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BOUNDARY LAYER PHENOMENON FOR THREE –POINT BOUNDARY VALUE PROBLEM FOR THE NONLINEAR SINGULARLY PERTURBED SYSTEMS

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This paper deals with the three–point boundary value problem for the nonlinear singularly perturbed second-order systems. Especially, we focus on an analysis of the solutions in the right endpoint of considered interval from an appearance of the boundary layer point of view. We use the method of lower and upper solutions combined with analysis of the integral equation associated with the class of nonlinear systems considered here.

Keywords: singularly perturbed systems, three–point boundary value problem, method of lower and upper solutions, controller

Classification: 93C10, 34E15, 34A34, 34A40, 34B10

1. MOTIVATION AND INTRODUCTION

We will consider the nonlinear singularly perturbed systems described by differential equation of the form

$$\epsilon y'' + ky = f(x, y), \quad x \in \langle a, b\rangle, \quad k < 0 \tag{1}$$

with three–point boundary value conditions

$$y'(a) = 0, \quad y(b) = y(c), \quad a < c < b \tag{2}$$

where ϵ is a small perturbation parameter ($0 < \epsilon \ll 1$). The dependence upon the variable x of the continuous function f represents the effects of outer disturbances.

Singularly perturbed systems (SPSs) normally occur due to the presence of small "parasitic" parameters, armature inductance in a common model for most DC motors, small time constants, etc.

Boundary value problems (1), (2) can arise in the study of the steady–states of a heated bar with a thermostat described by partial differential equation

$$\frac{\partial y}{\partial t} = \epsilon \frac{\partial^2 y}{\partial x^2} + ky - f(x, y)$$

with stationary condition $\partial y/\partial t = 0$, where a controller at $x = b$ maintains a fixed temperature according to the temperature detected by a sensor at $x = c$, while the rate of temperature change at the left end of the bar is zero. In this case, we consider a uniform bar of length $b-a$ with non-uniform temperature lying on the x-axis from $x=a$ to $x=b$. The parameter ϵ represents the thermal diffusivity. Thus, the singular perturbation problems are of common occurrence in modeling the heat-transport problems with large Peclet number [5, 10, 13]. One of the typical behaviors of SPSs is the boundary layer phenomenon: the solutions vary rapidly within very thin layer regions near the boundary. The goal of this paper is to analyze the thermal boundary layer phenomena arising in such singularly perturbed systems. We give an accurate estimate for determining the rate of boundary layer growth.

The literature on control of nonlinear SPSs is extensive, at least starting with the pioneering work of P. Kokotovic *et al.* nearly 30 years ago [12] and continuing to the present including authors such as Z. Artstein [1, 2], V. Gaitsgory [3, 4, 6], etc.

In the past few years the multi–point boundary value problem (BVP) has received a wide attention see e. g. [8, 11] and the references therein. For example, Khan [11] have studied a four-point boundary value problem of type $y(c)-\nu_1 y(a)=0, y(b)-\nu_2 y(d)=0$ where the constants ν_1, ν_2 are not simultaneously equal to 1 and $\epsilon=1$.

Recently in [14], we have shown that for every $\epsilon>0$ sufficiently small ($\epsilon\in(0,\epsilon_0\rangle$) there is a unique solution y_ϵ of BVP (1), (2) such that $[x, y_\epsilon(x)]\subset D(u)$ and y_ϵ converges uniformly to the solution u of reduced problem $ku=f(x,u)$ for $\epsilon\to 0^+$ on every compact subset $K\subset\langle a,b)$. Consequently, $y_\epsilon(b)=y_\epsilon(c)\to u(c)$ for $\epsilon\to 0^+$. For definition of the set $D(u)$ see below.

In this paper we focus our attention on the detailed analysis of the behavior of the solutions y_ϵ for (1), (2) in the point $x=b$ when a small parameter ϵ tends to zero. We show that the solutions y_ϵ of (1), (2) remain close to u on K with an arising fast transient of y_ϵ to $y_\epsilon(b)$ ($|y'_\epsilon(b)|\to\infty$ for $u(b)\neq u(c)$ and $\epsilon\to 0^+$), which is the so-called boundary layer phenomenon [7, 12]. Boundary layers are formed due to the nonuniform convergence of the exact solution y_ϵ to the solution u of reduced problem in the neighborhood of the right end b.

We will assume that the following conditions are satisfied throughout this paper:

(H1) The solution u of a reduced problem $ku=f(x,u)$ is a C^3 function defined on the interval $\langle a,b\rangle$.

