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BOUNDS OF GENERAL FRÉCHET CLASSES

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This paper deals with conditions of compatibility of a system of copulas and with bounds of general Fréchet classes. Algebraic search for the bounds is interpreted as a solution to a linear system of Diophantine equations. Classical analytical specification of the bounds is described.

Keywords: copula, Fréchet class, Diophantine equation

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1. INTRODUCTION

The *copula* is a real multivariate cumulative distribution function with all one-dimensional *margins* (1-margins) being uniform on $\langle 0,1\rangle$. Recall that copulas join one-dimensional marginal distribution functions to form multivariate distribution functions. For $n \geq 2$, let $\mathbf{X} = (X_1, \ldots, X_n)^T$ be a random vector with joint distribution function F. The copula of the random vector $\mathbf{X}$ is the joint distribution function C of the vector $(F_1(X_1), \ldots, F_n(X_n))^T$ if F_i's are the continuous marginal distribution functions of $\mathbf{X}$. Let us note that copula has no discrete part. So, it is always continuous as a composition of absolutely continuous and singular parts.

Theorem 1.1. (Sklar) Let F be an n-dimensional distribution function with margins $F_1, F_2, \ldots, F_n$. Then there exists an n-copula C such that for all $x \in \mathbb{R}^n$,

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)). \tag{1}$$

If $F_1, F_2, \ldots, F_n$ are all continuous, then C is unique; otherwise, C is uniquely determined on $\mathrm{Ran}F_1 \times \mathrm{Ran}F_2 \times \ldots \times \mathrm{Ran}F_n$. Conversely, if C is an n-copula and $F_1, F_2, \ldots, F_n$ are distribution functions, then the function F defined by (1) is an n-dimensional distribution function with 1-margins $F_1, F_2, \ldots, F_n$.

A copula can be regarded as an abstract structure which fully represents relationship of random variables. As mentioned in [5], there are two main reasons to be interested in copulas. Firstly, it is a way of studying scale-free measures of dependence. On the other hand, copulas are starting points for constructing families of multivariate distributions, sometimes with a view to simulation (see e. g. [2]). Recent interest in copulas was induced by applicability in finance and insurance.

Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ be from $\mathbb{R}^n$ and $J \subset S_n = \{1, 2, \ldots, n\}$. We will let $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}^J$ denote such vector $(c_1, \ldots, c_n)$ that $c_k = a_k$ if $k \in J$, $c_k = b_k$ if $k \notin J$.

Example 1.2. If $n = 4$, there is

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}^{\{2,3\}} = (b_1, a_2, a_3, b_4)\,. \tag{2}$$

For a real function F of n variables and all related points in Dom$F$, the F-*volume* of an n-*box* $\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ is given by

$$V_F(\langle \boldsymbol{a}, \boldsymbol{b} \rangle) = \sum_{J \subset S_n} (-1)^{|J|} F\left(\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}^J \right) \tag{3}$$

where $a_k \leq b_k$ for all $k = 1, \ldots, n$ and $|J|$ is the *cardinality* of J.

An n-copula is fully determined by performance on $\langle 0, 1 \rangle^n$. So, we can consider the n-copula as a function $C : \langle 0, 1 \rangle^n \to \langle 0, 1 \rangle$ with the next properties:

(i) $C(\boldsymbol{u}) = 0$ whenever $\boldsymbol{u} = (u_1, \ldots, u_n) \in \langle 0, 1 \rangle^n$ has at least one component equal to 0 (C is grounded),

(ii) $C(\boldsymbol{u}) = u_k$ whenever all components of $\boldsymbol{u} \in \langle 0, 1 \rangle^n$ are equal to 1 except for the kth one (uniformity of 1-margins),

(iii) $V_C(\langle \boldsymbol{a}, \boldsymbol{b} \rangle) \geq 0$ for any such $\boldsymbol{a}$ and $\boldsymbol{b}$ from $\langle 0, 1 \rangle^n$ that $\boldsymbol{a} \leq \boldsymbol{b}$ (C is n-increasing).

For $\upsilon = \{j_1, j_2, \ldots, j_k\} \subset S_n$ and an n-copula C, a υ-margin of C is its k-margin $C_\upsilon : \langle 0, 1 \rangle^k \to \langle 0, 1 \rangle$, $C_\upsilon(\boldsymbol{u}_\upsilon) = C\left(\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{1}_n \end{pmatrix}^\upsilon \right)$, where $\boldsymbol{u}_\upsilon = (u_{j_1}, u_{j_2}, \ldots, u_{j_k})$, $j_i < j_{i+1}$ and $\boldsymbol{1}_n$ is the vector of ones. We will call υ the *determinative set* of the margin C_υ. Any k-margin of a copula is a k-copula. For any n-copula C and permutation $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ of $(1, 2, \ldots, n)$, the function $C^{(\boldsymbol{\sigma})} : \langle 0, 1 \rangle^n \to \langle 0, 1 \rangle$ given by

$$C^{(\boldsymbol{\sigma})}(\boldsymbol{u}) = C(u_{\sigma_1}, \ldots, u_{\sigma_n}) \tag{4}$$

is also n-copula. So, for k-margins of a copula, it is sufficient to debate only about the variable indexed by the determinative set, not about their order. We consider only increasing indexing of variables of υ-margins.

Conditional distribution of a copula does not necessarily have to be also copula. As we aim to use the consecutive apparatus also for conditional distributions in the next section, we start to debate about general probability distributions $F(\boldsymbol{u})$ on $\langle 0, 1 \rangle^n$. Note that along with a given distributions on $\langle 0, 1 \rangle^n$ with cumulative distribution function F, there are 2^n *associated* distributions, each of them defined by

$$F^\varsigma(\boldsymbol{u}) = \lim_{\boldsymbol{t}_\varsigma \to \boldsymbol{u}_\varsigma^+} V_F\left(\left\langle \begin{pmatrix} \boldsymbol{1}_n - \boldsymbol{t} \\ \boldsymbol{0}_n \end{pmatrix}^\varsigma, \begin{pmatrix} \boldsymbol{1}_n \\ \boldsymbol{u} \end{pmatrix}^\varsigma \right\rangle \right) \tag{5}$$

for a subset ς of S_n and $\boldsymbol{t} \in \langle 0, 1 \rangle^n$. The limit can be omitted if F is continuous.

Example 1.3. Together with a 3-copula C, we have the associated copulas

$$\begin{aligned}
C^{\{1\}}(\boldsymbol{u}) &= C_{\{2,3\}}(u_2,u_3) - C(1-u_1,u_2,u_3)\\
C^{\{2\}}(\boldsymbol{u}) &= C_{\{1,3\}}(u_1,u_3) - C(u_1,1-u_2,u_3)\\
C^{\{3\}}(\boldsymbol{u}) &= C_{\{1,2\}}(u_1,u_2) - C(u_1,u_2,1-u_3)\\
C^{\{1,2\}}(\boldsymbol{u}) &= u_3 - C_{\{1,3\}}(1-u_1,u_3) - C_{\{2,3\}}(1-u_2,u_3) + C(1-u_1,1-u_2,u_3)\\
C^{\{1,3\}}(\boldsymbol{u}) &= u_2 - C_{\{1,2\}}(1-u_1,u_2) - C_{\{2,3\}}(u_2,1-u_3) + C(1-u_1,u_2,1-u_3)\\
C^{\{2,3\}}(\boldsymbol{u}) &= u_1 - C_{\{1,2\}}(u_1,1-u_2) - C_{\{1,3\}}(u_1,1-u_3) + C(u_1,1-u_2,1-u_3)\\
C^{S_3}(\boldsymbol{u}) &= u_1+u_2+u_3-2+C_{\{1,2\}}(1-u_1,1-u_2)+C_{\{1,3\}}(1-u_1,1-u_3)\\
&\quad + C_{\{2,3\}}(1-u_2,1-u_3) - C(1-u_1,1-u_2,1-u_3) \qquad (6)
\end{aligned}$$

where $\boldsymbol{u} = (u_1,u_2,u_3) \in \langle 0,1\rangle^3$.

