}{8}\vee y\tfrac{1}{8}\big)+\big(x\tfrac{5}{8}\vee y\tfrac{7}{8}\big)+\big(x\tfrac{2}{8}\big)+\big(x\tfrac{1}{8}\vee y\tfrac{6}{8}\big)\Big)\\ =&\ \tfrac{9}{8}x+\tfrac{14}{8}y-\Big(\big(x\tfrac{1}{8}\vee y\tfrac{1}{8}\big)+\big(x\tfrac{5}{8}\vee y\tfrac{7}{8}\big)+x\tfrac{2}{8}+\big(x\tfrac{1}{8}\vee y\tfrac{6}{8}\big)\Big)\end{aligned}$$

and

$$\epsilon_{(I_1,I_2)}=\tfrac{9}{8}+\tfrac{14}{8}-\big(\tfrac{1}{8}+\tfrac{7}{8}+\tfrac{2}{8}+\tfrac{6}{8}\big)=\tfrac{7}{8}.$$

Similarly, if $I_1 = \{1,2\}$ and $I_2 = \{4\}$ we obtain

$$\Lambda_U^{(I_1,I_2)}(x,y) = \tfrac{9}{8}x + y - \Big(\big(x\tfrac{1}{8} \vee y\tfrac{1}{8}\big) + \big(x\tfrac{5}{8} \vee y\tfrac{1}{8}\big) + x\tfrac{2}{8} + \big(x\tfrac{1}{8} \vee y\tfrac{6}{8}\big)\Big)$$

and

$$\epsilon_{(I_1,I_2)} = \tfrac{9}{8} + 1 - \big(\tfrac{1}{8} + \tfrac{5}{8} + \tfrac{2}{8} + \tfrac{6}{8}\big) = \tfrac{3}{8}.$$

Example 2.5. For the symmetric logistic model we have

$$l^{(I_1,I_2)}(x,y) = -\log F(a_1^{(I_1,I_2)}(x,y), \dots, a_d^{(I_1,I_2)}(x,y)) = \Big(\sum_{j=1}^{d} (a_j^{(I_1,I_2)}(x,y))^{-1/\theta}\Big)^{\theta}$$

with $\theta \in (0,1]$, $x, y > 0$. Therefore,

$$\begin{aligned}\Lambda_U^{(I_1,I_2)}(x,y) &= l^{(I_1,\emptyset)}(x^{-1},x^{-1}) + l^{(\emptyset,I_2)}(y^{-1},y^{-1}) - l^{(I_1,I_2)}(x^{-1},y^{-1}) \\ &= \Big(\sum_{j\in I_1} x^{1/\theta}\Big)^{\theta} + \Big(\sum_{j\in I_2} y^{1/\theta}\Big)^{\theta} - \Big(\sum_{j\in I_1} x^{1/\theta} + \sum_{j\in I_2} y^{1/\theta}\Big)^{\theta} \\ &= |I_1|^{\theta} x + |I_2|^{\theta} y - \big(|I_1| x^{1/\theta} + |I_2| y^{1/\theta}\big)^{\theta}\end{aligned}$$

and

$$\epsilon_{(I_1,I_2)} = |I_1|^{\theta} + |I_2|^{\theta} - \big(|I_1| + |I_2|\big)^{\theta}.$$

Proposition 2.6. Under the conditions of Proposition 2.2 we have

(i) $0 \le \Lambda_U^{(I_1,I_2)}(x,y) \le x\epsilon_{I_1} \wedge y\epsilon_{I_2}$

(ii) $0 \le \epsilon_{(I_1,I_2)} \le \epsilon_{I_1} \wedge \epsilon_{I_2}$.

The result in (i) agrees with the one for the bivariate case. Observe that the boundary cases correspond to, respectively, independence and total dependence.

Remark 2.7. With the conventions $1/0 := \infty$ and $1/\infty := 0$, we can define $\Lambda_U^{(I_1,I_2)}(x,y)$ in $[0,\infty]^2 \backslash \{(\infty,\infty)\}$ and found $\Lambda_U^{(I_1,I_2)}(0,y) = 0 = \Lambda_U^{(I_1,I_2)}(x,0)$, $\Lambda_U^{(I_1,I_2)}(\infty,y) = y\epsilon_{I_2}$ and $\Lambda_U^{(I_1,I_2)}(x,\infty) = x\epsilon_{I_1}$.

Proposition 2.8. Under the conditions of Proposition 2.2 and Remark 2.7, for each $y \ge 0$, the partial derivative $\partial\Lambda_U^{(I_1,I_2)}/\partial x$ exists for almost all $x > 0$, and

$$0 \le \tfrac{\partial}{\partial x}\Lambda_U^{(I_1,I_2)}(x,y) \le |I_1|.$$

Similarly, for each $x \geq 0$, the partial derivative $\partial\Lambda_U^{(I_1,I_2)}/\partial x$ exists for almost all $y > 0$, and

$$0 \leq \tfrac{\partial}{\partial y}\Lambda_U^{(I_1,I_2)}(x,y) \leq |I_2|.$$

Also, the functions $x \mapsto \partial\Lambda_U^{(I_1,I_2)}(x,y)/\partial y$ and $y \mapsto \partial\Lambda_U^{(I_1,I_2)}(x,y)/\partial x$ are defined and non decreasing almost everywhere on $[0,\infty)$.

Proof. The function $\Lambda_U^{(I_1,I_2)}(x,y)$ is 2-increasing since a bivariate d.f. is 2-increasing. By Remark 2.7 we conclude that $\Lambda_U^{(I_1,I_2)}(x,y)$ is grounded. Hence, applying Lemma 2.1.5. in Nelsen [22] we have, for $(x,y),(x^*,y^*) \in [0,\infty]^2\backslash\{(\infty,\infty)\}$,

$$\begin{aligned}
&|\Lambda_U^{(I_1,I_2)}(x,y) - \Lambda_U^{(I_1,I_2)}(x^*,y^*)| \\
\leq\ & \lim_{t\to\infty} t|P(M(I_1) > 1 - \tfrac{x}{t}) - P(M(I_1) > 1 - \tfrac{x^*}{t})| \\
&+\lim_{t\to\infty} t|P(M(I_2) > 1 - \tfrac{y}{t}) - P(M(I_2) > 1 - \tfrac{y^*}{t})| \\
\leq\ & |I_1||x - x^*| + |I_2||y - y^*|.
\end{aligned}$$

Now, the proof is straightforward from Theorem 3 in Schmidt and Stadtmüller [25]. □

Remark 2.7 and Propositions 2.6.(i) and 2.8 extend, respectively, Theorems 1.i), 2.i) and 3 of Schmidt and Stadtmüller [25]. Moreover, the properties ii)-v) of Theorems 1 and 2 of Schmidt and Stadtmüller [25] are straightforward for $\Lambda_U^{(I_1,I_2)}(x,y)$.