Denote $D(u)=\{(x,y)|\quad a\le x\le b, |y-u(x)|\le d(x)\}$, where $d(x)$ is the positive continuous function on $\langle a,b\rangle$,

$$d(x)=\begin{cases}\delta & \text{for } a\le x\le b-\delta\\ \frac{2}{\delta}|u(b)-u(c)|(x-b)+2|u(b)-u(c)|+\delta & \text{for } b-\delta\le x\le b-\frac{\delta}{2}\\ |u(b)-u(c)|+\delta & \text{for } b-\frac{\delta}{2}\le x\le b,\end{cases}$$

where δ is a small positive constant.

(H2) $f(c,u(c))\neq f(b,u(c))$.

It is instructive for the future to keep in mind that this assumption implies that $u(c)\neq u(b)$ and $f(c,y_\epsilon(c))\neq f(b,y_\epsilon(b))$ for every sufficiently small ϵ, say $0<\epsilon<\epsilon_0$.

(H3) $f \in C^1(D(u))$ and there exists a positive constant w such that

$$\left|\frac{\partial f(x,y)}{\partial y}\right| \leq w < -k \quad \text{for every} \quad (x,y) \in D(u).$$

Notation.

$g_{1,\epsilon}(x) = k - \frac{\partial f(x,y_\epsilon(x))}{\partial y}$

$g_{2,\epsilon}(x) = \frac{\partial f(x,y_\epsilon(x))}{\partial x}$

$m = -k - w$

$\gamma_\epsilon(x) = \frac{1}{m}\left|\epsilon u'''(x) + g_{1,\epsilon}(x)u'(x) - g_{2,\epsilon}(x)\right|.$

Obviously, $\gamma_\epsilon(x) \geq 0$ and $\lim_{\epsilon\to 0^+}\gamma_\epsilon(x) = 0$ for $x \in \langle a,b)$ and $\lim_{\epsilon\to 0^+}\gamma_\epsilon(b) \neq 0$ for $u'(b) \neq \frac{\partial f(b,u(c))}{\partial x}\left(k - \frac{\partial f(b,u(c))}{\partial y}\right)^{-1}$. The equality $u(b) = u(c)$ implies $\lim_{\epsilon\to 0^+}\gamma_\epsilon(b) = 0$.

2. BOUNDARY LAYER PHENOMENON AT $x = b$

For an illustrative example we consider (1), (2) with $f(x,y) = x^2$, $a = 0$, $b = 2$, $c = 1$ and its solution

$$\begin{aligned} y_\epsilon(x) = &-\frac{3}{k}\cdot\frac{\mathrm{e}^{2\sqrt{-\frac{k}{\epsilon}}}}{\mathrm{e}^{4\sqrt{-\frac{k}{\epsilon}}} - \mathrm{e}^{3\sqrt{-\frac{k}{\epsilon}}} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}} + 1}\cdot \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}x} \\ &-\frac{3}{k}\cdot\frac{\mathrm{e}^{2\sqrt{-\frac{k}{\epsilon}}}}{\mathrm{e}^{4\sqrt{-\frac{k}{\epsilon}}} - \mathrm{e}^{3\sqrt{-\frac{k}{\epsilon}}} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}} + 1}\cdot \mathrm{e}^{-\sqrt{-\frac{k}{\epsilon}}x} + \frac{x^2}{k} - \frac{2\epsilon}{k^2}. \end{aligned}$$

Hence we have

1. $\lim\limits_{\epsilon\to 0^+} y_\epsilon(x_0) = \frac{f(x_0)}{k} = u(x_0)$ for every $x_0 \in \langle 0,2)$
2. $\lim\limits_{\epsilon\to 0^+} y_\epsilon(2) = \frac{f(1)}{k} = u(1)$
3. $\lim\limits_{\epsilon\to 0^+} |y'_\epsilon(2)| = \infty$ (a boundary layer phenomenon).

We precede the main result of this article with the following important lemmas.