Let $\boldsymbol{S}$ be a system of subsets of S_n. The *Fréchet class* $\mathcal{F}_n(C_v : v \in \boldsymbol{S})$ is the set of all n-copulas $\tilde{C}(\boldsymbol{u})$ with given v-margin equal to $C_v(\boldsymbol{u}_v)$ for each $v \in \boldsymbol{S}$. Except these given margins C_v, all margins of them and 1-margins are in fact fixed too.

Example 1.4. In the Fréchet class $\mathcal{F}_4(C_{\{1,2\}}, C_{\{2,3,4\}})$, all four 1-margins are uniform and $\{2,3\}$-, $\{2,4\}$- and $\{3,4\}$-margins are also given.

Fréchet classes are studied largely in context of construction of multivariate distributions. The most frequent are questions of uniqueness, subfamilies with desirable properties, boundaries and their nature (see e. g. [4]). We strive after upper and lower bounds of general Fréchet classes in this text.

2. SUFFICIENT BOUNDS AND COMPATIBILITY

For a function F of n variables, $\boldsymbol{a} \le \boldsymbol{b}$ and a partition $\mathcal{A} = \{A_1, A_2, \ldots, A_l\}$ of the set S_n, we call

$$V_F^{\mathcal{A}}(\langle \boldsymbol{a},\boldsymbol{b}\rangle) = \sum_{K\subset S_l} (-1)^{|K|} F\left(\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}^{\cup_{i\in K} A_i}\right) \qquad (7)$$

an *F-volume* of $\langle \boldsymbol{a},\boldsymbol{b}\rangle$ at the partition $\mathcal{A}$.

Example 2.1. For the partition $\mathcal{A} = \{\{1,2\},\{3\}\}$ of S_3, the F-volume of $\langle \boldsymbol{a},\boldsymbol{b}\rangle$ at the partition $\mathcal{A}$ is

$$V_F^{\mathcal{A}}(\langle \boldsymbol{a},\boldsymbol{b}\rangle) = F(\boldsymbol{b}) - F(a_1,a_2,b_3) - F(b_1,b_2,a_3) + F(\boldsymbol{a}). \qquad (8)$$

Theorem 2.2. For any cumulative distribution function F of n variables, $\boldsymbol{a} \le \boldsymbol{b}$ in $\mathbb{R}^n$ and a partition $\mathcal{A} = \{A_1, A_2, \ldots, A_l\}$ of the set S_n, it holds

$$V_F^{\mathcal{A}}(\langle \boldsymbol{a},\boldsymbol{b}\rangle) \ge 0. \qquad (9)$$

P r o o f. We will prove the proposition by induction with respect to cardinalities of blocks of partitions. For any $\boldsymbol{c} \leq \boldsymbol{d}$ both in $\mathbb{R}^n$, there is $V_F^{\{\{1\},\{2\},\dots,\{n\}\}}(\langle \boldsymbol{c},\ \boldsymbol{d}\rangle) \geq 0$ as a general property of cumulative distribution functions. We will proceed from the trivial partition $\{\{1\},\{2\},\dots,\{n\}\}$ to the partition $\mathcal{A}$ by incremental attaching singletons to respective kernels of the partition $\mathcal{A}$, not exceeding contents of its blocks.

Let $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ be such partition of S_n that for arbitrary A_l, a block of $\mathcal{A}$, any block of $\mathcal{B}$ is either disjoint with A_l or singleton or a temporal kernel of A_l. Moreover, let $V_F^{\mathcal{B}}(\langle \boldsymbol{c}, \boldsymbol{d}\rangle) \geq 0$ for any $\boldsymbol{c} \leq \boldsymbol{d} \in \mathbb{R}^n$. Let B_j be the kernel of A_l and $B_i = \{k\}$ be a singleton, both subsets of A_l. We can suppose in virtue of simplicity and without loss of generality that $B_j = \{1, 2, \dots, k-1\}$. Let the partition $\mathcal{B}'$ arise from $\mathcal{B}$ by fusion of the blocks B_i and Bj. We are going to prove that then

$$V_F^{\mathcal{B}'}(\langle \boldsymbol{c}, \boldsymbol{d}\rangle) = V_F^{\mathcal{B}}\left(\left\langle \begin{pmatrix} \boldsymbol{0}_n \\ \boldsymbol{c} \end{pmatrix}^{\{k\}}, \boldsymbol{d} \right\rangle\right) + \\ + V_F^{\mathcal{B}}\left(\left\langle \begin{pmatrix} \boldsymbol{0}_n \\ \boldsymbol{c} \end{pmatrix}^{B_j}, \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix}^{B_j} \right\rangle\right) \quad (10)$$

for any $\boldsymbol{c} \leq \boldsymbol{d} \in \mathbb{R}^n$ and the vector of zeros $\boldsymbol{0}_n$. Consequently, the left side L of (10) is nonnegative as a sum of two nonnegative terms R_1 and R_2 on the right side.

Every expression of the form (7) consists of 2^l summands. The half of them, ergo 2^{l-1}, are provided with positive sign and the same number with negative sign. So, L consists of $|\mathcal{B}'| = 2^{m-1}$ summands. Both terms R_1 and R_2 are formally sums of 2^m summands. The half of the summands is zero from definition because the lower endpoints of n-boxes in R_1 and R_2 are zero at components corresponding to one block of partition $\mathcal{B}$ ($B_i = \{k\}$ and B_j). With respect to all those zeros, each of terms L, R_1 and R_2 can be regarded as a sum of 2^{m-1} summands.

As $\{k\} \cup B_j = \{1, \dots, k-1, k\} \in \mathcal{B}'$, any summand of L has either form $F(c_1, \dots, c_{k-1}, c_k, \dots)$ or $F(d_1, \dots, d_{k-1},\ d_k, \dots)$, apart from its sign. The sign depends on the number of $\mathcal{B}'$-blocks covering the indices of components of its argument which are identical with $\boldsymbol{c}$, the lower endpoint of the n-box $\langle \boldsymbol{c}, \boldsymbol{d}\rangle$.