We now discuss the case of tail independence between $M(I_1)$ and $M(I_2)$ and hence extend our context beyond an MEV distribution.

Notice that, in case of tail dependence between r.v.'s $F_1(X_1)$ and $F_2(X_2)$, the mapping

$$t \mapsto P\Big(F_1(X_1) > 1 - \frac{x}{t}, F_2(X_2) > 1 - \frac{y}{t}\Big) \tag{7}$$

is regularly varying of order -1 at ∞, and so an homogeneity property holds for large t. However, if $(F_1(X_1), F_2(X_2))$ is tail independent, this latter does not hold and an adjusted homogeneity property can be obtained by assuming that (7) is regularly varying of order $-1/\eta$ at ∞, $\eta < 1$ (the case $\eta = 1$ corresponds to tail dependence). Coefficient η is the *coefficient of tail dependence* introduced in Ledford and Tawn (1996, 1997).

Thus being, if we assume that (7) is regularly varying of order $-1/\eta$ at ∞, i. e.,

$$\lim_{t\to\infty} \frac{P\big(F_1(X_1) > 1 - x/t, F_2(X_2) > 1 - y/t\big)}{P\big(F_1(X_1) > 1 - 1/t, F_2(X_2) > 1 - 1/t\big)} = c^*(x,y) \tag{8}$$

for $(x,y) \in [0,\infty)^2$, where c^* is homogeneous of order $1/\eta$ for some $\eta \in (0,1]$ and $c^*(1,1) = 1$, then $t \mapsto P\big(F_1(X_1) > 1 - 1/t, F_2(X_2) > 1 - 1/t\big)$ is regularly varying at ∞ with index $-1/\eta$ (choose $x = y$ in (8)), and hence we can write

$$P\big(F_1(X_1) > 1 - 1/t, F_2(X_2) > 1 - 1/t\big) = t^{-1/\eta}L(t)$$

where L is a slowly varying function at ∞ (i. e., $L(tx)/L(t) \to 1$, as $t \to \infty$, for any $x > 0$). Observe that η dominates the speed of convergence of $P\big(F_1(X_1) > 1-1/t, F_2(X_2) > 1-1/t\big)$ to 0. If $\eta < 1$ then $F_1(X_1)$ and $F_2(X_2)$ (and thus X_1 and X_2) are asymptotically independent (or tail independent). In this case, the tail dependence coefficient λ in (1) is null. Conversely, asymptotic dependence holds if $\eta = 1$ and $L(t) \to a > 0$, as $t \to \infty$, and we have $\lambda > 0$. If $\eta = 1/2$ we have (almost) independence (perfect independence if $L(t) = 1$ and (8) holds with $c^*(x,y) = xy$). The cases $\eta \in (0,1/2)$ and $\eta \in (1/2,1)$ correspond to asymptotically negative independence and to asymptotically positive independence, respectively. Roughly speaking, coefficient η governs a kind of a pre-asymptotic tail behavior that allows to better estimate the probability of extreme events in case of tail independence. A bivariate extreme value distribution (BEV) allows only tail dependence ($\eta = 1$) or independence ($\eta = 1/2$), since

$$\begin{aligned} & P(F_1(X_1) > 1-1/t, F_2(X_2) > 1-1/t) \\ \sim \;& (2 - l^{(\{1\},\{2\})}(1,1))/t + ((l^{(\{1\},\{2\})}(1,1))^2/2 - 1)/t^2 \end{aligned}$$

as $t \to \infty$. For a discussion on this topic see, for instance, Ledford and Tawn [14], Draisma et al. [3] and Drees and Müller [4].

Now assume that (8) holds for random pair $(M(I_1), M(I_2))$, i. e.,

$$\lim_{t\to\infty} \frac{P\big(M(I_1) > 1-x/t, M(I_2) > 1-y/t\big)}{P\big(M(I_1) > 1-1/t, M(I_2) > 1-1/t\big)} = c_{(I_1,I_2)}(x,y) \tag{9}$$

for $(x,y) \in [0,\infty)^2$, where $c_{(I_1,I_2)}$ is homogeneous of order $1/\eta_{(I_1,I_2)}$ for some $\eta_{(I_1,I_2)} \in (0,1]$ and $c_{(I_1,I_2)}(1,1) = 1$. Taking $x = y$ in (9), one obtains that $P\big(M(I_1) > 1 - 1/t, M(I_2) > 1-1/t\big)$ is regularly varying at ∞, i. e.,

$$P\big(M(I_1) > 1-1/t, M(I_2) > 1-1/t\big) = t^{-1/\eta_{(I_1,I_2)}} L_{(I_1,I_2)}(t), \tag{10}$$

where $L_{(I_1,I_2)}(t)$ is a slowly varying function at ∞. Coefficient $\eta_{(I_1,I_2)}$ is now a measure of the speed of convergence of $P\big(M(I_1) > 1-1/t, M(I_2) > 1-1/t\big)$ to 0 and is, therefore, a coefficient of tail dependence between $M(I_1)$ and $M(I_2)$, with analogous conclusions derived for η above. Similarly, in an MEV we obtain, as $t \to \infty$,

$$P(M(I_1) > 1-1/t, M(I_2) > 1-1/t) \sim (\epsilon_{I_1} + \epsilon_{I_2} - \epsilon_{I_1\cup I_2})/t + (\epsilon^2_{I_1\cup I_2} - \epsilon^2_{I_1} - \epsilon^2_{I_2})/(2t^2).$$

Hence it only occurs asymptotic dependence whenever $\epsilon_{(I_1,I_2)} = \epsilon_{I_1} + \epsilon_{I_2} - \epsilon_{I_1\cup I_2} > 0$ (with $\eta_{(I_1,I_2)} = 1$), and otherwise independence ($\eta_{(I_1,I_2)} = 1/2$).

In the next result we compute $\eta_{(I_1,I_2)}$ and found that it is given by the maximum coefficient $\eta_{\{i\},\{j\}}$, $\forall\, i \in I_1, j \in I_2$.

