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THE EXISTENCE OF LIMIT CYCLE FOR PERTURBED BILINEAR SYSTEMS

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In this paper, the feedback control for a class of bilinear control systems with a small parameter is proposed to guarantee the existence of limit cycle. We use the perturbation method of seeking in approximate solution as a finite Taylor expansion of the exact solution. This perturbation method is to exploit the "smallness" of the perturbation parameter ε to construct an approximate periodic solution. Furthermore, some simulation results are given to illustrate the existence of a limit cycle for this class of nonlinear control systems.

Keywords: perturbed bilinear system, feedback control, limit cycle

Classification: 70K05, 37G15

1. INTRODUCTION

Modern applications need to solve power conversion problems to achieve more efficiency in the control design and constitute a wide and useful applications classes of perturbed systems. The use of dither signals for stabilization of nonlinear control systems is a well-known and frequently used technique. The idea is that by injecting a suitably chosen high-frequency signal in the control loop, the nonlinear sector is effectively narrowed and the system can thereby be stabilized (see [2]). In these applications, the steady state is generally depicted by a periodic motion ([3],[4],[5],[6],[8]). So it is of importance to study this limit cycle, this is due to theoretical interests as well as to powerful tool for oscillator designs, using the classical method which consists the use of the Poincare–Bendixson map technique. In [7], the limit cycle phenomenon for a class of nonlinear discrete-time systems was investigated using analytic method. In general, we have to resort to approximate solutions. The goal of the perturbation method is to exploit the "smallness" of the perturbation parameter ε to construct an approximate solutions that are valid for sufficiently small ε. In this paper, a feedback control is proposed to guarantee the existence of a limit cycle for a class of perturbed bilinear systems. In addition, we use the perturbation method as in [3] to approximate the periodic solution. The effectiveness of these approximation is verified in numerical example.

2. PROBLEM FORMULATION AND MAIN RESULTS

In this paper, we consider the following bilinear control system:

$$\dot{x}(t) = Ax(t) + u(t)B(\varepsilon)x(t), \quad \forall\, t \geq t_0 \geq 0, \tag{1}$$

$$x(t_0) = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix} \text{ where } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2, \; u \in \mathbb{R},$$

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad B(\varepsilon) = \begin{pmatrix} -1+\varepsilon & 0 \\ 0 & -1+\varepsilon \end{pmatrix}$$

with $a > 0$, $\varepsilon \neq 1$ and $x(t_0) \neq 0$. Specially, the feedback control law is selected as follows:

$$u(t) = (1-\varepsilon)\bar{r}(x_1^2(t) + x_2^2(t)), \quad \text{with } \bar{r} > 0. \tag{2}$$

Thus, the closed-loop system is given by:

$$\begin{cases} \dot{x}_1(t) = -bx_2(t) - (-1+\varepsilon)^2\bar{r}x_1(t)\left[x_1^2(t) + x_2^2(t) - \dfrac{a}{(-1+\varepsilon)^2\bar{r}}\right] \\ \\ \dot{x}_2(t) = bx_1(t) - (-1+\varepsilon)^2\bar{r}x_2(t)\left[x_1^2(t) + x_2^2(t) - \dfrac{a}{(-1+\varepsilon)^2\bar{r}}\right], \; \forall\, t \geq t_0. \end{cases} \tag{3}$$

Obviously, $x = (0,0)$ is an equilibrium point of system (3), it means that the solution of system (3) is given by $x(t) = 0$ if $x(t_0) = 0$. To avoid the trivial case of $x(t_0) = 0$, in the following, we only consider the system (3) under the case of $x(t_0) \neq 0$.

Definition 2.1. Consider the system (3). The closed and bounded manifold $s(x) = 0$, in the $x_1 - x_2$ plane, is said to be an exponentially stable limit cycle if there exists a positive number α such that the manifold of $s(x) = 0$ along the trajectories of system (3) satisfies the following inequality:

$$|s(x(t))| \leq |s(x(t_0))| \exp[-\alpha(t - t_0)], \quad \forall\, t \geq t_0 \geq 0.$$

In this case, the positive number α is called the guaranteed convergence rate.

Now, we present the main result for the existence of the exponentially stable limit cycle of system (1) as follows.