First, let's consider a summand $(\pm)F(c_1, \dots, c_{k-1}, c_k, \dots)$ of L where the components of the argument $(c_1, \dots, c_{k-1},\ c_k, \dots)$ with indices greater than k are identical either with $\boldsymbol{c}$ or $\boldsymbol{d}$, according to blocks of $\mathcal{B}'$. But however the same summand is included in R_2 because on the $\mathcal{B}$-block B_j the first $k-1$ components of the argument are identical with $(c_1, \dots, c_{k-1},\ d_k, \dots, d_n)$, the upper endpoint of the n-box $\left\langle \begin{pmatrix} \boldsymbol{0}_n \\ \boldsymbol{c} \end{pmatrix}^{B_j}, \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix}^{B_j} \right\rangle$, and the kth component ($\mathcal{B}$-block B_i) is common with the lower endpoint $(0, \dots, 0, c_k, \dots, c_n)$. The blocks covering the last indices are the same for both partitions $\mathcal{B}'$ and $\mathcal{B}$. So, the number of blocks for this summand, covering the indices where the argument is identical with the lower endpoint, is the very same regardless of whether it is member L or R_2. The sign of $(\pm)F(c_1, \dots, c_{k-1},\ c_k, \dots)$ in L and R_2 is the same. Evidently, no summand of the form $(\pm)F(c_1, \dots, c_{k-1},\ c_k, \dots)$ is a member of R_1. As well, we can prove that a summand $(\pm)F(d_1, \dots, d_{k-1},\ d_k, \dots)$ of L (altogether 2^{m-2} such summands) is a part of R_1 including its sign, but it does not occur in R_2.

We still have the last 2^{m-2} summands $(\pm)F(c_1,\dots,c_{k-1},\, d_k,\dots)$ in R_1 or R_2 which have no occurrence in L. In R_1, the first $k-1$ components ($\mathcal{B}$-block B_j) of the argument are common with the lower endpoint $(c_1,\dots,c_{k-1},\, 0, c_{k+1},\dots,c_n)$ of the corresponding n-box, whereas the kth component ($\mathcal{B}$-block B_i) comes from the upper endpoint $\boldsymbol{d}$. Analyzing $(\pm)F(c_1,\dots,c_{k-1},\, d_k,\dots)$ in framework of R_2, no one of the first k components is common with the corresponding lower endpoint $(0,\dots,0,c_k,\dots,c_n)$. So, the signs of these summands are opposite in R_1 and R_2. They both cancel out in the total R_1+R_2 of the right side of (10). □

Corollary 2.3. Let C be an n-copula, $\boldsymbol{u}\in\langle 0,1\rangle^n$ and $\upsilon_1,\upsilon_2\subset S_n$. Then

$$C_{\upsilon_1\cap\upsilon_2}(\boldsymbol{u}_{\upsilon_1\cap\upsilon_2}) - C_{\upsilon_1}(\boldsymbol{u}_{\upsilon_1}) - C_{\upsilon_2}(\boldsymbol{u}_{\upsilon_2}) + C_{\upsilon_1\cup\upsilon_2}(\boldsymbol{u}_{\upsilon_1\cup\upsilon_2}) \geq 0. \tag{11}$$

Proof. The left side of (11) is C-*volume* of $\left\langle \begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{0}_n\end{pmatrix}^{(\upsilon_1-\upsilon_2)\cup(\upsilon_2-\upsilon_1)}, \begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{1}_n\end{pmatrix}^{\upsilon_1\cap\upsilon_2}\right\rangle$ at the partition $\mathcal{B}=\{\upsilon_1\cap\upsilon_2,\, \upsilon_1-\upsilon_2,\, \upsilon_2-\upsilon_1,\, S_n-(\upsilon_1\cup\upsilon_2)\}$ with potential exclusion of empty parts. So, by Theorem 2.2, it is nonnegative. □

The next theorem gives alternative definition of Fréchet class through appropriate bounds.

Theorem 2.4. Let $\boldsymbol{S}$ be a system of subsets of S_n and $\{C_\upsilon : \upsilon\in\boldsymbol{S}$ and C_υ is a $|\upsilon|$-copula$\}$ be a set of copulas. Any function $\tilde{C}:\langle 0,1\rangle^n\to\mathbb{R}$ belongs to the *Fréchet class* $\mathcal{F}_n(C_\upsilon:\upsilon\in\boldsymbol{S})$ if and only if $\tilde{C}$ is n-increasing and

$$\max\mathcal{L}|_{\boldsymbol{u}} \leq \tilde{C}(\boldsymbol{u}) \leq \min\mathcal{U}|_{\boldsymbol{u}} \tag{12}$$

for any $\boldsymbol{u}\in\langle 0,1\rangle^n$, where

$$\mathcal{U}|_{\boldsymbol{u}} = \{C_\upsilon(\boldsymbol{u}_\upsilon) : \upsilon\in\boldsymbol{S}\}\cup\{u_i : i\notin\cup\boldsymbol{S}\} \tag{13}$$

and

$$\mathcal{L}|_{\boldsymbol{u}} = \left\{C_\upsilon(\boldsymbol{u}_\upsilon) - |S_n-\upsilon| + \sum_{i\notin\upsilon} u_i : \upsilon\in\boldsymbol{S}\right\}\cup\{0\}. \tag{14}$$

Proof. Let $\tilde{C}\in\mathcal{F}_n(C_\upsilon:\upsilon\in\boldsymbol{S})$. Then $\tilde{C}$ is n-increasing and nonnegative all over $\langle 0,1\rangle^n$. As $\mathcal{U}|_{\boldsymbol{u}}$ is the set of values of margins of $\tilde{C}$ at $\boldsymbol{u}$ (including some 1-margins), $\tilde{C}(\boldsymbol{u})\leq\min\mathcal{U}|_{\boldsymbol{u}}$. Let $\upsilon\in\boldsymbol{S}$ and $S_n-\upsilon=\{i_1,i_2,\dots,i_m\}$. Then

$$\begin{aligned}
|S_n-\upsilon| - C_\upsilon(\boldsymbol{u}_\upsilon) - \sum_{i\notin\upsilon} u_i + \tilde{C}(\boldsymbol{u}) &= m - \tilde{C}_\upsilon(\boldsymbol{u}_\upsilon) - \sum_{i\notin\upsilon} u_i + \tilde{C}(\boldsymbol{u})\\
&= [1-\tilde{C}_\upsilon(\boldsymbol{u}_\upsilon) - u_{i_1} + \tilde{C}_{\upsilon\cup\{i_1\}}(\boldsymbol{u}_{\upsilon\cup\{i_1\}})]\\
&\quad + [1-\tilde{C}_{\upsilon\cup\{i_1\}}(\boldsymbol{u}_{\upsilon\cup\{i_1\}}) - u_{i_2} + \tilde{C}_{\upsilon\cup\{i_1,i_2\}}(\boldsymbol{u}_{\upsilon\cup\{i_1,i_2\}})]\\
&\quad + [1-\tilde{C}_{\upsilon\cup\{i_1,i_2\}}(\boldsymbol{u}_{\upsilon\cup\{i_1,i_2\}}) - u_{i_3} + \tilde{C}_{\upsilon\cup\{i_1,i_2,i_3\}}(\boldsymbol{u}_{\upsilon\cup\{i_1,i_2,i_3\}})] + \dots\\
&\quad + [1-\tilde{C}_{\upsilon\cup\{i_1,\dots,i_{m-2}\}}(\boldsymbol{u}_{\upsilon\cup\{i_1,\dots,i_{m-2}\}}) - u_{i_{m-1}} + \tilde{C}_{\upsilon\cup\{i_1,\dots,i_{m-1}\}}(\boldsymbol{u}_{\upsilon\cup\{i_1,\dots,i_{m-1}\}})]\\
&\quad + [1-\tilde{C}_{\upsilon\cup\{i_1,\dots,i_{m-1}\}}(\boldsymbol{u}_{\upsilon\cup\{i_1,\dots,i_{m-1}\}}) - u_{i_m} + \tilde{C}(\boldsymbol{u})]
\end{aligned} \tag{15}$$

is positive as a sum of positive parts in brackets by Corollary 2.3. Consequently, $\tilde{C}(u) \geq C_v(\boldsymbol{u}_v) - |S_n - v| + \sum_{i \notin v} u_i$ and thus $\tilde{C}(\boldsymbol{u}) \geq \max \mathcal{L}|_{\boldsymbol{u}}$.