Proposition 2.9. Suppose that (10) holds and

$$P\Big(\min_{i\in I, j\in J}(F_i(X_i), F_j(X_j)) > 1-1/t\Big) = t^{-1/\eta_{I,J}} L_{\eta_{I,J}}(t) \tag{11}$$

holds for all $\emptyset \neq I \subset I_1$ and $\emptyset \neq J \subset I_2$, where $L_{\eta_{I,J}}$ is a slowly varying function at ∞.

Then $\eta_{(I_1,I_2)} = \max\{\eta_{\{i\},\{j\}} : i \in I_1, j \in I_2\}$.

Proof. First observe that if $I' \subset I$ and $J' \subset J$ then

$$1 \geq t^{-1/\eta_{I',J'}} L_{\eta_{I',J'}}(t) \geq t^{-1/\eta_{I,J}} L_{\eta_{I,J}}(t).$$

We have that

$$\begin{aligned}
&P\Big(\bigvee_{i\in I_1} F_i(X_i) > 1-1/t, \bigvee_{j\in I_2} F_j(X_j) > 1-1/t\Big)\\
= \;& P\Big(\bigcup_{i\in I_1}\{F_i(X_i) > 1-1/t\}, \bigcup_{j\in I_2}\{F_j(X_j) > 1-1/t\}\Big)\\
= \;& \sum_{\emptyset\neq S\subseteq I_1} (-1)^{|S|+1} P\Big(\bigcap_{i\in S}\{F_i(X_i) > 1-1/t\}, \bigcup_{j\in I_2}\{F_j(X_j) > 1-1/t\}\Big) \qquad (12)\\
= \;& \sum_{\emptyset\neq S\subseteq I_1}\sum_{\emptyset\neq T\subseteq I_2} (-1)^{|S|+|T|} P\Big(\bigcap_{i\in S}\{F_i(X_i) > 1-1/t\}, \bigcap_{j\in T}\{F_j(X_j) > 1-1/t\}\Big)\\
= \;& \sum_{\emptyset\neq S\subseteq I_1}\sum_{\emptyset\neq T\subseteq I_2} (-1)^{|S|+|T|} t^{-1/\eta_{S,T}} L_{\eta_{S,T}}(t),
\end{aligned}$$

where in the last equality we have applied (11). Let

$$\eta = \max_{\emptyset\neq S\subseteq I_1 \emptyset\neq T\subseteq I_2} \eta_{S,T} \qquad (13)$$

From (12) and (13) we have that

$$\begin{aligned}
&P(\textstyle\bigvee_{i\in I_1} F_i(X_i) > 1-1/t, \bigvee_{j\in I_2} F_j(X_j) > 1-1/t)\\
= \;& t^{-1/\eta} L_\eta(t) \textstyle\sum_{\emptyset\neq S\subseteq I_1}\sum_{\emptyset\neq T\subseteq I_2}(-1)^{|S|+|T|} A_{S,T}(t)
\end{aligned}$$

where $A_{S,T}(t) = t^{-(1/\eta_{S,T}-1/\eta)} L^*_{\eta_{S,T}}(t)$ and $L^*_{\eta_{S,T}}(t) = L_{\eta_{S,T}}(t)/L_\eta(t)$ is a slowly varying function. Observe that, if $S' \subset S$ and $T' \subset T$, then $+\infty > A_{S',T'}(t) \geq A_{S,T}(t)$ and, by the definition of η, we have $A_{S,T}(t) = 1$ or $A_{S,T}(t) \to 0$ as $t \to \infty$, for all $S \subset I_1$ and $T \subset I_2$. Therefore,

$$P(M(I_1) > 1-1/t, M(I_2) > 1-1/t) \sim t^{-1/\eta} L_\eta(t).$$

Moreover, considering $\eta = \eta_{S_0,T_0}$ for some $S_0 \subset I_1, T_0 \subset I_2$, and so $A_{S_0,T_0}(t) = 1 \leq A_{\{i\},\{j\}}(t),\ \forall\ i \in S_0, j \in T_0$, we must have $A_{\{i\},\{j\}}(t) = 1,\ \forall\ i \in S_0, j \in T_0$. Then $\eta = \eta_{\{i\},\{j\}},\ \forall\ i \in S_0, j \in T_0$ and $\eta \leq \max_{i\in I_1, j\in I_2} \eta_{\{i\},\{j\}}$. But, by (13), $\eta \geq \max_{i\in I_1, j\in I_2} \eta_{\{i\},\{j\}}$ which leads to the result. □

In the following we present some examples where tail independence takes place.

Example 2.10. Consider $\{V_n\}_{n\geq 1}$ an i.i.d. sequence of r.v.'s with distribution $\mathtt{U}(0,1)$ and $\mathbf{X} = (X_1, X_2, X_3, X_4)$ a random vector such that, $X_1 = \min(V_3, V_2, V_1)$, $X_2 = \min(V_4, V_2, V_1)$, $X_3 = \min(V_4, V_3, V_1)$ and $X_4 = V_5$. Observe that, for $0 \leq x \leq 1$, $F_{X_1}(x) = 1-(1-x)^3 = F_{X_2}(x) = F_{X_3}(x)$ and $F_{X_4}(x) = x$ and hence $F_{X_1}^{-1}(x) = 1-(1-x)^{1/3} = F_{X_2}^{-1}(x) = F_{X_3}^{-1}(x)$ and $F_{X_4}^{-1}(x) = x$. Consider $I_1 = \{1,2\}$ and $I_2 = \{3,4\}$.

We have successively,

$$P(\min(F_1(X_1), F_3(X_3)) > 1-t^{-1}) = P(\min(F_2(X_2), F_3(X_3)) > 1-t^{-1}) = t^{-4/3},$$

and

$$P(\min(F_1(X_1), F_4(X_4)) > 1-t^{-1}) = P(\min(F_2(X_2), F_4(X_4)) > 1-t^{-1}) = t^{-2}.$$

Hence, by Proposition 2.9, we must derive $\eta_{(\{1,2\},\{3,4\})} = 3/4$.