Theorem 2.2. For the feedback bilinear systems (1), all of phase trajectories tend to the exponentially stable limit cycle

$$s(x) = x_1^2 + x_2^2 - \frac{a}{(-1+\varepsilon)^2\bar{r}}$$

in the $x_1 - x_2$ plane, with the guaranteed convergence rate

$$\alpha = \begin{cases} +\infty \text{ if } x_{01}^2 + x_{02}^2 = \dfrac{a}{(-1+\varepsilon)^2\bar{r}} \\ 2a \text{ if } x_{01}^2 + x_{02}^2 > \dfrac{a}{(-1+\varepsilon)^2\bar{r}} \\ 2a(-1+\varepsilon)^2(x_{01}^2 + x_{02}^2) \text{ if } x_{01}^2 + x_{02}^2 < \dfrac{a}{(-1+\varepsilon)^2\bar{r}}. \end{cases}$$

Furthermore, the states $x_1(t)$ and $x_2(t)$ exponentially track, respectively, the trajectories

$$\sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\cos\left[b(t-t_0)+\tan^{-1}\left(\frac{x_{02}}{x_{01}}\right)\right]$$

and

$$\sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\sin\left[b(t-t_0)+\tan^{-1}\left(\frac{x_{02}}{x_{01}}\right)\right]$$

in the time domain, with the guaranteed convergence rate $\frac{\alpha}{2}$.

P r o o f. Define a smooth manifold $s(x)=0$ and a continuous function

$$\theta(x)=\tan^{-1}\left[\frac{x_2}{x_1}\right]$$

with

$$s(x)=x^Tx-\frac{a}{(-1+\varepsilon)^2\bar{r}}.$$

Then, the time derivatives of $s^2(x)$ and $\theta(x)$ along the trajectories of system (3) is given by:

$$\frac{\mathrm{d}s^2(x(t))}{\mathrm{d}t}=2s(x(t))(2x_1\dot{x}_1+2x_2\dot{x}_2)=-4(-1+\varepsilon)^2\bar{r}(x_1^2+x_2^2)s^2(x(t)), \tag{4}$$

$$\frac{\mathrm{d}\theta(x(t))}{\mathrm{d}t}=\frac{\dot{x}_2x_1-\dot{x}_1x_2}{x_1^2+x_2^2}=b, \tag{5}$$

which implies that

$$\theta(x(t))=b(t-t_0)+\tan^{-1}\left(\frac{x_{02}}{x_{01}}\right). \tag{6}$$

In the following, there are three cases to discuss the trajectories of the feedback control system of (3).

Case 1: $x_1^2(t_0)+x_2^2(t_0)=\dfrac{a}{(-1+\varepsilon)^2\bar{r}}$ (or equivalently $s(x(t_0))=0$).

In this case, from (4) it follows that

$$\frac{\mathrm{d}s^2(x(t))}{\mathrm{d}t}=0,$$

which implies that

$$x_1^2(t)+x_2^2(t)=\frac{a}{(-1+\varepsilon)^2\bar{r}}. \tag{7}$$

Hence, we obtain:

$$x_1(t)=\sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\cos\left[b(t-t_0)+\tan^{-1}\left(\frac{x_{02}}{x_{01}}\right)\right],\quad \forall\, t\geq t_0, \tag{8}$$

and

$$x_2(t) = \sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\sin\left[b(t-t_0)+\tan^{-1}\left(\frac{x_{02}}{x_{01}}\right)\right], \quad \forall\, t \geq t_0, \tag{9}$$

and

$$s(x(t)) = 0. \tag{10}$$

Case 2: $x_1^2(t_0) + x_2^2(t_0) > \dfrac{a}{(-1+\varepsilon)^2\bar{r}}$ (or equivalently $s(x(t_0)) > 0$).

In this case, from (4), we obtain:

$$\frac{\mathrm{d}s^2(x(t))}{\mathrm{d}t} \leq -4as^2(x(t)).$$

Let

$$U(t) = s^2(x(t)),$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) \leq -4aU(t).$$

Hence

$$\int_{t_0}^{t}\frac{\mathrm{d}U(t)}{U(t)} \leq \int_{t_0}^{t} -4a\,\mathrm{d}t.$$

It follows that

$$U(t) \leq U(t_0)\exp[(-4a(t-t_0))], \quad \forall\, t \geq t_0.$$

This implies that

$$s^2(x(t)) \leq |s^2(x(t_0))|\exp[(-4a(t-t_0))], \quad \forall\, t \geq t_0.$$

Then

$$s(x(t)) \leq |s(x(t_0))|\exp[(-2a(t-t_0))], \quad \forall\, t \geq t_0.$$

So, we have

$$\left|\sqrt{x_1^2(t)+x_2^2(t)} - \sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\right|^2 \leq \left|s(x(t_0))\right|\exp[(-2a(t-t_0))], \quad \forall\, t \geq t_0.$$