Lemma 2.1. Let the assumptions (H1) and (H3) hold. Let $[x, y_\epsilon(x)] \subset D(u)$ for $\epsilon \in (0,\epsilon_0\rangle$ and $x \in \langle a,b\rangle$ where y_ϵ is the solution of (1), (2). Then we have on $\langle a,b\rangle$ the estimate

$$|y'_\epsilon(x) - u'(x)| \leq v_{L,\epsilon}(x) + v_{R,\epsilon}(x) + \gamma_{\epsilon,\max} \tag{3}$$

where

$$\begin{aligned} v_{L,\epsilon}(x) &= |u'(a)|\,\mathrm{e}^{\sqrt{\frac{m}{\epsilon}}(a-x)} \\ v_{R,\epsilon}(x) &= |u'(b) - y'_\epsilon(b)|\,\mathrm{e}^{\sqrt{\frac{m}{\epsilon}}(x-b)} \\ \gamma_{\epsilon,\max} &= \max\{\gamma_\epsilon(x);\ x \in \langle a,b\rangle\}. \end{aligned}$$

P r o o f. Differentiating (1) with respect to the variable x we obtain for y'_ϵ, $\epsilon \in (0, \epsilon_0\rangle$ linear differential equation

$$\epsilon z'' + g_{1,\epsilon}(x) z = g_{2,\epsilon}(x) \tag{4}$$

with the Dirichlet boundary condition

$$z(a) = 0, \ z(b) = y'_\epsilon(b). \tag{5}$$

First we show that $z_\epsilon = y'_\epsilon$ is an unique solution of Dirichlet BVP (4), (5) for y_ϵ, $\epsilon \in (0, \epsilon_0\rangle$. Assume to the contrary, that Z_1, Z_2 are two solutions of (4), (5) for $\epsilon \in (0, \epsilon_0\rangle$ fixed. Denote $Z(x) = Z_1(x) - Z_2(x)$. Then Z is a solution of the homogeneous Dirichlet problem

$$\epsilon z'' + g_{1,\epsilon}(x) z = 0,$$

$$z(a) = 0, \ z(b) = 0.$$

Thus there is $x_0 \in (a, b)$ such that $Z(x_0) \neq 0$, $Z'(x_0) = 0$ and $Z(x_0) Z''(x_0) \leq 0$ which contradicts to the assumption (H3). To prove Lemma 2.1 it is sufficient to show that for every y_ϵ, $\epsilon \in (0, \epsilon_0\rangle$ there is a solution z_ϵ of (4), (5) satisfying (3). We apply the method of lower and upper solutions [9]. As usual, a function α_ϵ is called a lower solution of the Dirichlet BVP (4), (5) if $\alpha_\epsilon \in C^2(\langle a, b\rangle)$ and satisfies

$$\epsilon \alpha''_\epsilon(x) + g_{1,\epsilon}(x) \alpha_\epsilon \geq g_{2,\epsilon}(x) \tag{6}$$

$$\alpha_\epsilon(a) \leq 0, \ \alpha_\epsilon(b) \leq y'_\epsilon(b).$$

An upper solution $\beta_\epsilon \in C^2(\langle a, b\rangle)$ of the problem (4), (5) is defined similarly by reversing the inequalities. If $\alpha_\epsilon \leq \beta_\epsilon$ on $\langle a, b\rangle$ then there exists a solution z_ϵ with $\alpha_\epsilon \leq z_\epsilon \leq \beta_\epsilon$ on $\langle a, b\rangle$.

Define

$$\alpha_\epsilon(x) = u'(x) - v_{L,\epsilon}(x) - v_{R,\epsilon}(x) - \gamma_{\epsilon,\max}$$

and

$$\beta_\epsilon(x) = u'(x) + v_{L,\epsilon}(x) + v_{R,\epsilon}(x) + \gamma_{\epsilon,\max}.$$

It is easy to check that $\alpha_\epsilon(a) \leq 0 \leq \beta_\epsilon(a)$, $\alpha_\epsilon(b) \leq y'_\epsilon(b) \leq \beta_\epsilon(b)$ and $\alpha_\epsilon(x) \leq \beta_\epsilon(x)$ for $x \in \langle a, b\rangle$. Now we show that the inequality (6) holds. For β_ϵ we proceed analogously.

$$\begin{aligned}
\epsilon \alpha''_\epsilon(x) &+ g_{1,\epsilon}(x) \alpha_\epsilon(x) - g_{2,\epsilon}(x) \\
&= \epsilon u'''(x) - \epsilon v''_{L,\epsilon}(x) - \epsilon v''_{R,\epsilon}(x) \\
&+ g_{1,\epsilon}(x) \left(u'(x) - v_{L,\epsilon}(x) - v_{R,\epsilon}(x) - \gamma_{\epsilon,\max} \right) - g_{2,\epsilon}(x) \\
&\geq \epsilon u'''(x) - \epsilon v''_{L,\epsilon}(x) - \epsilon v''_{R,\epsilon}(x) \\
&+ g_{1,\epsilon}(x) u'(x) + m v_{L,\epsilon}(x) + m v_{R,\epsilon}(x) + m \gamma_{\epsilon,\max} - g_{2,\epsilon}(x) \\
&= \epsilon u'''(x) + g_{1,\epsilon}(x) u'(x) - g_{2,\epsilon}(x) + m \gamma_{\epsilon,\max} \geq 0.
\end{aligned}$$