On the other hand, let a function $\tilde{C} : \langle 0,1\rangle^n \to \mathbb{R}$ be n-increasing and (12) hold for any $\boldsymbol{u} \in \langle 0,1\rangle^n$. Values of $\tilde{C}$ are inside $\langle 0,1\rangle$ as any value of copula from $\mathcal{U}|_{\boldsymbol{u}}$ is less than 1 and $\mathcal{L}|_{\boldsymbol{u}}$ contains 0. It remains to prove that $\tilde{C}$ is grounded and has uniform 1-margins.

Let the kth component u_k of $\boldsymbol{u}$ be 0. As the union of determinative sets of all given margins from $\mathcal{U}|_{\boldsymbol{u}}$ (including the 1-margins) $\{u_i : i \notin \cup \boldsymbol{S}\}$ is S_n, k belongs at least to one determinative set of these margins. But since this margin is a copula, its value at $\boldsymbol{u}$ is 0. As a consequence of $\tilde{C}(\boldsymbol{u}) \leq \min \mathcal{U}|_{\boldsymbol{u}}$, there is $\tilde{C}(\boldsymbol{u}) = 0$ and $\tilde{C}$ is grounded.

Let all components of $\boldsymbol{u}$, except for the kth one, be equal to 1. As k belongs to determinative set of some margin from $\mathcal{U}|_{\boldsymbol{u}}$, the value of this copula at $\boldsymbol{u}$ is u_k and hence $\tilde{C}(\boldsymbol{u}) \leq u_k$. On the other hand, there is $C_v(\boldsymbol{u}_v) = u_k$ for $k \in v \in \boldsymbol{S}$ and $u_i = 1$ for $i \notin v$. Thus $C_v(\boldsymbol{u}_v) - |S_n - v| + \sum_{i \notin v} u_i = u_k - |S_n - v| + |S_n - v| = u_k$. For $k \notin v \in \boldsymbol{S}$, we get $C_v(\boldsymbol{u}_v) - |S_n - v| + \sum_{i \notin v} u_i = 1 - |S_n - v| + |S_n - v| - 1 + u_k = u_k$. Consequently, $u_k = \max \mathcal{L}|_{\boldsymbol{u}} \leq \tilde{C}(\boldsymbol{u})$. □

Of course, the lower or upper bounds in (12) are not copulas in general.

Example 2.5. Considering the Fréchet class $\mathcal{F}_n(\emptyset)$ in Theorem 2.4, there is $\max \mathcal{L}|_{\boldsymbol{u}} = W^n(\boldsymbol{u}) = \max\{0, u_1 + u_2 + \ldots + u_n - n + 1\}$ and is well known that for $n > 2$ this is not copula.

On the other hand, $W^2(u_1,u_2) = \max\{0, u_1 + u_2 - 1\}$ together with $\Pi^2(u_1,u_2) = u_1u_2$ are 2-copulas. The Fréchet class $\mathcal{F}_n(\Pi^2(u_2,u_3), \Pi^2(u_1,u_3), W^2(u_1,u_2))$ contains also the 3-copula $C(u_1,u_2,u_3) = W^2(u_1,u_2)\cdot u_3$ but $U(u_1,u_2,u_3) = \min\{\Pi^2(u_2,u_3), \Pi^2(u_1,u_3), W^2(u_1,u_2)\}$ is not copula as the U-volume $V_U\left(\left\langle\left(\frac{3}{4},\frac{3}{4},\frac{3}{4}\right),\ (1,1,1)\right\rangle\right) = -\frac{1}{8}$.

A set $\{C_v : v \in \boldsymbol{S}\}$ of copulas is *compatible* if the corresponding *Fréchet class* $\mathcal{F}_n(C_v : v \in \boldsymbol{S})$ is nonempty. According to the proof of Theorem 2.4 we can establish necessary conditions for compatibility. The next claim is suitable for easy distinction of incompatibility.

Corollary 2.6. Let $\boldsymbol{S}$ be a system of subsets of S_n, $\mathcal{U} = \{C_v : v \in \boldsymbol{S}$ and C_v is a $|v|$-copula$\}$ be a set of copulas and $\boldsymbol{\Upsilon}(\boldsymbol{S})$ be the system of all such $\tau \subset S_n$ that are either singletons or subsets of the sets $v \in \boldsymbol{S}$. Let $\mathcal{L}$ be the set of all functions of the form

$$\begin{aligned} L_C^{\Theta}(\boldsymbol{u}) = C_{v_1}(\boldsymbol{u}_{v_1}) - C_{v_2\cap v_1}(\boldsymbol{u}_{v_2\cap v_1}) + C_{v_2}(\boldsymbol{u}_{v_2}) \\ - C_{v_3\cap(v_1\cup v_2)}\left(\boldsymbol{u}_{v_3\cap(v_1\cup v_2)}\right) + \ldots + C_{v_{k-1}}(\boldsymbol{u}_{v_{k-1}}) \\ - C_{v_k\cap(v_1\cup\ldots\cup v_{k-1})}\left(\boldsymbol{u}_{v_k\cap(v_1\cup\ldots\cup v_{k-1})}\right) + C_{v_k}(\boldsymbol{u}_{v_k}) \end{aligned} \tag{16}$$

for $\boldsymbol{u} \in \langle 0,1\rangle^n$ and such a sequence $\Theta = (v_1,\ldots,v_k)$ of sets $v_i \in \boldsymbol{\Upsilon}(\boldsymbol{S})$ that $\bigcup_{i=1}^{k} v_i = S_n$.

If the set $\mathcal{U}$ is compatible then $L_C^{\Theta}(\boldsymbol{u}) \leq C_\tau(\boldsymbol{u}_\tau)$ for each $\boldsymbol{u} \in \langle 0,1\rangle^n$, $L_C^{\Theta} \in \mathcal{L}$ and $\tau \in \boldsymbol{\Upsilon}(\boldsymbol{S})$.