It yields,

$$\left|\sqrt{x_1^2(t)+x_2^2(t)} - \sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\right| \leq \sqrt{|s(x(t_0))|}\exp[(-a(t-t_0))], \quad \forall\, t \geq t_0. \tag{11}$$

Consequently, by (6) and (11), we obtain the following exponential estimations

$$\left|x_1(t) - \sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\cos\left[b(t-t_0)+\tan^{-1}\left(\frac{x_{02}}{x_{01}}\right)\right]\right|$$

$$\leq \sqrt{s(x(t_0))}\exp[(-a(t-t_0))], \quad \forall\, t \geq t_0,$$

and

$$\left| x_2(t) - \sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\sin\left[b(t-t_0) + \tan^{-1}\left(\frac{x_{02}}{x_{01}}\right)\right]\right|$$

$$\leq \sqrt{s(x(t_0))}\exp[(-a(t-t_0))], \quad \forall\, t \geq t_0.$$

Case 3: $x_1^2(t_0) + x_2^2(t_0) < \dfrac{a}{(-1+\varepsilon)^2\bar{r}}$ (or equivalently $s(x(t_0)) < 0$).

In this case, from (4), we obtain:

$$\frac{\mathrm{d}s^2(x(t))}{\mathrm{d}t} \leq -4(-1+\varepsilon)^2\bar{r}(x_{01}^2 + x_{02}^2)s^2(x(t)).$$

Let

$$U(t) = s^2(x(t)).$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) \leq -4(-1+\varepsilon)^2\bar{r}(x_{01}^2 + x_{02}^2)U(t).$$

Hence

$$\int_{t_0}^{t}\frac{\mathrm{d}U(t)}{U(t)} \leq \int_{t_0}^{t} -4(-1+\varepsilon)^2\bar{r}(x_{01}^2 + x_{02}^2)\,\mathrm{d}t.$$

It follows that

$$U(t) \leq U(t_0)\exp[(-4(-1+\varepsilon)^2\bar{r}(x_{01}^2 + x_{02}^2)(t-t_0))], \quad \forall\, t \geq t_0.$$

This implies that

$$s^2(x(t)) \leq |s^2(x(t_0))|\exp[(-4(-1+\varepsilon)^2\bar{r}(x_{01}^2 + x_{02}^2)(t-t_0))], \quad \forall\, t \geq t_0.$$

Then

$$|s(x(t))| \leq |s(x(t_0))|\exp[(-2(-1+\varepsilon)^2\bar{r}(x_{01}^2 + x_{02}^2)(t-t_0))], \quad \forall\, t \geq t_0.$$

So

$$\left|\sqrt{x_1^2(t) + x_2^2(t)} - \sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\right|^2$$

$$\leq \left|s(x(t_0))\right|\exp[(-2(-1+\varepsilon)^2\bar{r}(x_{01}^2 + x_{02}^2)(t-t_0))], \quad \forall\, t \geq t_0.$$

It yields

$$\left|\sqrt{x_1^2(t)+x_2^2(t)}-\sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\right|$$
$$\leq \sqrt{|s(x(t_0))|}\exp[(-(-1+\varepsilon)^2\bar{r}(x_{01}^2+x_{02}^2)(t-t_0))], \quad \forall t \geq t_0. \tag{12}$$

Consequently, by (6) and (12), we obtain the following exponential estimations

$$\left|x_1(t)-\sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\cos\left[b(t-t_0)+\tan^{-1}\left(\frac{x_{02}}{x_{01}}\right)\right]\right|$$
$$\leq \sqrt{|s(x(t_0))|}\exp[(-(-1+\varepsilon)^2\bar{r}(x_{01}^2+x_{02}^2)((t-t_0))], \quad \forall t \geq t_0,$$

and

$$\left|x_2(t)-\sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}}\sin\left[b(t-t_0)+\tan^{-1}\left(\frac{x_{02}}{x_{01}}\right)\right]\right|$$
$$\leq \sqrt{|s(x(t_0))|}\exp[(-(-1+\varepsilon)^2\bar{r}(x_{01}^2+x_{02}^2)((t-t_0))], \quad \forall t \geq t_0.$$

This completes the proof. □

Remark 2.3. Obviously, by theorem 2.2, such bilinear feedback system of (1) can be represented as nonlinear oscillators with

$$\text{the amplitude } \sqrt{\frac{a}{(-1+\varepsilon)^2\bar{r}}} \text{ and the frequency } b.$$

Such oscillations are generally independent of the initial condition and limit cycles of such oscillation are not influenced by a small parameter variation.