The Lemma 2.1 is proven. □

Lemma 2.2. Let the assumptions (H1) and (H3) hold. Then the set

$$\{\epsilon\,|y'_\epsilon(b)|\,;\ \epsilon \in (0, \epsilon_0\rangle\}$$

is bounded.

Proof. By Lagrange's Theorem and from Diff. Eq. (1) we obtain

$$|y'_\epsilon(b) - y'_\epsilon(a)| = |y''_\epsilon(\theta_\epsilon)|\,(b-a) = \frac{1}{\epsilon}\,|f(\theta_\epsilon, y_\epsilon(\theta_\epsilon)) - k y_\epsilon(\theta_\epsilon)|\,(b-a) \le \frac{C^*_\delta}{\epsilon}(b-a)$$

where $\theta_\epsilon \in (a,b)$ and $C^*_\delta = \max\{|f(x,y) - ky|;\ (x,y) \in D(u)\}$.

Hence $\epsilon\,|y'_\epsilon(b)| \le C^*_\delta(b-a)$ for $\epsilon \in (0,\epsilon_0\rangle$. □

3. MAIN RESULT

Our main result is the following.

Theorem 3.1. Under the assumptions (H1)-(H3) the problem (1), (2) has for every ϵ, $\epsilon \in (0,\epsilon_0\rangle$ the unique solution y_ϵ in $D(u)$ which converges uniformly to the solution u of reduced problem for $\epsilon \to 0^+$ on an arbitrary compact subset K of $\langle a, b)$ and the set

$$\{|y'_\epsilon(x)|\,;\ x \in \langle a,b\rangle, \epsilon \in (0,\epsilon_0\rangle\}$$

is unbounded.

More precisely,

$$|y'_\epsilon(b)| = O\left(\frac{1}{\sqrt{-k\epsilon}}\right) \quad \text{i. e. } |y'_\epsilon(b)| \to \infty \text{ for } \epsilon \to 0^+. \tag{7}$$

Proof. The existence, uniqueness in $D(u)$ and asymptotic behavior of the solutions for (1), (2) on the compact subset $K \subset \langle a, b)$ has been proven in [14]. It remains to prove (7), a boundary layer phenomenon at $x = b$.

Assume to the contrary that the set

$$\{|y'_\epsilon(x)|\,;\ x \in \langle a,b\rangle, \epsilon \in (0,\epsilon_0\rangle\}$$

is bounded. Consequently,

$$\left|\frac{\mathrm{d}f(x, y_\epsilon(x))}{\mathrm{d}x}\right| = \left|\frac{\partial f(x,y_\epsilon(x))}{\partial x} + \frac{\partial f(x,y_\epsilon(x))}{\partial y} y'_\epsilon\right| \le \tilde{C}_\delta, \tag{8}$$

on $\langle a,b\rangle$, $\tilde{C}_\delta > 0$ is constant. The problem (1), (2) is equivalent to the nonlinear integral equation

$$\begin{aligned} y_\epsilon(x) = {} & \frac{I}{\Lambda} \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(x-a)} + \frac{I}{\Lambda} \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-x)} \\ & + \int\limits_a^x \frac{\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(x-s)} - \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(s-x)}}{2\sqrt{-\frac{k}{\epsilon}}} \cdot \frac{f(s, y_\epsilon(s))}{\epsilon}\mathrm{d}s, \end{aligned} \tag{9}$$

where

$$I = \int_a^c \frac{e^{\sqrt{-\frac{k}{\epsilon}}(c-s)} - e^{\sqrt{-\frac{k}{\epsilon}}(s-c)}}{2\sqrt{-\frac{k}{\epsilon}}} \cdot \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s$$
$$- \int_a^b \frac{e^{\sqrt{-\frac{k}{\epsilon}}(b-s)} - e^{\sqrt{-\frac{k}{\epsilon}}(s-b)}}{2\sqrt{-\frac{k}{\epsilon}}} \cdot \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s,$$
$$\Lambda = e^{\sqrt{-\frac{k}{\epsilon}}(b-a)} + e^{\sqrt{-\frac{k}{\epsilon}}(a-b)} - e^{\sqrt{-\frac{k}{\epsilon}}(c-a)} - e^{\sqrt{-\frac{k}{\epsilon}}(a-c)}.$$