P r o o f. Let $\tilde{C} \in \mathcal{F}_n(C_\upsilon : \upsilon \in \boldsymbol{S})$, $L_C^\Theta(\boldsymbol{u}) \in \mathcal{L}$ where $\Theta = (\upsilon_1, \dots, \upsilon_k)$, $\varsigma_0 = \emptyset$ and $\varsigma_j = \bigcup_{i=1}^{j} \upsilon_i$ for $j = 1, \dots, k$. By Corollary 2.3, it holds

$$\begin{aligned} &C_{\upsilon_{i+1}\cap\varsigma_i}(\boldsymbol{u}_{\upsilon_{i+1}\cap\varsigma_i}) - C_{\upsilon_{i+1}}(\boldsymbol{u}_{\upsilon_{i+1}}) \\ &\qquad = \tilde{C}_{\upsilon_{i+1}\cap\varsigma_i}(\boldsymbol{u}_{\upsilon_{i+1}\cap\varsigma_i}) - \tilde{C}_{\upsilon_{i+1}}(\boldsymbol{u}_{\upsilon_{i+1}}) \geq \tilde{C}_{\varsigma_i}(\boldsymbol{u}_{\varsigma_i}) - \tilde{C}_{\varsigma_{i+1}}(\boldsymbol{u}_{\varsigma_{i+1}}) \end{aligned} \tag{17}$$

for all $i = 1, 2, \dots, k-1$. Consequently, $1 = C_{\varsigma_0\cap\upsilon_1}(\boldsymbol{u}_{\varsigma_0\cap\upsilon_1})$ and thus

$$\begin{aligned} 1 - L_C^\Theta(\boldsymbol{u}) &= \sum_{i=0}^{k-1} \left(C_{\upsilon_{i+1}\cap\varsigma_i}(\boldsymbol{u}_{\upsilon_{i+1}\cap\varsigma_i}) - C_{\upsilon_{i+1}}(\boldsymbol{u}_{\upsilon_{i+1}})\right) \\ &\geq \sum_{i=0}^{k-1} \left(\tilde{C}_{\varsigma_i}(\boldsymbol{u}_{\varsigma_i}) - \tilde{C}_{\varsigma_{i+1}}(\boldsymbol{u}_{\varsigma_{i+1}})\right) = 1 - \tilde{C}(\boldsymbol{u}) \end{aligned} \tag{18}$$

for any $\boldsymbol{u} \in \langle 0, 1\rangle^n$.

On the other hand, $\tilde{C}(\boldsymbol{u}) \leq C_\tau(\boldsymbol{u}_\tau)$ for each $\tau \in \boldsymbol{\Upsilon}(\boldsymbol{S})$ by definition. □

Example 2.7. It is known that $M^n(\boldsymbol{u}) = \min\{u_1, \dots, u_n\}$ is n-copula for each natural n. The question is if there is such a 2-copula C that the set $\{C(u_1, u_2),\ M^2(u_1, u_3),\ M^2(u_2, u_3)\}$ is compatible. According to Corollary 2.6, it holds

$$L_C^{(\{1,3\},\{2,3\})}(u_1, u_2, u_3) = M^2(u_1, u_3) - u_3 + M^2(u_2, u_3) \leq C(u_1, u_2). \tag{19}$$

If $u_1 \leq u_2$, let us $u_3 = u_1$ in (19) with result $u_1 \leq C(u_1, u_2)$. Symmetrically, $u_2 \leq C(u_1, u_2)$ for $u_2 \leq u_1$. It is therefore necessarily $C(u_1, u_2) = \min\{u_1, u_2\} = M^2(u_1, u_2)$. The corresponding Fréchet class contains $M^3(u_1, u_2, u_3)$ (and nothing more).

3. ALGEBRAIC BOUNDS

The series of *implicit* inequalities

$$F^\varsigma\left(\left(\begin{array}{c} \boldsymbol{1}_n - \boldsymbol{u} \\ \boldsymbol{u} \end{array}\right)^\varsigma\right) = \lim_{\boldsymbol{t}_\varsigma \to \boldsymbol{u}_\varsigma^-} V_F\left(\left\langle\left(\begin{array}{c} \boldsymbol{t} \\ \boldsymbol{0}_n \end{array}\right)^\varsigma, \left(\begin{array}{c} \boldsymbol{1}_n \\ \boldsymbol{u} \end{array}\right)^\varsigma\right\rangle\right) \geq 0 \tag{20}$$

for all $\varsigma \subset S_n$ comes from the fact that associated distribution functions F^ς (F-volumes) are nonnegative. After expansion, they are expressed by margins of F. If we use the implicit inequalities of a member $\tilde{C}$ of a Fréchet class $\mathcal{F}_n(C_\upsilon : \upsilon \in \boldsymbol{S})$ to describe this class, the margins in inequalities are either fixed $C_\upsilon(\boldsymbol{u}_\upsilon)$ or free $\tilde{C}_\upsilon(\boldsymbol{u}_\upsilon)$ depending on whether $\upsilon \in \boldsymbol{\Upsilon}(\boldsymbol{S})$ or $\upsilon \notin \boldsymbol{\Upsilon}(\boldsymbol{S})$. Transforming the implicit inequalities by their linear combinations with nonnegative coefficients, we can eliminate all proper free margins of $\tilde{C}(\boldsymbol{u})$. After this resultant adaptation, an explicit inequality defines upper or lower bound of the Fréchet class if it contains the term $\tilde{C}(\boldsymbol{u})$. An explicit system is regarded as a *minimal sufficient explicit system* of inequalities if any explicit inequality can be obtained as a nonnegative linear combination of its members. So, if any copula fulfils a minimal sufficient explicit system, it fulfils any explicit inequality.

Example 3.1. Let us consider a Fréchet class $\mathcal{F}_3(\emptyset)$. Only 1-margins are fixed by definition of copulas. For a member $\tilde{C}$ of the Fréchet class, the corresponding system of implicit inequalities

$$\begin{array}{lrrrr|rrr|r}
I_1: & & & & & & & & \tilde{C}(\boldsymbol{u}) \geq 0 \\
I_2: & & & & & & & \tilde{C}_{\{2,3\}} & -\tilde{C}(\boldsymbol{u}) \geq 0 \\
I_3: & & & & & & \tilde{C}_{\{1,3\}} & & -\tilde{C}(\boldsymbol{u}) \geq 0 \\
I_4: & & & & & \tilde{C}_{\{1,2\}} & & & -\tilde{C}(\boldsymbol{u}) \geq 0 \\
I_5: & & & & u_3 & & -\tilde{C}_{\{1,3\}} & -\tilde{C}_{\{2,3\}} & +\tilde{C}(\boldsymbol{u}) \geq 0 \\
I_6: & & & u_2 & & -\tilde{C}_{\{1,2\}} & & -\tilde{C}_{\{2,3\}} & +\tilde{C}(\boldsymbol{u}) \geq 0 \\
I_7: & & u_1 & & & -\tilde{C}_{\{1,2\}} & -\tilde{C}_{\{1,3\}} & & +\tilde{C}(\boldsymbol{u}) \geq 0 \\
I_8: & 1 & -u_1 & -u_2 & -u_3 & +\tilde{C}_{\{1,2\}} & +\tilde{C}_{\{1,3\}} & +\tilde{C}_{\{2,3\}} & -\tilde{C}(\boldsymbol{u}) \geq 0
\end{array} \tag{21}$$

corresponds to (20), where $\boldsymbol{u} = (u_1, u_2, u_3) \in \langle 0, 1\rangle^3$ and $\tilde{C}_v$ is an abbreviation for $\tilde{C}_v(\boldsymbol{u}_v)$. The framed part in (21) must be eliminated by nonnegative linear combination coefficients $\boldsymbol{x} = (x_2, \ldots, x_8)$ associated with inequalities $I_2, \ldots, I_8$. We try to find sufficient and the best possible explicit form of boundaries obtained in this way. The equivalent task is to find sufficient solutions of the system of linear *Diophantine* equations $\boldsymbol{x} \cdot \mathbf{A} = (0, 0, 0)$ where

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \tag{22}$$

is the matrix of coefficients at margins in framed part of (21).