3. THE PERTURBATION METHOD

Consider the system (3). Suppose we want to solve the state equation (3) for a given initial state

$$\begin{cases} x_1(t_0)=\eta_1(\varepsilon) \\ x_2(t_0)=\eta_2(\varepsilon) \end{cases} \tag{13}$$

where, for more generality, we allow the initial state to depend "smoothly" on ε. The solution of (3)−(13) will depend on the parameter ε, a point that we shall emphasize by writing the solution as $(x_1(t,\varepsilon), x_2(t,\varepsilon))$.
The goal of the perturbation method is to exploit the "smallness" of the perturbation parameter ε to construct approximate solution that will be valid for sufficiently small $|\varepsilon|$. The simplest approximation results by setting $\varepsilon = 0$ in (3)−(13) to obtain the nominal or unperturbed problem:

$$\begin{cases} \dot{x}_1(t) = -bx_2(t) - \bar{r}x_1(t)\left[x_1^2(t)+x_2^2(t)-\frac{a}{\bar{r}}\right] \\ \dot{x}_2(t) = bx_1(t) - \bar{r}x_2(t)\left[x_1^2(t)+x_2^2(t)-\frac{a}{\bar{r}}\right], \ \forall t \geq t_0, \end{cases} \tag{14}$$

with initial state

$$\begin{cases} x_1(t_0) = \eta_1(0) \\ x_2(t_0) = \eta_2(0). \end{cases}$$

Suppose the problem has an unique solution $(x_{10}(t), x_{20}(t))$ defined on $[t_0, t_1]$ and

$$(x_{10}(t), x_{20}(t)) \in \mathbb{R}^2, \quad \forall\, t \in [t_0, t_1].$$

Suppose further that

$$(t, x, \varepsilon) \longrightarrow Ax(t) + u(t)B(\varepsilon)x(t)$$

and η are twice continuously differentiable in their arguments for (t, x, ε) in

$$[t_0, t_1] \times \mathbb{R}^2 \times [-\varepsilon_0, \varepsilon_0].$$

From continuity of solutions with respect to initial states and parameters (see theorem 2 in [3] pp. 97) we know that, for sufficiently small $|\varepsilon|$, the problem (3) – (13) has an unique solution $(x_1(t, \varepsilon), x_2(t, \varepsilon))$ defined on $[t_0, t_1]$.

Approximating $(x_1(t, \varepsilon), x_2(t, \varepsilon))$ by $(x_{10}(t), x_{20}(t))$ can be justified by using Taylor's theorem (see [1]) for $(x_1(t, \varepsilon), x_2(t, \varepsilon))$ to show that

$$\|(x_1(t, \varepsilon), x_2(t, \varepsilon)) - (x_{10}(t), x_{20}(t))\| \leq k|\varepsilon|, \quad \forall\, |\varepsilon| < \varepsilon_1, \quad \forall\, t \in [t_0, t_1],$$

for some $k > 0$ and $\varepsilon_1 \leq \varepsilon_0$.

If we succeed the showing this bound on the approximation error, we then say that the error is of order $O(\varepsilon)$ and write

$$(x_1(t, \varepsilon), x_2(t, \varepsilon)) - (x_{10}(t), x_{20}(t)) = O(\varepsilon).$$

This order of magnitude will be used frequently in this section, it is defined as follows.