Differentiating the integral equation (9) with respect to the variable x we obtain

$$y'_\epsilon(x) = \frac{I\sqrt{-\frac{k}{\epsilon}}}{\Lambda} e^{\sqrt{-\frac{k}{\epsilon}}(x-a)} - \frac{I\sqrt{-\frac{k}{\epsilon}}}{\Lambda} e^{\sqrt{-\frac{k}{\epsilon}}(a-x)}$$
$$+ \int_a^x \frac{e^{\sqrt{-\frac{k}{\epsilon}}(x-s)} + e^{\sqrt{-\frac{k}{\epsilon}}(s-x)}}{2} \cdot \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s.$$

Hence

$$y'_\epsilon(b) = \frac{I\sqrt{-\frac{k}{\epsilon}}}{\Lambda} \left(e^{\sqrt{-\frac{k}{\epsilon}}(b-a)} - e^{\sqrt{-\frac{k}{\epsilon}}(a-b)} \right)$$
$$+ \frac{1}{2} \int_a^b \left(e^{\sqrt{-\frac{k}{\epsilon}}(b-s)} + e^{\sqrt{-\frac{k}{\epsilon}}(s-b)} \right) \frac{f(s, y_\epsilon(s))}{\epsilon} \mathrm{d}s. \tag{10}$$

Integrating all integrals in (10) by parts and after little algebraic arrangement we obtain

$$y'_\epsilon(b) = \frac{\sqrt{-\frac{k}{\epsilon}}}{k} \Bigg[\left(f(c, y_\epsilon(c)) - f(b, y_\epsilon(b)) \right) \sigma_\epsilon$$
$$+ \frac{\sigma_\epsilon}{2} \Bigg(\int_a^b \left(e^{\sqrt{-\frac{k}{\epsilon}}(b-s)} + e^{\sqrt{-\frac{k}{\epsilon}}(s-b)} \right) \frac{\mathrm{d}f(s, y_\epsilon(s))}{\mathrm{d}s} \mathrm{d}s$$
$$- \int_a^c \left(e^{\sqrt{-\frac{k}{\epsilon}}(c-s)} + e^{\sqrt{-\frac{k}{\epsilon}}(s-c)} \right) \frac{\mathrm{d}f(s, y_\epsilon(s))}{\mathrm{d}s} \mathrm{d}s \Bigg)$$
$$+ \frac{1}{2} \int_a^b \left(-e^{\sqrt{-\frac{k}{\epsilon}}(b-s)} + e^{\sqrt{-\frac{k}{\epsilon}}(s-b)} \right) \frac{\mathrm{d}f(s, y_\epsilon(s))}{\mathrm{d}s} \mathrm{d}s \Bigg]$$

where

$$\sigma_\epsilon = \frac{e^{\sqrt{-\frac{k}{\epsilon}}(b-a)} - e^{\sqrt{-\frac{k}{\epsilon}}(a-b)}}{\Lambda} \to 1^+ \text{ for } \epsilon \to 0^+. \tag{11}$$

Taking into consideration (8), the integrals

$$\int_a^b e^{\sqrt{-\frac{k}{\epsilon}}(s-b)} \frac{df(s, y_\epsilon(s))}{ds} ds, \quad \int_a^c e^{\sqrt{-\frac{k}{\epsilon}}(s-c)} \frac{df(s, y_\epsilon(s))}{ds} ds$$

are $O(\sqrt{\epsilon})$ by the mean value theorem for integrals.