The problem of linear Diophantine equations was recently studied especially in the area of artificial intelligence. Given an integer matrix $\mathbf{A} \in \mathbb{Z}^{k \times m}$, the set of nonnegative real solutions $\boldsymbol{x}$ of

$$\boldsymbol{x} \cdot \mathbf{A} = \mathbf{0}_m \tag{23}$$

is a convex polyhedral cone (see e. g. [7]). It is well-known that this cone is the convex hull of its extreme rays which are finitely many. Each extreme ray is a set $\{\alpha \boldsymbol{r} : \alpha \in \langle 0, \infty)\}$ where $\mathbf{0}_k \neq \boldsymbol{r} \in \mathbb{N}^k$ is a minimal solution of (23) in component-wise ordering, with minimal number of nonzero components (i. e. at most $m + 1$ components where m is the rank of $\mathbf{A}$). If $\mathcal{R} = \{\boldsymbol{r}_1, \ldots, \boldsymbol{r}_p\}$ is the set of such solutions, then the set of all real solutions of (23) is $\{\sum_{i=1}^{p} \alpha_i \boldsymbol{r}_i : \alpha_i \in \langle 0, \infty)\}$. It should be noted that the computational problem belongs to the NP-complete class and it is applicable for cluster or parallel computing. There were developed many competitive ways of solution for this task. A simple algorithm (from [7]) recovering $\mathcal{R}$ for the system (23) with matrix $\mathbf{A}$, which is assumed of full column rank with m columns, follows.

Algorithm 3.2.

```
for each combination C of m+1 rows of matrix A do
   compute s[1],...,s[m+1] with
      s[k] = det(Ck) where Ck is C leaving out row k;
   if not all s[k] are null and no two components of
      (s[1],-s[2],...,(-1)^m*s[m+1]) are of opposite sign
   then abs(s[1],-s[2],...,(-1)^m*s[m+1])/gcd(s[1],...,s[m+1])
      are the components of a minimal solution corresponding
      to the rows of the matrix A which are in C, the other
      components being null.  Here, abs and gcd are operators of
      absolute value and greatest common divisor.
```

So, the minimal sufficient system of explicit inequalities consists of inequalities of the original system (20) with no free proper margins and of linear combinations of the rest implicit inequalities where the combination coefficients are appointed by the minimal solutions $\boldsymbol{r}_i$, $i = 1, \dots, p$. The inequalities with a negative coefficient at $\tilde{F}(\boldsymbol{u})$ define single upper bounds, those with a positive coefficient define single lower bounds and those without $\tilde{F}(\boldsymbol{u})$ are only conditions of compatibility. The final upper or lower bound is the minimum or maximum of the set of single upper or lower bounds, respectively.

As the number of implicit inequalities (20) is also exponential in n, the complexity and number of solutions increase rapidly.

Example 3.3. The next table contains the counts of all algebraic single upper and lower bounds for Fréchet classes of all $(n-2)$-margins.

n	3	4	5	6
upper bounds	3	16	1430	473605
lower bounds	2	23	1309	453779

The technique, described in the next section, somewhen requires not to eliminate all proper free margins but to let some ones in formulations of single bounds. The columns of the matrix of Diophantine system are restricted only on the rest free margins then.

Example 3.4. (Example 3.1 continued) The matrix

$$\mathbf{X} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix} \tag{24}$$

contains in rows all solutions $\boldsymbol{r}_i \in \mathcal{R}$ of the Diophantine system with the matrix (22). These vectors are in fact coefficients (at I_2-I_8) defining the minimal sufficient explicit system of inequalities (except the first one) I_1, $I_2 + I_3 + I_5$, $I_2 + I_4 + I_6$, $I_3 + I_4 + I_7$, $(I_2 + I_5 + I_6 + I_8$, $I_3 + I_5 + I_7 + I_8$, $I_4 + I_6 + I_7 + I_8)$, $I_5 + I_6 + I_7 + 2I_8$ for $\mathcal{F}_3(\emptyset)$

$$\begin{array}{ccccr} & & & & \tilde{C}(\boldsymbol{u}) \geq 0 \\ & & & u_3 & -\tilde{C}(\boldsymbol{u}) \geq 0 \\ & & u_2 & & -\tilde{C}(\boldsymbol{u}) \geq 0 \\ & u_1 & & & -\tilde{C}(\boldsymbol{u}) \geq 0 \\ 2 & -u_1 & -u_2 & -u_3 & +\tilde{C}(\boldsymbol{u}) \geq 0 \end{array} \tag{25}$$

where I_1 was explicit already in the original system (21) and the combinations in parenthesis are omitted because they eliminate also $\tilde{C}(\boldsymbol{u})$ and make only conditions of compatibility. So, we have got bounds

$$\max\{0, u_1 + u_2 + u_3 - 2\} \leq \tilde{C}(\boldsymbol{u}) \leq \min\{u_1, u_2, u_3\} \tag{26}$$

for any $\tilde{C} \in \mathcal{F}_3(\emptyset)$ and $\boldsymbol{u} \in \langle 0, 1\rangle^3$.

The next theorem (see e.g. [5]) includes generalization of this result.

Theorem 3.5. (Fréchet-Hoeffding) If C is any n-copula, then for every $\boldsymbol{u} \in \langle 0, 1\rangle^n$,

$$\max\{0, u_1 + u_2 + \ldots + u_n - n + 1\} \leq C(\boldsymbol{u}) \leq \min\{u_1, u_2, \ldots, u_n\}. \tag{27}$$

It is easy to show that members of $\boldsymbol{\Upsilon}(\boldsymbol{S})$ from Corollary 2.6 are algebraic single upper bounds and functions $L_C^{\Theta}(\boldsymbol{u})$ including those in (14) are algebraic single lower bounds of the Fréchet class $\mathcal{F}_n(C_\upsilon : \upsilon \in \boldsymbol{S})$.

Algebraic single bounds might be also helpful in determining the necessary conditions of compatibility for a system of concrete fixed margins. Any single upper bound must dominate any single lower bound all over $\langle 0, 1\rangle^n$. This makes stronger the Corollary 2.6. But the number of comparisons might grow hugely (see Example 3.3).

Technique presented in this section allows us to set bounds to Fréchet class better than boundaries in the Theorem 2.4, since $\mathcal{U}|_{\boldsymbol{u}}$ and $\mathcal{L}|_{\boldsymbol{u}}$ include only some of the algebraic boundaries.

Example 3.6. It is easy to show that one of the single upper bounds of a Fréchet class $\mathcal{F}_n(C_{\{1,2\}}, C_{\{1,3\}}, C_{\{2,3\}})$ is $\mathfrak{u}(u_1, u_2, u_3) = 1 - u_1 - u_2 - u_3 + C_{\{1,2\}}(u_1, u_2) + C_{\{1,3\}}(u_1, u_3) + C_{\{2,3\}}(u_2, u_3)$ (see e.g. [4] or [1]). Following Example 2.5, we have got $\mathfrak{u}(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}) = 1 - \frac{3}{4} - \frac{3}{4} - \frac{3}{4} + W^2(\frac{3}{4}, \frac{3}{4}) + \Pi^2(\frac{3}{4}, \frac{3}{4}) + \Pi^2(\frac{3}{4}, \frac{3}{4}) = \frac{3}{8} < \frac{1}{2} = U(\frac{3}{4}, \frac{3}{4}, \frac{3}{4})$.