Definition 3.1. $\delta_1(\varepsilon) = O(\delta_2(\varepsilon))$ if there exist positive constants k and c such that

$$|\delta_1(\varepsilon)| \leq k|\delta_2(\varepsilon)|, \quad \forall\, |\varepsilon| < c.$$

Consider the following bilinear control systems (1). Suppose we want to construct a finite Taylor series with $N = 3$. Let

$$x_i = x_{i0} + \varepsilon x_{i1} + \varepsilon^2 x_{i2} + \varepsilon^3 x_{iR}, \quad i = 1, 2, \tag{15}$$

and

$$\eta_i = \eta_{i0} + \varepsilon \eta_{i1} + \varepsilon^2 \eta_{i2} + \varepsilon^3 \eta_{iR}, \quad i = 1, 2.$$

Substitution of (15) in (3), yields

$$\begin{aligned} \dot{x}_{10} + \varepsilon \dot{x}_{11} + \varepsilon^2 \dot{x}_{12} + \varepsilon^3 \dot{x}_{1R} = &-b[x_{20} + \varepsilon x_{21} + \varepsilon^2 x_{22} + \varepsilon^3 x_{2R}] \\ &-(-1+\varepsilon)^2 \bar{r}(x_{10} + \varepsilon x_{11} + \varepsilon^2 x_{12} + \varepsilon^3 x_{1R}) \end{aligned}$$

$$\times \left[(x_{10} + \varepsilon x_{11} + \varepsilon^2 x_{12} + \varepsilon^3 x_{1R})^2 + (x_{20} + \varepsilon x_{21} + \varepsilon^2 x_{22} + \varepsilon^3 x_{2R})^2 - \frac{a}{(-1+\varepsilon)^2 \bar{r}} \right]$$

and

$$\begin{aligned} \dot{x}_{20} + \varepsilon \dot{x}_{21} + \varepsilon^2 \dot{x}_{22} + \varepsilon^3 \dot{x}_{2R} = b[x_{10} + \varepsilon x_{11} + \varepsilon^2 x_{12} + \varepsilon^3 x_{1R}] \\ -(-1+\varepsilon)^2 \bar{r} (x_{20} + \varepsilon x_{21} + \varepsilon^2 x_{22} + \varepsilon^3 x_{2R}) \end{aligned}$$

$$\times \left[(x_{10} + \varepsilon x_{11} + \varepsilon^2 x_{12} + \varepsilon^3 x_{1R})^2 + (x_{20} + \varepsilon x_{21} + \varepsilon^2 x_{22} + \varepsilon^3 x_{2R})^2 - \frac{a}{(-1+\varepsilon)^2 \bar{r}} \right].$$

Matching coefficients of ε^0, we obtain

$$\begin{aligned} \dot{x}_{10} &= -b x_{20} - \bar{r} x_{10} \left[x_{10}^2 + x_{20}^2 - \frac{a}{\bar{r}} \right], \quad x_{10}(0) = \eta_{10}, \\ \dot{x}_{20} &= b x_{10} - \bar{r} x_{20} \left[x_{10}^2 + x_{20}^2 - \frac{a}{\bar{r}} \right], \quad x_{20}(0) = \eta_{20}, \end{aligned}$$

which is the unperturbed problem at $\varepsilon = 0$.

Matching coefficients of ε, we obtain

$$\begin{aligned} \dot{x}_{11} &= -b x_{21} - 2\bar{r} x_{10}^2 x_{11} - 2\bar{r} x_{10} x_{20} x_{21} - \bar{r} x_{11} x_{10}^2 - \bar{r} x_{11} x_{20}^2 + 2\bar{r} x_{10}^3 + 2\bar{r} x_{10} x_{20}^2 \\ & \quad + a x_{11}, \quad x_{11}(0) = \eta_{11}, \\ \dot{x}_{21} &= b x_{11} - 2\bar{r} x_{20} x_{10} x_{11} - 2\bar{r} x_{20}^2 x_{21} - \bar{r} x_{21} x_{10}^2 - \bar{r} x_{21} x_{20}^2 + 2\bar{r} x_{20} x_{10}^2 + 2\bar{r} x_{20}^3 \\ & \quad + a x_{21}, \quad x_{21}(0) = \eta_{21}, \end{aligned}$$

