KYBERNETIKA — VOLUME 47 (2011), NUMBER 4, PAGES 532–540

A NOTE ON BICONIC COPULAS

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We describe a class of bivariate copulas having a fixed diagonal section. The obtained class contains both the Fréchet upper and lower bounds and it allows to describe non-trivial tail dependence coefficients along both the diagonals of the unit square.

Keywords: copulas, diagonal section, probability mass distribution, tail dependence

Classification: 62E10

1. INTRODUCTION

Copulas are d–dimensional probability distribution functions that concentrate the probability mass on $\mathbb{I}^d$ ($\mathbb{I} := [0,1]$) and whose univariate margins are uniformly distributed on $\mathbb{I}$. Such functions have been largely used in multivariate statistical models, due to the fact that they capture the scale-invariant dependence of random vectors [9, 21].

In view of possible applications, methods for obtaining bivariate copulas have received a great attention in the literature. In particular, several authors have focused on "geometric methods", i. e. on methods that introduce new copulas "utilizing some information of a geometric nature, such as a description of the support or the shape of the graphs of horizontal, vertical, or diagonal sections" (compare with [21, chapter 3]). Recently, such an approach has been used, for instance, in [2, 3, 16, 17, 23, 24, 25].

In particular, powerful geometric constructions have been obtained by using the so-called "semilinear method", first considered in [7]. Roughly speaking, a semilinear copula is a copula whose restriction on some specific section of its domain is an affine function. Classes of semilinear copulas have been considered in [1, 12, 15].

In this note, we study a class of bivariate copulas having a suitable diagonal section δ such that the following property is satisfied: *for all* $x_0 \in]0,1[$ C *is linear on the segments connecting the diagonal point* (x_0, x_0) *with the corner points* $(1,0)$ *and* $(0,1)$.

Such a class of copulas has some distinguished features:

- it is comprehensive, i. e. it contains both the Fréchet upper bound $M(x,y) = \min(x,y)$ and the Fréchet lower bound $W(x,y) = \max(x+y-1, 0)$;

- it allows to describe non-trivial tail dependence coefficients along both the diagonals of the unit square.

The idea of such a construction was first introduced by Jwaid and De Baets [11] under the name *biconic copulas* (or *double conic copulas*); then it has been investigated in [13] (see also [14]), where different other aggregation functions of the same type have been considered.

The results presented here complement the previous cited investigations. Moreover, the description of the class of biconic copulas is obtained by using a recent characterization of copulas by means of Dini derivatives [5].

2. THE CLASS OF BICONIC COPULAS

Let $\delta\colon \mathbb{I} \to \mathbb{I}$ be a *diagonal*, i.e., for every $t \in \mathbb{I}$ $\delta(t) = C(t,t)$ for some copula C. As known [21], any diagonal δ is characterized by the following analytical properties: (a) $\delta(t)$ increasing, (b) $W(t,t) \leq \delta(t) \leq t$ for every $t \in \mathbb{I}$, (c) δ is 2–Lipschitz.

Starting with a diagonal δ, we are interested in considering the functions $F_\delta\colon \mathbb{I}^2 \to \mathbb{I}$ such that F_δ is linear on the segments connecting any point $(x_0, x_0) \in \mathbb{I}^2$ with the corner points $(1,0)$ and $(0,1)$.

Such a F_δ is expressed in the following way:

$$F_\delta(x,y) = \begin{cases} \delta\left(\dfrac{y}{y+1-x}\right)(y+1-x)\,, & x \geq y, \\ \delta\left(\dfrac{x}{x+1-y}\right)(x+1-y)\,, & x < y, \end{cases} \tag{1}$$

with the convention $\delta\left(\frac{0}{0}\right) := 0$. In a short form,

$$F_\delta(x,y) = \delta\left(\tfrac{x\wedge y}{x\wedge y+1-x\vee y}\right)(x\wedge y + 1 - x\vee y)\,.$$

In fact, for every $(x_0, y_0) \in \,]0,1[^2$ and for every $s \in [x_0, 1]$, we have

$$F_\delta\left(s, \frac{x_0}{1-x_0}(1-s)\right) = \delta\left(x_0\right)\cdot\frac{1-s}{1-x_0},$$

i. e., for $s \in [x_0, 1]$ the mapping $s \mapsto F_\delta\big(s, \frac{x_0}{1-x_0}(1-s)\big)$ is an affine function. Analogously, it can be shown that, for every $s \in \big[0, x_0\big]$, $s \mapsto F_\delta\big(s, \frac{x_0-1}{x_0}s+1\big)$ is an affine function.