4. ANALYTIC CONSTRUCTION OF BOUNDS

Let us consider a Fréchet class $\mathcal{F}_n(C_\upsilon : \upsilon \in \boldsymbol{S})$ where all determinative sets from $\boldsymbol{S}$ have at least two elements and none of them is a subset of another. For some $\varsigma = \{j_1, \ldots, j_k\} \subset S_n$, let $\boldsymbol{T} \subset \boldsymbol{S}$ be a set of such $\upsilon \in \boldsymbol{S}$ that $\varsigma \subset \upsilon$ and $|\boldsymbol{T}| \geq 2$. Let us denote $\tau = \cup\boldsymbol{T} \notin \boldsymbol{S}$. Suppose, we can suitably set bounds $\mathfrak{u}(\boldsymbol{u}_{\tau-\varsigma}, \boldsymbol{t}_\varsigma)$ and $\mathfrak{l}(\boldsymbol{u}_{\tau-\varsigma}, \boldsymbol{t}_\varsigma)$ to the conditional distribution function $\tilde{C}_{\tau-\varsigma|\varsigma}(\boldsymbol{u}_{\tau-\varsigma}|\boldsymbol{t}_\varsigma)$ (the condition is $\boldsymbol{u}_\varsigma = \boldsymbol{t}_\varsigma$, where $\boldsymbol{t}_\varsigma$ is an element of $\langle 0,1\rangle^k$). Then it holds true

$$\beta(\boldsymbol{u}_\tau) = \underset{\boldsymbol{t}_\varsigma\in\langle 0,u_{j_1}\rangle\times\ldots\times\langle 0,u_{j_k}\rangle}{\int\cdots\int} \mathfrak{l}(\boldsymbol{u}_{\tau-\varsigma},\boldsymbol{t}_\varsigma)\mathrm{d}C_\varsigma(\boldsymbol{t}_\varsigma)$$
$$\leq \tilde{C}_\tau(\boldsymbol{u}_\tau) = \underset{\boldsymbol{t}_\varsigma\in\langle 0,u_{j_1}\rangle\times\ldots\times\langle 0,u_{j_k}\rangle}{\int\cdots\int} \tilde{C}_{\tau-\varsigma|\varsigma}(\boldsymbol{u}_{\tau-\varsigma}|\boldsymbol{t}_\varsigma)\mathrm{d}C_\varsigma(\boldsymbol{t}_\varsigma)$$
$$\leq \underset{\boldsymbol{t}_\varsigma\in\langle 0,u_{j_1}\rangle\times\ldots\times\langle 0,u_{j_k}\rangle}{\int\cdots\int} \mathfrak{u}(\boldsymbol{u}_{\tau-\varsigma},\boldsymbol{t}_\varsigma)\mathrm{d}C_\varsigma(\boldsymbol{t}_\varsigma) = \alpha(\boldsymbol{u}_\tau). \quad (28)$$

If we let set out the free margin $\tilde{C}_\tau(\boldsymbol{u}_\tau)$ in expressions of the algebraic single bounds of the Fréchet class in addition to fixed margins, we can next replace $\tilde{C}_\tau(\boldsymbol{u}_\tau)$ in these expressions by $\beta(\boldsymbol{u}_\tau)$ or $\alpha(\boldsymbol{u}_\tau)$ in order to rise the number of single bounds.

The associated distributions of $\tilde{C}_\tau(\boldsymbol{u}_\tau)$ might contribute by additional single bounds. It was shown in [4], [1] and by many other authors that the single bounds obtained by this method can narrow down the bounds. One can use multilevel version of this method. Aforementioned bounds $\mathfrak{u}(\boldsymbol{u}_{\tau-\varsigma},\ \boldsymbol{t}_\varsigma)$ and $\mathfrak{l}(\boldsymbol{u}_{\tau-\varsigma},\boldsymbol{t}_\varsigma)$ in (28) could be constructed in the same analytical way.

Example 4.1. Let us demonstrate these principles on a Fréchet class $\mathcal{F}_5(C_{\{1,2,3\}}, C_{\{1,2,4\}}, C_{\{1,5\}}, C_{\{3,4\}})$. After solution the relevant Diophantine system, the upper algebraic bound is the minimum of 11 single upper bounds and the lower bound is the maximum of 10 single lower bounds for any element $\tilde{C}(\boldsymbol{u})$ of the Fréchet class.

The union of all determinative sets including $\varsigma = \{3\}$ is $\tau = \{1,2,3\} \cup \{3,4\} = \{1,2,3,4\}$. The distribution function $C_{\{1,2,4\}|3}(u_1,u_2,u_4|t)$ conditional in the 3-rd variable has algebraic bounds

$$\begin{aligned}\mathfrak{u}_1(u_1,u_2,t,u_4) &= \min\left\{C_{\{1,2\}|3}(u_1,u_2|t), C_{\{4\}|3}(u_4|t)\right\} \\ \mathfrak{l}_1(u_1,u_2,t,u_4) &= \max\{0, C_{\{1,2\}|3}(u_1,u_2|t) + C_{\{4\}|3}(u_4|t) - 1\}.\end{aligned} \quad (29)$$

The free margin $\tilde{C}_\tau(u_1,u_2,u_3,u_4)$ might be bounded in the sense of (28) by

$$\beta_1(\boldsymbol{u}_\tau) = \int_0^{u_3} \mathfrak{l}_1(u_1,u_2,t,u_4)\mathrm{d}t \leq \tilde{C}_\tau(\boldsymbol{u}_\tau) \leq \int_0^{u_3} \mathfrak{u}_1(u_1,u_2,t,u_4)\mathrm{d}t = \alpha_1(\boldsymbol{u}_\tau). \quad (30)$$

Not eliminating free margin $\tilde{C}_\tau(\boldsymbol{u}_\tau)$ in the expression of algebraic bounds of $\tilde{C}(\boldsymbol{u})$, we get

$$\max\left\{0, \tilde{C}_\tau(\boldsymbol{u}_\tau) + u_5 - 1\right\} \leq \tilde{C}(\boldsymbol{u}) \leq \min\left\{\tilde{C}_\tau(\boldsymbol{u}_\tau), u_5\right\}. \quad (31)$$

Conditions (30) and (31) define additional single bounds for $\tilde{C}(\boldsymbol{u})$

$$\begin{aligned}a_1(\boldsymbol{u}) &= \min\left\{\alpha_1(\boldsymbol{u}_\tau), u_5\right\}, \\ b_1(\boldsymbol{u}) &= \max\left\{0, \beta_1(\boldsymbol{u}_\tau) + u_5 - 1\right\}.\end{aligned} \quad (32)$$