while matching coefficients ε^2, we have

$$\begin{aligned} \dot{x}_{12} &= -b x_{22} - \bar{r} x_{10} x_{11}^2 - 2\bar{r} x_{10}^2 x_{12} - \bar{r} x_{10} x_{21}^2 - 2\bar{r} x_{10} x_{20} x_{22} \\ & \quad - 2\bar{r} x_{10} x_{11}^2 - 2\bar{r} x_{11} x_{20} x_{21} - \bar{r} x_{12} x_{10}^2 - \bar{r} x_{12} x_{20}^2 - \bar{r} x_{10}^3 - \bar{r} x_{10} x_{20}^2 + 4\bar{r} x_{10}^2 x_{11} \\ & \quad + 4\bar{r} x_{10} x_{20} x_{21} + 2\bar{r} x_{11} x_{10}^2 + 2\bar{r} x_{11} x_{20}^2 + a x_{12}, \quad x_{12}(0) = \eta_{12}, \\ \dot{x}_{22} &= b x_{12} - \bar{r} x_{20} x_{11}^2 - 2\bar{r} x_{20} x_{10} x_{12} - \bar{r} x_{20} x_{21} - 2\bar{r} x_{20}^2 x_{22} - 2\bar{r} x_{21} x_{10} x_{11} \\ & \quad - 2\bar{r} x_{21}^2 x_{20} - \bar{r} x_{22} x_{10}^2 - \bar{r} x_{22} x_{20}^2 + 4\bar{r} x_{20} x_{10} x_{11} + 4\bar{r} x_{20} x_{10} x_{11} + 4\bar{r} x_{20}^2 x_{21} \\ & \quad + 2\bar{r} x_{21} x_{10}^2 + 2\bar{r} x_{21} x_{20}^2 + a x_{22}, \quad x_{22}(0) = \eta_{22}. \end{aligned}$$

Having calculated the terms (x_{10}, x_{20}), (x_{11}, x_{21}) and (x_{12}, x_{22}), our task now is to show that $(x_{10}, x_{20}) + (x_{11}, x_{21})\varepsilon + (x_{12}, x_{22})\varepsilon^2$ is indeed $O(\varepsilon^3)$.

Note that, in [3], the author gave a result of existence and uniqueness for a class of perturbed systems. Now, using the theorem in ([3] pp. 388), we have the following proposition to approximate the solution.

Proposition 3.2. Suppose that

- The functions $(t, x, \varepsilon) \longrightarrow Ax(t) + u(t)B(\varepsilon)x(t)$ and η have continuous derivatives up to order 3 with respect to their arguments for

$$(t, x, \varepsilon) \in [t_0, t_1] \times \mathbb{R}^2 \times [-\varepsilon_0, \varepsilon_0].$$

- The nominal problem (14) has an unique solution $(x_{10}(t), x_{20}(t))$ defined on $[t_0, t_1]$ and $(x_{10}(t), x_{20}(t)) \in \mathbb{R}^2$ for all $t \in [t_0, t_1]$.

Then there exists $\varepsilon^* > 0$ such that $\forall |\varepsilon| < \varepsilon^*$, the problem (3)−(13) has an unique solution

$$(x_1(t,\varepsilon), x_2(t,\varepsilon)) \text{ defined on } [t_0, t_1]$$

which satisfies

$$(x_1(t,\varepsilon), x_2(t,\varepsilon)) - [(x_{10}, x_{20}) + (x_{11}, x_{21})\varepsilon + (x_{12}, x_{22})\varepsilon^2] = O(\varepsilon^3).$$

We give now an example to illustrate the applicability of the main result.

Example 3.3. Consider the following bilinear control system:

$$\dot{x}(t) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} x(t) + u(t) \begin{pmatrix} -1+\varepsilon & 0 \\ 0 & -1+\varepsilon \end{pmatrix} x(t), \quad \forall t \geq 0 \tag{16}$$

with initial state

$$\begin{cases} x_1(t_0) = -2 + \varepsilon \\ x_2(t_0) = 2 - 2\varepsilon. \end{cases}$$

- If we let $\varepsilon = 0$ can be obtained as in [7], then (16) becomes

$$\dot{x}(t) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} x(t) + u(t) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} x(t), \quad \forall t \geq 0. \tag{17}$$

By theorem 2.2, with the control law

$$u(t) = 5x_1^2(t) + 5x_2^2(t)$$

and $x(0) = [-2\ 2]^T$, we conclude that the phase trajectories of system (17) tend to the exponentially stable limit cycle

$$s(x) = x_1^2 + x_2^2 - \frac{3}{5}$$

in the $x_1 - x_2$ plane, with the guaranteed convergence rate $\alpha = 6$. Furthermore, the states $x_1(t)$ and $x_2(t)$ exponentially track, respectively, the trajectories

$$\sqrt{\frac{3}{5}} \cos\left[2t + \frac{3\pi}{4}\right]$$

and

$$\sqrt{\frac{3}{5}} \sin\left[2t + \frac{3\pi}{4}\right],$$

in the time domain, with the guaranteed convergence rate $\frac{\alpha}{2} = 3$. Some state trajectories of the feedback-controlled system are depicted in Figures 1 and 2.