In the following, we denote by $\mathcal{C}_{\mathrm{BC}}$ the class of all copulas of type (1) (i. e., biconic copulas). We are interested in the characterization of the diagonals δ such that $F_\delta \in \mathcal{C}_{\mathrm{BC}}$.

First, notice that the basic copulas W and M belong to $\mathcal{C}_{\mathrm{BC}}$. On the contrary, the independence copula $\Pi(x,y) = xy$ does not belong to this class. Moreover, as known [20], there exists a diagonal δ such that F_δ is not a copula.

The characterization of the elements of $\mathcal{C}_{\mathrm{BC}}$ is given by the following result, which has been proved for the first time in [13]. A different proof will be presented in Section 3.

Theorem 2.1. Let δ be a diagonal. A function F_δ of type (1) belongs to $\mathcal{C}_{\mathrm{BC}}$ if, and only if, δ is convex.

We denote by $\mathcal{D}_{BC}$ be the class of all diagonals generating a copula in $\mathcal{C}_{\mathrm{BC}}$. As evident by eq. (1), every element of $\mathcal{C}_{\mathrm{BC}}$ is described by its diagonal section. In particular, properties of $\mathcal{D}_{BC}$ may be translated into properties of $\mathcal{C}_{\mathrm{BC}}$. In fact, the following results can be easily proved.

Proposition 2.2. The class $\mathcal{C}_{\mathrm{BC}}$ is a convex set. Moreover, if F_{δ_1} and F_{δ_2} belong to $\mathcal{C}_{\mathrm{BC}}$, then, for every $\lambda \in \mathbb{I}$, $\lambda F_{\delta_1} + (1-\lambda)F_{\delta_2}$ is the copula of type (1) generated by $\lambda\delta_1 + (1-\lambda)\delta_2$.

Moreover, $\mathcal{C}_{\mathrm{BC}}$ is a closed set with respect to pointwise convergence.

Proposition 2.3. The class $\mathcal{C}_{\mathrm{BC}}$ admits pointwise upper and lower bound given, respectively, by M and W. Moreover, if $\delta_1, \delta_2 \in \mathcal{D}_{BC}$ and $\delta_1 \le \delta_2$, then $F_{\delta_1} \le F_{\delta_2}$.

Copulas constructed from diagonal sections are often useful when one wants to have a family of copulas with a variety of tail dependence coefficients. In particular, to every copula C four tail dependence coefficients [27] can be associated in order to describe the tail behavior of C in each of the four corners of $\mathbb{I}^2$. For a copula $F_\delta \in \mathcal{C}_{\mathrm{BC}}$, the tail dependence coefficients assume the following values

$$\lambda_{U,U} = \delta'(1^-), \quad \lambda_{L,L} = \delta'(0^+), \quad \lambda_{U,L} = \lambda_{L,U} = 1 - 2\delta\left(\frac{1}{2}\right).$$

Example 2.4. Let us consider the family $\{\delta_\alpha\}_{\alpha\in[1,2]}$ of diagonals in $\mathcal{D}_{BC}$ given by $\delta_\alpha(t) = t^\alpha$. Then, the copula F_δ generated via (1) is given by

$$F_\delta(x,y) = (x \wedge y)^\alpha \, (x \wedge y + 1 - x \vee y)^{1-\alpha}.$$

The four tail dependence coefficients associated with F_δ are given by

$$\lambda_{U,U} = \alpha, \quad \lambda_{L,L} = 0, \quad \lambda_{U,L} = \lambda_{L,U} = 1 - 2\left(\frac{1}{2}\right)^\alpha.$$

When $\delta\left(\frac{1}{2}\right) < \frac{1}{2}$, every $F_\delta \in \mathcal{C}_{\mathrm{BC}}$ has non-zero tail dependence coefficients along the opposite diagonal section of $\mathbb{I}^2$. Moreover, a suitable choice of the diagonal δ allows to obtain copulas such that all the four tail dependence coefficients are non-zero.