In fact, applying (12), after some calculations we have

$$P(M(I_1) > 1-t^{-1}x, M(I_2) > 1-t^{-1}y)$$

$$= \begin{cases} 2t^{-4/3}xy^{1/3} + 2t^{-2}xy - t^{-4/3}x^{4/3} - 2t^{-7/3}xy^{4/3} - 2t^{-7/3}x^{4/3}y & , x \leq y \\ t^{-4/3}yx^{1/3} + 2t^{-2}xy - 3t^{-7/3}x^{1/3}y^2 - t^{-7/3}x^{4/3}y & , x > y. \end{cases}$$

According to (10), coefficient $\eta_{(I_1,I_2)}$ can be obtained by taking $x = y = 1$ in the expression above, and by (9) we obtain

$$c_{(\{1,2\},\{3,4\})}(x,y) = \begin{cases} 2xy^{1/3} - x^{4/3}, & x \leq y \\ yx^{1/3}, & x > y. \end{cases}$$

which is homogeneous of order $4/3$.

Similarly, if we consider $I_1 = \{1,2,3\}$ and $I_2 = \{4\}$ we obtain $\eta_{(\{1,2,3\},\{4\})} = 1/2$ and $c_{(\{1,2,3\},\{4\})}(x,y) = xy$, and if $I_1 = \{1\}$ and $I_2 = \{2,3,4\}$ we have $\eta_{(\{1\},\{2,3,4\})} = 3/4$ and

$$c_{(\{1\},\{2,3,4\})}(x,y) = \begin{cases} xy^{1/3}, & x \leq y \\ 2yx^{1/3} - y^{4/3}, & x > y \end{cases} = c_{(\{1,2\},\{3,4\})}(y,x).$$

Example 2.11. Consider $\mathbf{X} = (X_1, \ldots X_d)$ a standard d-variate Gaussian random vector with positive definite correlation matrix.The bivariate tail-dependence structure is given by

$$P(F_i(X_i) > 1-1/t, F_j(X_j) > 1-1/t) \sim C_{\rho_{i,j}} t^{-2/(1+\rho_{i,j})} (\log(t))^{-\rho_{i,j}/(1+\rho_{i,j})}, \text{ as } t \to \infty,$$

for $i,j \in \{1,\ldots,d\}$, $i<j$, where $\rho_{i,j} = corr(X_i, X_j) \notin \{-1,1\}$ and

$$C_{\rho_{i,j}} = (1+\rho_{i,j})^{3/2}(1-\rho_{i,j})^{-1/2}(4\pi)^{-\rho_{i,j}/(1+\rho_{i,j})}.$$

Hence (11) holds for $I = \{i\}$ and $J = \{j\}$ with $\eta_{i,j} = (1+\rho_{i,j})/2$ (see Ledford and Tawn [14], Draisma et al. [3]). According to Hua and Joe [10], (11) also holds for non-empty sets $I_1, I_2 \subset \{1,\ldots,d\}$. If we consider $\rho_{(I_1,I_2)} = \max\{\rho_{i,j} : i \in I_1, j \in I_2\}$ then, by Proposition 2.9, we find $\eta_{(I_1,I_2)} = (1+\rho_{(I_1,I_2)})/2$, provided the left-hand side of (10) is non-null.

3. ESTIMATION

Several estimators for the bivariate stable tail dependence function in (6) or even for the more general d-variate stable tail dependence function

$$\lim_{t\to\infty} tP\Big(\big\{F_1(X_1) > 1 - \frac{x_1}{t}\big\} \text{ or } \ldots \text{ or } \big\{F_d(X_d) > 1 - \frac{x_d}{t}\big\}\Big) \tag{14}$$

have been considered in literature. For a survey, see Krajina [13]. Parametric and semi-parametric estimators perform quite well under right model assumptions but can be disastrous otherwise (see, for instance, Frahm et al. [8]). Nonparametric estimators avoid wrong model assumptions but usually have to deal with a bias-variance trade-off arising from the following two sources. The first one is the number of block maxima, m. Thus, the larger m the smaller the variance but the larger the bias. The second source is the fact that the estimators are based on asymptotic results that depend on a sequence of positive integers, $\{k_n\}$, going to infinity at a lower rate than n. For instance, the estimator based on (6) by plugging-in the respective empirical counterparts given by

$$\begin{aligned} &\frac{n}{k_n} P_n\Big(\big\{\widehat{F}_1(X_1) > 1 - \frac{k_n}{n}x\big\} \text{ or } \big\{\widehat{F}_2(X_2) > 1 - \frac{k_n}{n}y\big\}\Big) \\ = \;& \frac{1}{k_n}\sum_{i=1}^{n} \mathbf{1}_{\{\{\widehat{F}_1(X_1)>1-\frac{k_n}{n}x\} \text{ or } \{\widehat{F}_2(X_2)>1-\frac{k_n}{n}y\}\}}, \end{aligned}$$

where $\widehat{F}_l(u) = n^{-1}\sum_{k=1}^{n} \mathbf{1}_{\{X_k \le u\}}$ is the empirical d.f. of F_l, $l = 1, 2$, is consistent and asymptotically normal if $\{k_n\}$ is an intermediate sequence, i. e., $k_n \to \infty$ and $k_n/n \to 0$, as $n \to \infty$ (Huang [11]). The choose of the value k in the sequence $\{k_n\}$ that allows the better trade-off between bias and variance is of major difficulty, since small values of k come along with a large variance whenever an increasing k results in a strong bias. Therefore, simulation studies have been carried out in order to find the best value of k that allows this compromise.

3.1. Some Properties

The following result suggests a nonparametric estimation procedure for the d-variate stable tail dependence function in (14) that only evolves a sample mean.

Proposition 3.1. If F is an MEV distribution with copula function C_F given in (2), we have, for $l(x_1, \ldots, x_d) = -\log C_F(\exp(-x_1^{-1}), \ldots, \exp(-x_d^{-1}))$,

$$l(x_1, \ldots, x_d) = \frac{E(F_1(X_1)^{x_1} \vee \ldots \vee F_d(X_d)^{x_d})}{1 - E(F_1(X_1)^{x_1} \vee \ldots \vee F_d(X_d)^{x_d})}.$$

P r o o f. Observe that the d.f. of $F_1(X_1)^{x_1} \vee \ldots \vee F_d(X_d)^{x_d}$ is given by

$$\begin{aligned} P(F_1(X_1)^{x_1} \vee \ldots \vee F_d(X_d)^{x_d} \le u) = & \; C_F\big(u^{1/x_1}, \ldots, u^{1/x_d}\big) \\ = & \; \exp\big(-l\big(-1/\log u^{1/x_1}, \ldots, -1/\log u^{1/x_d}\big)\big) \\ = & \; \exp\big(-(-\log u) l(x_1, \ldots, x_d)\big) \\ = & \; u^{l(x_1,\ldots,x_d)}. \end{aligned}$$

Hence

$$E(F_1(X_1)^{x_1} \vee \ldots \vee F_d(X_d)^{x_d}) = \int_0^1 u^{l(x_1,\ldots,x_d)} l(x_1, \ldots, x_d) \mathrm{d}u = \frac{l(x_1, \ldots, x_d)}{1 + l\big(x_1, \ldots, x_d\big)}. \quad (15)$$

□

Remark 3.2. Observe that the d-variate stable tail dependence function in (14) corresponds to $-\log C_F(\exp(-x_1), \ldots, \exp(-x_d))$.