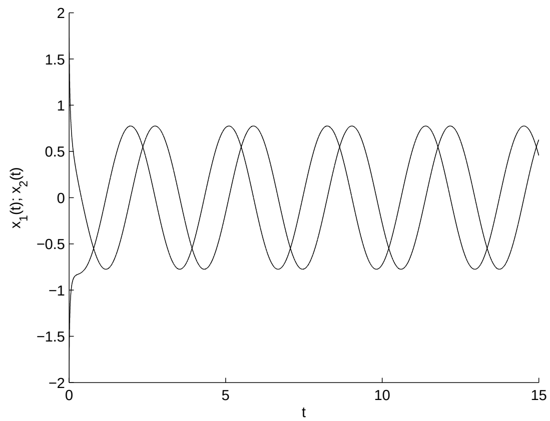

Fig. 1. $x_1(t)$ and $x_2(t)$ of the feedback-controlled system.

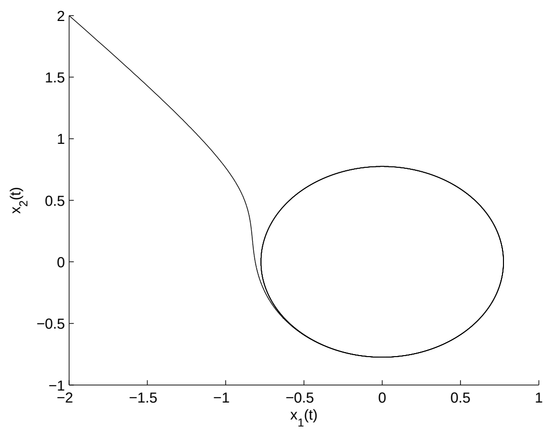

Fig. 2. Typical phase trajectories of the feedback-controlled system.

- If we let $\varepsilon = \frac{1}{2}$, then (16) becomes

$$\dot{x}(t) = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} x(t) + u(t) \begin{pmatrix} -\dfrac{1}{2} & 0 \\ 0 & -\dfrac{1}{2} \end{pmatrix} x(t), \quad \forall\, t \geq 0. \quad (18)$$

By theorem 2.2 with the control law

$$u(t) = \frac{5}{2}x_1^2(t) + \frac{5}{2}x_2^2(t)$$

and $x(0) = [1\ 1]^T$, we conclude that the phase trajectories of system (18) tend to the exponentially stable limit cycle

$$s(x) = x_1^2 + x_2^2 - \frac{12}{5}$$

in the $x_1 - x_2$ plane, with the guaranteed convergence rate $\alpha = 3$. Furthermore, the states $x_1(t)$ and $x_2(t)$ exponentially track, respectively, the trajectories

$$\frac{2\sqrt{3}}{\sqrt{5}} \cos\left[2t + \frac{\pi}{4}\right]$$

and

$$\frac{2\sqrt{3}}{\sqrt{5}} \sin\left[2t + \frac{\pi}{4}\right],$$

in the time domain, with the guaranteed convergence rate $\frac{\alpha}{2} = \frac{3}{2}$.

Some state trajectories of the feedback-controlled system are depicted in Figures 3 and 4.

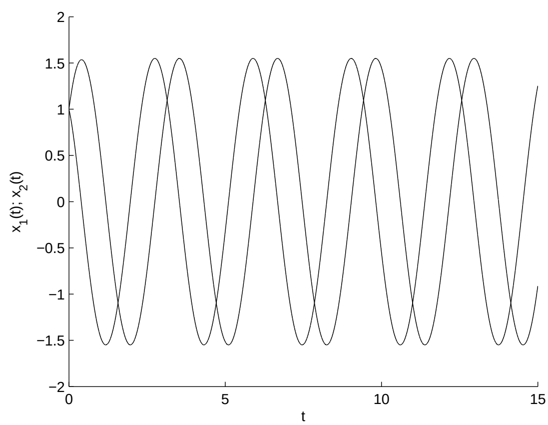

Fig. 3. $x_1(t)$ and $x_2(t)$ of the feedback-controlled system.