By analogy with (29) and (30), we can set bounds

$$\begin{aligned}\beta_2'(\boldsymbol{u}_\tau) &= \int_0^{1-u_3} \max\left\{0, C^{\{3,4\}}_{\{1,2\}|3}(u_1,u_2|t) + C^{\{3,4\}}_{\{4\}|3}(1-u_4|t) - 1\right\} \mathrm{d}t \\ &\le \tilde{C}^{\{3,4\}}_\tau(u_1,u_2,1-u_3,1-u_4) \\ &\le \int_0^{1-u_3} \min\left\{C^{\{3,4\}}_{\{1,2\}|3}(u_1,u_2|t), C^{\{3,4\}}_{\{4\}|3}(1-u_4|t)\right\} \mathrm{d}t = \alpha_2'(\boldsymbol{u}_\tau)\end{aligned} \tag{33}$$

for the associated copula $\tilde{C}^{\{3,4\}}_\tau$ at the argument $(u_1,u_2,1-u_3,1-u_4)$

$$\begin{aligned}&\tilde{C}^{\{3,4\}}_\tau(u_1,u_2,1-u_3,1-u_4) \\ &\qquad = C_{\{1,2\}}(\boldsymbol{u}_{\{1,2\}}) - C_{\{1,2,3\}}(\boldsymbol{u}_{\{1,2,3\}}) - C_{\{1,2,4\}}(\boldsymbol{u}_{\{1,2,4\}}) + \tilde{C}_\tau(\boldsymbol{u}_\tau)\end{aligned} \tag{34}$$

and consequently for $\tilde{C}_\tau(\boldsymbol{u}_\tau)$

$$\begin{aligned}\alpha_2(\boldsymbol{u}_\tau) &= \alpha_2'(\boldsymbol{u}_\tau) - C_{\{1,2\}}(\boldsymbol{u}_{\{1,2\}}) + C_{\{1,2,3\}}(\boldsymbol{u}_{\{1,2,3\}}) + C_{\{1,2,4\}}(\boldsymbol{u}_{\{1,2,4\}}) \\ &\le \tilde{C}_\tau(\boldsymbol{u}_\tau) \\ &\le \beta_2'(\boldsymbol{u}_\tau) - C_{\{1,2\}}(\boldsymbol{u}_{\{1,2\}}) + C_{\{1,2,3\}}(\boldsymbol{u}_{\{1,2,3\}}) + C_{\{1,2,4\}}(\boldsymbol{u}_{\{1,2,4\}}) = \beta_2(\boldsymbol{u}_\tau)\end{aligned} \tag{35}$$

by (33) and (34). By analogy with (30), (31) and (32), we have got the next single bounds

$$\begin{aligned}a_2(\boldsymbol{u}) &= \min\left\{\alpha_2(\boldsymbol{u}_\tau), u_5\right\}, \\ b_2(\boldsymbol{u}) &= \max\left\{0, \beta_2(\boldsymbol{u}_\tau) + u_5 - 1\right\}.\end{aligned} \tag{36}$$

Searching for another single bounds browsing $\varsigma \subset S_5$ and copulas associated with corresponding $\tilde{C}_\tau$, we will come to $\varsigma = \{1\}$ with $\tau = S_5$. Inside conditional distribution $u_1 = t$, we come at algebraic bounds

$$\begin{aligned}\mathfrak{u}_3(t,u_2,u_3,u_4,u_5) &= \min\left\{C_{\{2,3\}|1}(u_2,u_3|t), C_{\{2,4\}|1}(u_2,u_4|t), C_{\{5\}|1}(u_5|t)\right\} \\ \mathfrak{l}_3(t,u_2,u_3,u_4,u_5) &= \max\{0, C_{\{2,3\}|1}(u_2,u_3|t) + C_{\{2,4\}|1}(u_2,u_4|t) + \\ &\quad + C_{\{5\}|1}(u_5|t) - C_{\{2\}|1}(u_2|t) - 1\}\end{aligned} \tag{37}$$

of $\tilde{C}_{\{2,3,4,5\}|1}(u_2,u_3,u_4,u_5|t)$. It leads directly to single bounds

$$\begin{aligned}a_3(\boldsymbol{u}) &= \int_0^{u_1} \mathfrak{u}_3(t,u_2,u_3,u_4,u_5)\mathrm{d}t \\ b_3(\boldsymbol{u}) &= \int_0^{u_1} \mathfrak{l}_3(t,u_2,u_3,u_4,u_5)\mathrm{d}t\end{aligned} \tag{38}$$

of $\tilde{C}(\boldsymbol{u}) = \int_0^{u_1} \tilde{C}_{\{2,3,4,5\}|1}(u_2,u_3,u_4,u_5|t)\mathrm{d}t$ by (28).

Moreover, we can set bounds

$$\begin{aligned}\alpha_3(t,u_2,u_3,u_4,u_5) &= \min\{\tilde{C}_{\{2,3,4\}|1}(u_2,u_3,u_4|t), C_{\{5\}|1}(u_5|t)\} \\ \beta_3(t,u_2,u_3,u_4,u_5) &= \max\{0, \tilde{C}_{\{2,3,4\}|1}(u_2,u_3,u_4|t) + C_{\{5\}|1}(u_5|t) - 1\}\end{aligned} \tag{39}$$

for free margin $\tilde{C}_{\{2,3,4,5\}|1}(u_2, u_3, u_4, u_5|t)$ where

$$\tilde{C}_{\{2,3,4\}|1}(u_2, u_3, u_4|t) = \int_0^{u_2} \tilde{C}_{\{3,4\}|1,2}(u_3, u_4|t, s)\mathrm{d}C_{\{2\}|1}(s) \tag{40}$$

and

$$\begin{aligned}\beta_4(t, s, u_3, u_4) = \max\{0, C_{\{3\}|1,2}(u_3|t, s) + C_{\{4\}|1,2}(u_4|t, s) - 1\}& \\ \leq \tilde{C}_{\{3,4\}|1,2}(u_3, u_4|t, s)& \\ \leq \min\{C_{\{3\}|1,2}(u_3|t, s), C_{\{4\}|1,2}(u_4|t, s)\} = \alpha_4(t, s, u_3, u_4).&\end{aligned} \tag{41}$$

Consequently by (39), (40) and (41), additional bounds for $\tilde{C}_{\{2,3,4,5\}|1}(u_2, u_3, u_4, u_5|t)$ are

$$\begin{aligned}\mathfrak{u}_4(t, u_2, u_3, u_4, u_5) &= \min\left\{\int_0^{u_2} \alpha_4(t, s, u_3, u_4)\mathrm{d}C_{\{2\}|1}(s), C_{\{5\}|1}(u_5|t)\right\} \\ \mathfrak{l}_4(t, u_2, u_3, u_4, u_5) &= \max\left\{0, \int_0^{u_2} \beta_4(t, s, u_3, u_4)\mathrm{d}C_{\{2\}|1}(s) + C_{\{5\}|1}(u_5|t) - 1\right\}.\end{aligned} \tag{42}$$

It defines the next single bounds for $\tilde{C}(\boldsymbol{u})$

$$\begin{aligned}a_4(\boldsymbol{u}) &= \int_0^{u_1} \mathfrak{u}_4(t, u_2, u_3, u_4, u_5)\mathrm{d}t \\ b_4(\boldsymbol{u}) &= \int_0^{u_1} \mathfrak{l}_4(t, u_2, u_3, u_4, u_5)\mathrm{d}t.\end{aligned} \tag{43}$$

Search for other single bounds is left on the forgiving reader.

5. CONCLUSION

In the second section it is shown that membership to Fréchet class is in addition to n-increase designed by simple bounds. The third part solves the problem of delimitation of algebraic bounds by means of system of Diophantine equations. The question remains whether the special form of the matrix of such systems allows a computationally more efficient solution. The fourth part is devoted to the analytical improvement of algebraic boundaries.

It was demonstrated by other authors that some single bounds might be inactive for concrete Fréchet classes. It would be interesting to determine a hierarchy of all these single bounds and allocate a minimal sufficient system of them.

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