By applying Proposition 3.1 with x_j replaced by x_j^{-1}, $j = 1, \ldots, d$, we get the following corollary.

Corollary 3.3. Under the conditions of Proposition 3.1, we have

$$x\epsilon_{I_1} \equiv l^{(I_1,\emptyset)}(x^{-1}, x^{-1}) = \frac{E(M(I_1)^{1/x})}{1 - E(M(I_1)^{1/x})},$$

$$y\epsilon_{I_2} \equiv l^{(\emptyset,I_2)}(y^{-1}, y^{-1}) = \frac{E(M(I_2)^{1/y})}{1 - E(M(I_2)^{1/y})}$$

and

$$l^{(I_1,I_2)}(x^{-1}, y^{-1}) = \frac{E(M(I_1)^{1/x} \vee M(I_2)^{1/y})}{1 - E(M(I_1)^{1/x} \vee M(I_2)^{1/y})}.$$

Consider the estimators derived from Proposition 3.1 and Corollary 3.3 by plugging-in the respective sample means, respectively,

$$\widetilde{l}(x_1, \ldots, x_d) = \frac{\overline{F_1(X_1)^{x_1} \vee \ldots \vee F_d(X_d)^{x_d}}}{1 - \overline{F_1(X_1)^{x_1} \vee \ldots \vee F_d(X_d)^{x_d}}}, \quad (16)$$

and

$$\begin{aligned} x\widetilde{\epsilon}_{I_1} = \frac{\overline{M(I_1)^{1/x}}}{1 - \overline{M(I_1)^{1/x}}}, \quad y\widetilde{\epsilon}_{I_2} = \frac{\overline{M(I_2)^{1/y}}}{1 - \overline{M(I_2)^{1/y}}} \\ \text{and} \;\; \widetilde{l}^{(I_1,I_2)}(x^{-1}, y^{-1}) = \frac{\overline{M(I_1)^{1/x} \vee M(I_2)^{1/y}}}{1 - \overline{M(I_1)^{1/x} \vee M(I_2)^{1/y}}}, \end{aligned} \quad (17)$$

where

$$\overline{M(I_1)^{1/x}} = \frac{1}{n}\sum_{i=1}^{n}\bigvee_{j\in I_1} F_j(X_j^{(i)})^{1/x}, \quad \overline{M(I_2)^{1/y}} = \frac{1}{n}\sum_{i=1}^{n}\bigvee_{j\in I_2} F_j(X_j^{(i)})^{1/y}$$

and

$$\overline{M(I_1)^{1/x}\vee M(I_2)^{1/y}} = \frac{1}{n}\sum_{i=1}^{n}\Big(\bigvee_{j\in I_1} F_j(X_j^{(i)})^{1/x}\vee\bigvee_{j\in I_2} F_j(X_j^{(i)})^{1/y}\Big).$$

We will consider two situations: the first one for known margins and the second one for unknown margins.

3.1.1. The case of known margins

In case the margins are known, it is quite straightforward to deduce the consistency and asymptotic normality of estimators (16) and (17) by the well-known Delta Method.

Proposition 3.4. Under the conditions of Proposition 3.1, we have

$$\sqrt{n}(\widetilde{l}(x_1,\ldots,x_d) - l(x_1,\ldots,x_d)) \to N(0,\sigma^2),$$

where $\widetilde{l}(x_1,\ldots,x_d)$ is the estimator derived from Proposition 3.1 by plugging-in the respective sample mean given in (16) and

$$\sigma^2 = \frac{l(x_1,\ldots,x_d)\big(1+l(x_1,\ldots,x_d)\big)^2}{\big(2+l(x_1,\ldots,x_d)\big)}.$$

Proof. Let Y_i, $i=1,\ldots,n$, be independent copies of $Y = F_1(X_1)^{x_1}\vee\ldots\vee F_d(X_d)^{x_d}$. We have that $\sqrt{n}(\overline{Y}-\mu_Y)\to N(0,\sigma_Y^2)$, where $\mu_Y = E(F_1(X_1)^{x_1}\vee\ldots\vee F_d(X_d)^{x_d})$ and $\sigma_Y^2 = Var(F_1(X_1)^{x_1}\vee\ldots\vee F_d(X_d)^{x_d})$. By a similar reasoning of (15) we derive

$$E((F_1(X_1)^{x_1}\vee\ldots\vee F_d(X_d)^{x_d})^2) = \tfrac{l(x_1,\ldots,x_d)}{2+l(x_1,\ldots,x_d)}$$

and hence,

$$Var((F_1(X_1)^{x_1}\vee\ldots\vee F_d(X_d)^{x_d})^2) = \frac{l(x_1,\ldots,x_d)}{\big(2+l(x_1,\ldots,x_d)\big)\big(1+l(x_1,\ldots,x_d)\big)^2}.$$

Let $g(x) = (1-x)^{-1}-1$. We have $[g'(\mu_Y)]^2 = (1-\mu_Y)^{-4}$ and, by the Delta Method, $\sqrt{n}(g(\overline{Y}) - x\epsilon_{I_1}) \to N(0,\sigma_Y^2(1-\mu_Y)^{-4})$. □