Now, we approximate $(x_1(t), x_2(t))$ by $[(x_{10}, x_{20}) + \frac{1}{2}(x_{11}, x_{21}) + \frac{1}{4}(x_{12}, x_{22})]$ because

$$(x_1(t), x_2(t)) - \left[(x_{10}, x_{20}) + \frac{1}{2}(x_{11}, x_{21}) + \frac{1}{4}(x_{12}, x_{22})\right] = O\left(\frac{1}{8}\right)$$

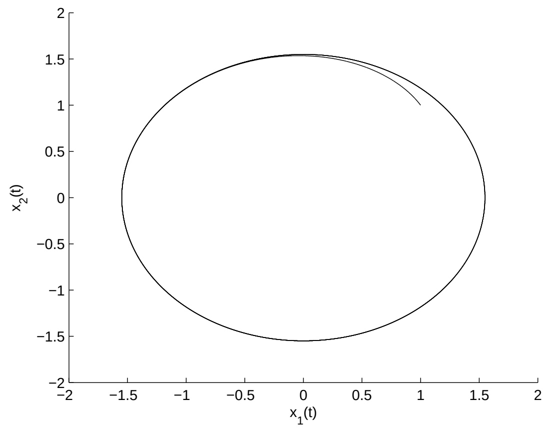

Fig. 4. Typical phase trajectories of the feedback-controlled system.

with (x_{10}, x_{20}) is the solution of (17), (x_{11}, x_{21}) is the solution of system

$$\begin{aligned}\dot{x}_{11} &= -2x_{21} - 10x_{10}^2x_{11} - 10x_{10}x_{20}x_{21} - 5x_{11}x_{10}^2 - 5x_{11}x_{20}^2 + 10x_{10}^3 + 10x_{10}x_{20}^2 \\ &\quad +3x_{11}, \quad x_{11}(0) = 6, \\ \dot{x}_{21} &= 2x_{11} - 10x_{20}x_{10}x_{11} - 10x_{20}^2x_{21} - 5x_{21}x_{10}^2 - 5x_{21}x_{20}^2 + 10x_{20}x_{10}^2 + 10x_{20}^3 \\ &\quad +3x_{21}, \quad x_{21}(0) = -2,\end{aligned}$$

and (x_{12}, x_{22}) is the solution of system:

$$\begin{aligned}\dot{x}_{12} &= -2x_{22} - 5x_{10}x_{11}^2 - 10x_{10}^2x_{12} - 5x_{10}x_{21}^2 - 10x_{10}x_{20}x_{22} - 10x_{10}x_{11}^2 \\ &\quad -10x_{11}x_{20}x_{21} - 5x_{12}x_{10}^2 - 5x_{12}x_{20}^2 - 5x_{10}^3 - 5x_{10}x_{20}^2 + 20x_{10}^2x_{11} + 20x_{10}x_{20}x_{21} \\ &\quad +10x_{11}x_{10}^2 + 10x_{11}x_{20}^2 + 3x_{12}, \quad x_{12}(0) = 0, \\ \dot{x}_{22} &= 2x_{12} - 5x_{20}x_{11}^2 - 10x_{20}x_{10}x_{12} - 5x_{20}x_{21} - 10x_{20}^2x_{22} - 10x_{21}x_{10}x_{11} \\ &\quad -10x_{21}^2x_{20} - 5x_{22}x_{10}^2 - 5x_{22}x_{20}^2 + 20x_{20}x_{10}x_{11} + 20x_{20}x_{10}x_{11} + 20x_{20}^2x_{21} \\ &\quad +10x_{21}x_{10}^2 + 10x_{21}x_{20}^2 + 3x_{22}, \quad x_{22}(0) = 0.\end{aligned}$$

4. CONCLUSION

In this paper, a feedback control is constructed to guarantee the existence of limit cycle for a class of bilinear systems. Furthermore, we gave a numerical value of the approximation error

$$(x_1(t, \varepsilon), x_2(t, \varepsilon)) - (x_{10}(t), x_{20}(t)) = O(\varepsilon)$$

for a given numerical value of ε when the error is $O(\varepsilon)$. Knowing that the error is $O(\varepsilon)$ means that its norm is less than $k|\varepsilon|$ for some positive constant k which is independent of ε and this means that the bound decreases monotonically as ε decreases. The perturbation method of seeking in approximate solution as a finite Taylor expansion of the exact solution where it is shown that an approximate periodic solution can be detected. The effectiveness of these approximation is verified in numerical examples. Also, Some simulation results are given to illustrate the existence of a limit cycle for this class of nonlinear control systems.

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