Thus we have

$$\begin{aligned} y_\epsilon'(b) = \frac{\sqrt{-\frac{k}{\epsilon}}}{k} \Bigg[& \left(f(c, y_\epsilon(c)) - f(b, y_\epsilon(b))\right) \sigma_\epsilon \\ & + \frac{1}{2}(\sigma_\epsilon - 1) \int_a^b e^{\sqrt{-\frac{k}{\epsilon}}(b-s)} \frac{df(s, y_\epsilon(s))}{ds} ds \\ & - \frac{1}{2}\sigma_\epsilon \int_a^c e^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \frac{df(s, y_\epsilon(s))}{ds} ds + O\left(\sqrt{\epsilon}\right) \Bigg]. \end{aligned} \tag{12}$$

From (11) we can write

$$\sigma_\epsilon - 1 = e^{\sqrt{-\frac{k}{\epsilon}}(c-b)} \omega_\epsilon \to 0^+ \text{ for } \epsilon \to 0^+$$

where

$$\omega_\epsilon = \frac{1}{\Lambda} \left(e^{\sqrt{-\frac{k}{\epsilon}}(b-a)} + e^{\sqrt{-\frac{k}{\epsilon}}(a+b-2c)} - 2e^{\sqrt{-\frac{k}{\epsilon}}(a-c)} \right) \to 1^+ \text{ for } \epsilon \to 0^+.$$

Thus from (12) we have

$$\begin{aligned} y_\epsilon'(b) = -\frac{1}{\sqrt{-k\epsilon}} \Bigg[& \left(f(c, y_\epsilon(c)) - f(b, y_\epsilon(b))\right) \sigma_\epsilon \\ & + \frac{1}{2}(\omega_\epsilon - \sigma_\epsilon) \int_a^c e^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \frac{df(s, y_\epsilon(s))}{ds} ds \\ & + \frac{1}{2}\omega_\epsilon \int_c^b e^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \frac{df(s, y_\epsilon(s))}{ds} ds + O\left(\sqrt{\epsilon}\right) \Bigg]. \end{aligned}$$

The integral

$$\int_c^b e^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \frac{df(s, y_\epsilon(s))}{ds} ds$$

is $O(\sqrt{\epsilon})$ by the analogous argument as above and

$$\int_a^c e^{\sqrt{-\frac{k}{\epsilon}}(c-s)} \left| \frac{df(s, y_\epsilon(s))}{ds} \right| ds \le (c-a)\tilde{C}_\delta e^{\sqrt{-\frac{k}{\epsilon}}(c-a)}. \tag{13}$$

Using (13), we have

$$
\begin{aligned}
&\left|\frac{1}{2}(\omega_\epsilon-\sigma_\epsilon)\int\limits_a^c \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-s)}\frac{\mathrm{d}f\left(s,y_\epsilon(s)\right)}{\mathrm{d}s}\mathrm{d}s\right| \\
&\leq \frac{1}{2}(\omega_\epsilon-\sigma_\epsilon)(c-a)\tilde{C}_\delta \mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(c-a)} \\
&= \frac{1}{2}(c-a)\tilde{C}_\delta\frac{1}{\Lambda}\left(\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}\frac{(b-c)}{2}}-\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}\frac{(c-b)}{2}}\right)^2 = O\left(\mathrm{e}^{\sqrt{-\frac{k}{\epsilon}}(a-c)}\right).
\end{aligned}
$$

Hence

$$
y'_\epsilon(b) = -\frac{1}{\sqrt{-k\epsilon}}\left[\left(f\left(c,y_\epsilon(c)\right)-f\left(b,y_\epsilon(b)\right)\right)\sigma_\epsilon + O\left(\sqrt{\epsilon}\right)\right] \tag{14}
$$

which gives a contradiction.

Now we show that (7) holds. In Lemma 2.1 we have introduced the functions $v_{L,\epsilon}(x)$ and $v_{R,\epsilon}(x)$. For these functions we obtain the estimates (for the second one by using Lemma 2.2)

$$
v_{L,\epsilon}(x) = |u'(a)|\,O\left(1\right),\ v_{R,\epsilon}(x) = O\left(\epsilon^{-1}\right)\mathrm{e}^{\sqrt{\frac{m}{\epsilon}}(x-b)} \to 0^+
$$

uniformly for $\epsilon \to 0^+$ on every compact subset K of the interval $\langle a,b)$. Further, $\gamma_{\epsilon,\max} = O(1)$ for $\epsilon \to 0^+$ and u is a C^3 function on $\langle a,b\rangle$ (the assumption (H1)). Thus on the basis of Lemma 2.1, the set

$$
\{|y'_\epsilon(x)|\,;\ x\in K, \epsilon\in(0,\epsilon_0\rangle\}
$$

is bounded. Combining this with (14) we obtain (7). The proof of Theorem 3.1 is complete. □

Remark 3.2. As we can see from (14) the assumption (H2) is essential for an appearance the boundary layer phenomenon for singularly perturbed system (1), (2) at the point $x = b$.

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