Corollary 3.5. Under the conditions of Proposition 3.1, we have

$$\sqrt{n}(x\widetilde{\epsilon}_{I_1} - x\epsilon_{I_1}) \to N(0,\sigma_1^2),$$

$$\sqrt{n}(y\widetilde{\epsilon}_{I_2} - y\epsilon_{I_2}) \to N(0,\sigma_2^2)$$

and

$$\sqrt{n}(\widetilde{l}^{(I_1,I_2)}(x^{-1},y^{-1}) - l^{(I_1,I_2)}(x^{-1},y^{-1})) \to N(0,\sigma_3^2),$$

where $x\widetilde{\epsilon}_{I_1}$, $y\widetilde{\epsilon}_{I_2}$ and $\widetilde{l}^{(I_1,I_2)}(x^{-1},y^{-1})$ are given in (17) and

$$\sigma_1^2 = \frac{x\epsilon_{I_1}\left(1+x\epsilon_{I_1}\right)^2}{\left(2+x\epsilon_{I_1}\right)},$$

$$\sigma_2^2 = \frac{y\epsilon_{I_2}\left(1+y\epsilon_{I_2}\right)^2}{\left(2+y\epsilon_{I_2}\right)}$$

and

$$\sigma_3^2 = \frac{l^{(I_1,I_2)}\left(x^{-1},y^{-1}\right)\left(1+l^{(I_1,I_2)}\left(x^{-1},y^{-1}\right)\right)^2}{\left(2+l^{(I_1,I_2)}\left(x^{-1},y^{-1}\right)\right)}.$$

Based on the definition in (4), a natural estimator for the upper-tail dependence function is

$$\widetilde{\Lambda}_U^{(I_1,I_2)}(x,y) = x\widetilde{\epsilon_{I_1}} + y\widetilde{\epsilon_{I_2}} - \widetilde{l}^{(I_1,I_2)}(x^{-1},y^{-1}),$$

with $x\widetilde{\epsilon_{I_1}}$, $y\widetilde{\epsilon_{I_2}}$ and $\widetilde{l}^{(I_1,I_2)}(x^{-1},y^{-1})$ stated in (17). Hence we have the following estimator for the extremal coefficient of dependence between $\mathbf{X}_{I_1}$ and $\mathbf{X}_{I_2}$:

$$\widetilde{\epsilon}_{(I_1,I_2)} = \widetilde{\epsilon_{I_1}} + \widetilde{\epsilon_{I_2}} - \widetilde{\epsilon}_{I_1\cup I_2}$$

where $\widetilde{\epsilon}_{I_1\cup I_2} = \widetilde{l}^{(I_1,I_2)}(1,1)$.

Proposition 3.6. Estimators $\widetilde{l}(x_1,\dots,x_d)$ and $\widetilde{\Lambda}_U^{(I_1,I_2)}(x,y)$ are strong consistent. Consequently, the same holds for $\widetilde{\epsilon}_{(I_1,I_2)}$.

P r o o f. Just observe that, as the sample mean $\overline{M(I_1)^{1/x}}$ converges almost surely to the mean value $E(M(I_1)^{1/x})$, i. e., $\overline{M(I_1)^{1/x}} \overset{a.s.}{\longrightarrow} E(M(I_1)^{1/x})$, then $x\widetilde{\epsilon_{I_1}} = g(\overline{M(I_1)^{1/x}}) \overset{a.s.}{\longrightarrow} x\epsilon_{I_1} = g(E(M(I_1)^{1/x}))$, where $g(x) = (1-x)^{-1} - 1$. Analogously for $y\widetilde{\epsilon_{I_2}}$, $\widetilde{l}^{(I_1,I_2)}(x^{-1},y^{-1})$ and $\widetilde{l}(x_1,\dots,x_d)$. Now, the strong consistency of $\widetilde{\Lambda}_U^{(I_1,I_2)}(x,y)$ is straightforward from

$$\begin{aligned} & |\widetilde{\Lambda}_U^{(I_1,I_2)}(x,y) - \Lambda_U^{(I_1,I_2)}(x,y)| \\ \le \ & |x\widetilde{\epsilon_{I_1}} - x\epsilon_{I_1}| + |y\widetilde{\epsilon_{I_2}} - y\epsilon_{I_2}| + |\widetilde{l}^{(I_1,I_2)}(x^{-1},y^{-1}) - l^{(I_1,I_2)}(x^{-1},y^{-1})|. \end{aligned}$$

□

3.1.2. The case of unknown margins

In case of unknown margins, we can replace F_j by an estimate $\widehat{F}_j$, $j = 1, \ldots, d$. A very common approach in multivariate extreme vale statistics is to use a modified version of the empirical d.f. of F_j, $j = 1, \ldots, d$, e. g.,

$$\widehat{F}_j(u) = \frac{1}{n+1} \sum_{k=1}^{n} \mathbf{1}_{\{X_j^{(k)} \leq u\}},$$

where the denominator $n + 1$ concerns estimation accuracy. Other modifications can also be used (see, for instance, Beirlant et al. [1]). Under this approach we denote, respectively, $\widehat{l}(x_1, \ldots, x_d)$, $x\widehat{\epsilon}_{I_1}$, $y\widehat{\epsilon}_{I_2}$ and $\widehat{l}^{(I_1,I_2)}(x^{-1}, y^{-1})$, the estimators in (16) – (17) replacing F_j with $\widehat{F}_j$, i. e., by considering,

$$\overline{\bigvee_{j\in A} \widehat{F}_j(X_j)^{x_j}} = \frac{1}{n} \sum_{i=1}^{n} \bigvee_{j\in A} \widehat{F}_j(X_j^{(i)})^{x_j}, \tag{18}$$

for each $A \subseteq \{1, \ldots, d\}$.

We still have asymptotic normality of (18) from the following result stated in Fermanian et al. [6] (Theorem 6).

Theorem 3.7. (Fermanian et al. [6], Theorem 6) Let F have continuous marginals and let copula C_F in (2) have continuous partial derivatives. Then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \{J(\widehat{F}_1(X_1^{(i)}), \ldots, \widehat{F}_d(X_d^{(i)})) - E(J(F_1(X_1^{(i)}), \ldots, F_d(X_d^{(i)})))\}$$

$$\to \quad \int_{[0,1]^d} \mathbb{G}(u_1, \ldots, u_d) \, \mathrm{d}J(u_1, \ldots, u_d)$$

in distribution in $\ell^\infty([0,1]^d)$, where the limiting process and $\mathbb{G}$ are centered Gaussian, and $J : [0,1]^d \to \mathbb{R}$ is of bounded variation, continuous from above and with discontinuities of the first kind (Neuhaus [23]).

The asymptotic normality of $\widehat{l}(x_1, \ldots, x_d)$, $x\widehat{\epsilon}_{I_1}$, $y\widehat{\epsilon}_{I_2}$ and $\widehat{l}^{(I_1,I_2)}(x^{-1}, y^{-1})$ is now derived from a general version of the Delta Method as considered in Schmidt and Stadtmüller [25] (Theorem 13).

Strong consistency of

$$\widehat{\Lambda}_U^{(I_1,I_2)}(x, y) = x\widehat{\epsilon_{I_1}} + y\widehat{\epsilon_{I_2}} - \widehat{l}^{(I_1,I_2)}(x^{-1}, y^{-1}),$$

and hence of

$$\widehat{\epsilon}_{(I_1,I_2)} = \widehat{\epsilon_{I_1}} + \widehat{\epsilon_{I_2}} - \widehat{\epsilon}_{I_1\cup I_2}, \tag{19}$$

where $\widehat{\epsilon}_{I_1\cup I_2} = \widehat{l}^{(I_1,I_2)}(1, 1)$ can also be stated.

Proposition 3.8. Estimators $\widehat{l}(x_1,\dots,x_d)$ and $\widehat{\Lambda}_U^{(I_1,I_2)}(x,y)$ are strong consistent. Therefore, the same holds for $\widehat{\epsilon}_{(I_1,I_2)}$.

Proof. The proof runs along the same lines as the one of Proposition 3.6. We only prove the more general case $\widehat{l}(x_1,\dots,x_d)\xrightarrow{a.s.} l(x_1,\dots,x_d)$. Observe that

$$\begin{aligned}
&\Big|\frac{1}{n}\sum_{i=1}^{n}\bigvee_{j\in\{1,\dots,d\}}\widehat{F_j}(X_j^{(i)})^{x_j}-E\big(\bigvee_{j\in\{1,\dots,d\}}F_j(X_j)^{x_j}\big)\Big|\\
\le\ &\Big|\frac{1}{n}\sum_{i=1}^{n}\bigvee_{j\in\{1,\dots,d\}}\widehat{F_j}(X_j^{(i)})^{x_j}-\frac{1}{n}\sum_{i=1}^{n}\bigvee_{j\in\{1,\dots,d\}}F_j(X_j^{(i)})^{x_j}\Big|\\
&+\Big|\frac{1}{n}\sum_{i=1}^{n}\bigvee_{j\in\{1,\dots,d\}}F_j(X_j^{(i)})^{x_j}-E\big(\bigvee_{j\in\{1,\dots,d\}}F_j(X_j)^{x_j}\big)\Big|,
\end{aligned}$$

where the second term converges *almost surely* to zero by the *Strong Law of Large Numbers.*

For the first term we have, successively,

$$\begin{aligned}
&|\tfrac{1}{n}\textstyle\sum_{i=1}^{n}\bigvee_{j\in\{1,\dots,d\}}\widehat{F}_j(X_j^{(i)})^{x_j}-\tfrac{1}{n}\sum_{i=1}^{n}\bigvee_{j\in\{1,\dots,d\}}F_j(X_j^{(i)})^{x_j}|\\
&\le\tfrac{1}{n}\textstyle\sum_{i=1}^{n}\bigvee_{j\in\{1,\dots,d\}}|\widehat{F}_j(X_j^{(i)})^{x_j}-F_j(X_j^{(i)})^{x_j}|\\
&\le\tfrac{1}{n}\textstyle\sum_{i=1}^{n}\sum_{j\in\{1,\dots,d\}}|\widehat{F}_j(X_j^{(i)})^{x_j}-F_j(X_j^{(i)})^{x_j}|,
\end{aligned}$$

which converges *almost surely* to zero according to Gilat and Hill ([9], proof of Theorem 1.1). □

3.2. An application to financial data

In this section we show that tail dependence is present in financial data. Our analysis is based on negative log-returns of daily closing values of the stock market indexes, CAC 40 (France), FTSE100 (UK), SMI (Swiss), XDAX (German), Dow Jones (USA), Nasdaq (USA), SP500 (USA), HSI (China), Nikkei (Japan). The period covered is January 1993 to March 2004. Since we do not have a sample of maximum values, we consider the monthly maximums in each market and group the indexes in Europe (CAC 40, FTSE100, SMI, XDAX), USA (Dow Jones, Nasdaq) and Far East (HSI, Nikkei). The scatter plots in Figure 1 show the presence of dependence between the monthly maximums in Europe and USA, Europe and Far East, USA and Far East, respectively. We are interested in assessing the amount of tail dependence between the three big world markets referred: Europe, USA and Far East, and this can be achieved through the *extremal coefficient of dependence* $\epsilon_{(I_1,I_2)}$, defined in (5). As we do not know the margins distribution, we use estimator $\widehat{\epsilon}_{(I_1,I_2)}$ in (19) based on ranks. In Table 1 are the obtained estimates for several groups, I_1 and I_2. One can see that the Far East market has less influence (lower

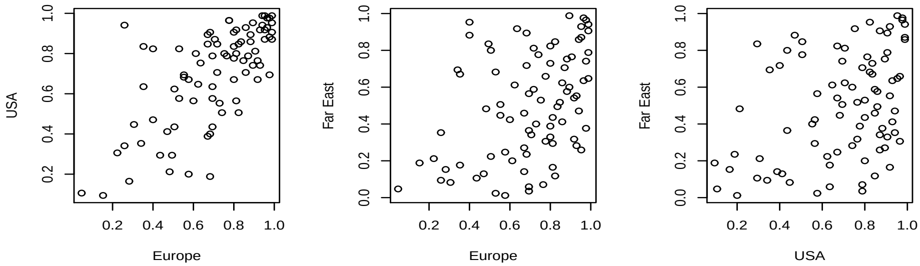

Fig. 1. Scatter plots of the monthly maximums (84 data points) in Europe versus USA, Europe versus Far East and USA versus Far East.

values of the coefficient) but Europe and USA have a stronger effect on each other and on the respective group of foreign markets. Observe that the difference between these two magnitudes of dependence is almost in the proportion 1:2.

I_1	I_2	$\widehat{\epsilon}_{(I_1,I_2)}$
Europe	USA	1.0083
Europe	Far East	0.5688
USA	Far East	0.3644
Europe	USA ∪ Far East	1.1259
USA	Europe ∪ Far East	0.9215
Far East	USA ∪ Europe	0.4820

Tab. 1. Estimates of the *extremal coefficient of dependence* $\widehat{\epsilon}_{(I_1,I_2)}$ for the indicated groups, I_1 and I_2.

4. CONCLUSION

In this work we introduce a new upper-tail dependence concept for a random vector which extends the relation of Huang [11]. Our approach weakens the usual imposed multivariate tail dependence and can be treated with bivariate techniques. The new function gives rise to the *extremal coefficient of dependence* once it is expressed through the extremal coefficient in Tiago de Oliveira [29] and Smith [26]. We also enlarge our discussion to tail independence in the sense of Ledford and Tawn [14, 15]. At this point we are beyond MEV distributions which only admit tail dependence or (exact) independence.

In calculating moments we arrive at simple estimators whose asymptotic normality is stated. These can also be applied to the well-known *stable tail dependence function*. We also prove strong consistency of the proposed estimators for our measures. We end with an illustrative application to financial data presenting tail dependence. In a future work we intend to carry out simulation studies to analyze the finite sample properties of our estimators, as well as assess their performance by comparing to other existing methods.

ACKNOWLEDGEMENT

The authors are grateful to the referees for their corrections and rigorous reports. Helena Ferreira was partially supported by the research unit "Centro de Matemática" of the University of Beira Interior and the research project PTDC/MAT/108575/2008 through the Foundation for Science and Technology (FCT) co-financed by FEDER/COMPETE. Marta Ferreira was partially supported by the Research Centre of Mathematics of the University of Minho through the FCT Pluriannual Funding Program and by PTDC/FEDER grants from Portugal.

(Received January 11, 2012)

REFERENCES

[1] J. Beirlant, Y. Goegebeur, J. Segers, and J. Teugels: Statistics of Extremes: Theory and Application. John Wiley, Chichester 2004.

[2] S. Coles, J. Heffernan, and J. Tawn: Dependence measures for extreme value analysis. Extremes *2* (1999), 339–366.

[3] G. Draisma, H. Drees, A. Ferreira, and L. de Haan: Bivariate tail estimation: dependence in asymptotic independence. Bernoulli *10* (2004), 251–280.

[4] H. Drees and P. Müller: Fitting and validation of a bivariate model for large claims. Insurance Math. Econom. *42* (2008), 638–650.

[5] P. Embrechts, F. Lindskog, and A. McNeil: Modelling dependence with copulas and applications to risk management. In: Handbook of Heavy Tailed Distibutions in Finance (S. Rachev, ed.), Elsevier, Amsterdam 2003, pp. 329–384.

[6] J. D. Fermanian, D. Radulović, and M. Wegkamp: Weak convergence of empirical copula processes. Bernoulli *10* (2004), 5, 847–860.

[7] G. Frahm: On the extremal dependence coefficient of multivariate distributions. Statist. Probab. Lett. *76* (2006), 1470–1481.

[8] G. Frahm, M. Junker, and R. Schmidt: Estimating the tail-dependence coefficient: properties and pitfalls. Insurance Math. Econom. *37* (2005), 1, 80–100.

[9] D. Gilat and T. Hill: One-sided refinements of the strong law of large numbers and the Glivenko-Cantelli theorem. Ann. Probab. *20* (1992), 1213–1221.

[10] L. Hua and H. Joe: Tail order and intermediate tail dependence of multivariate copulas. J. Multivariate Anal. *102* (2011), 10, 1454–1471.

[11] X. Huang: Statistics of Bivariate Extreme Values. Ph.D. Thesis, Tinbergen Institute Research Series 22, Erasmus University Rotterdam 1992.

[12] H. Joe: Multivariate Models and Dependence Concepts. Chapman and Hall, London 1997.

[13] A. Krajina: An M-Estimator of Multivariate Tail Dependence. Tilburg University Press 2010.

[14] A. Ledford and J. Tawn: Statistics for near independence in multivariate extreme values. Biometrika *83* (1996), 169–187.

[15] A. Ledford and J. Tawn: Modelling Dependence within joint tail regions. J. R. Statist. Soc. Ser. B Stat. Methodol. *59* (1997), 475–499.

[16] H. Li: Tail Dependence of Multivariate Pareto Distributions. WSU Mathematics Technical Report 2006-6, Washington 2006.

[17] H. Li: Tail dependence comparison of survival Marshall–Olkin copulas. Methodol. Comput. Appl. Probab. *10* (2008), 1, 39–54.

[18] H. Li: Orthant tail dependence of multivariate extreme value distributions. J. Multivariate Anal. *100* (2009), 1, 243–256.

[19] H. Li and Y. Sun: Tail dependence for heavy-tailed scale mixtures of multivariate distributions. J. Appl. Probab. *46* (2009), 4, 925–937.

[20] A. W. Marshall and I. Olkin: A multivariate exponential distribution. J. Amer. Statist. Assoc. *62* (1967), 30–44.

[21] R. B. Nelsen: Nonparametric measures of multivariate association. In: Distribution with fixed marginals and related topics, IMS Lecture Notes – Monograph Series, Vol. 28 (L. Rüschendorf et al., eds.) Hayward, Institute of Mathematical Statistics 1996, pp. 223–232.

[22] R. B. Nelsen: An Introduction to Copulas. Second edition. Springer, New York 2006.

[23] G. Neuhaus: On the weak convergence of stochastic processes with multidimensional time parameter. Ann. Math. Statist. *42* (1971), 1285–1295.

[24] F. Schmid and R. Schmidt: Multivariate conditional versions of Spearman's rho and related measures of tail dependence. J. Multivariate Anal. *98* (2007), 1123–1140.

[25] R. Schmidt and U. Stadtmüller: Nonparametric estimation of tail dependence. Scand. J. Statist. *33* (2006), 307–335.

[26] R. L. Smith: Max-stable processes and spatial extremes. Preprint, Univ. North Carolina, USA 1990.

[27] R. L. Smith and I. Weissman: Characterization and estimation of the multivariate extremal index. Manuscript, UNC 1996.

[28] M. Sibuya: Bivariate extreme statistics. Ann. Inst. Statist. Math. *11* (1960), 195–210.

[29] J. Tiago de Oliveira: Structure theory of bivariate extremes: extensions. Est. Mat. Estat. e Econ. *7* (1962/63), 165–195.

[30] E. F. Wolff: N-dimensional measures of dependence. Stochastica *4* (1980), 3, 175–188.

Helena Ferreira, Department of Mathematics, University of Beira Interior, Covilhã. Portugal.
e-mail: helena.ferreira@ubi.pt

Marta Ferreira, Department of Mathematics and Applications, University of Minho, Braga. Portugal.
e-mail: msferreira@math.uminho.pt