At the end of this section, we are interested in the distribution of the probability mass of a copula $F_\delta \in \mathcal{C}_{\mathrm{BC}}$.

A copula F_δ may have a singular component along the main diagonal of the unit square. In fact, the right derivatives $\frac{\partial F_\delta}{\partial x}$ and $\frac{\partial F_\delta}{\partial y}$ may have jump discontinuities along the line $y = x$ (compare with [10, Theorem 1.1]). Specifically, the following result holds.

Proposition 2.5. No absolutely continuous copula belongs to $\mathcal{C}_{\mathrm{BC}}$.

Proof. Let $F_\delta \in \mathcal{C}_{\mathrm{BC}}$. By using [10, Theorem 1.1], the probability mass of F_δ concentrated along the segment joining $(0,0)$ with (a,a) is given by

$$\int_0^a \left((1-2t)\,\delta'(t) + 2\delta(t)\right) \mathrm{d}t,$$

where δ' is intended to be the right side (or left side) derivative of δ. If F_δ is absolutely continuous, then $\delta(a) = 0$ for every $a \in [0, 1/2]$. But, the unique copula $F_\delta \in \mathcal{C}_{\mathrm{BC}}$ satisfying this property is W, which is singular. Therefore, the assertion follows. □

The fact that copulas in $\mathcal{C}_{\mathrm{BC}}$ have a singular component is a property often encountered in the study of copulas with given diagonal section. Such a singular component often causes analytical difficulties; nevertheless, it can be useful for some specific applications, like default models. Suppose, for instance, that X and Y are jointly connected by a copula $F_\delta \in \mathcal{C}_{\mathrm{BC}}$, $F_\delta \neq W$. Then $X = Y$ with non-zero probability. Translated into the language of default modeling this implies non-zero probability of the occurrence of joint defaults of X and Y (for more details on this point of view, see [18]).

3. PROOF OF THEOREM 2.1

In this section, we present the proof of Theorem 2.1. It requires a recent characterization of bivariate copulas [5], which uses the concept of Dini derivative. Therefore, it is necessary to recall such results.

Definition 3.1. Let $a, b \in \mathbb{R}$, $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be a function. Let x be a point in $[a, b)$. The limit

$$D^+ f(x) = \limsup_{h \to 0^+} \frac{f(x+h) - f(x)}{h},$$

is called *right side upper Dini derivative* of f at x.

Lemma 3.2. (Durante and Jaworski [5, Theorem 2.3]) A function $C\colon \mathbb{I}^2 \to \mathbb{I}$ is a copula if, and only if, C satisfies the following conditions:

(C1) $C(x, 0) = C(0, x) = 0$ for every $x \in \mathbb{I}$,

(C2) $C(x, 1) = C(1, x) = x$ for every $x \in \mathbb{I}$,

(C3) C is continuous;

(C4) there exists a countable set $\mathcal{Z} \subset \mathbb{I}$ such that, for every $x \in \mathbb{I} \setminus \mathcal{Z}$,

(i) $D^+ C_y(x)$ is finite for every $y \in \mathbb{I}$,

(ii) $D^+ C_z(x) \geq D^+ C_y(x)$ whenever $0 \leq y < z \leq 1$,

where $C_y(x) = C(x, y)$ denotes the horizontal section of C at y.

By applying the above result to $\mathcal{C}_{\mathrm{BC}}$ and by doing some standard calculations, the following statement can be obtained.

Lemma 3.3. Let δ be a diagonal. A function F_δ of type (1) belongs to $\mathcal{C}_{\mathrm{BC}}$ if, and only if, for all $x \in \mathbb{I} \setminus \mathcal{Z}$, where $\mathcal{Z}$ is a countable set, $h_x \colon \mathbb{I} \to \mathbb{R}$ given by

$$h_x(t) = \begin{cases} tD^+\delta(t) - \delta(t), & 0 \le t \le x, \\ (1-t)\, D^+\delta(t) + \delta(t), & x < t < 1, \end{cases} \tag{2}$$

is increasing on $\mathbb{I}$.

Proof. Let $F_\delta = F$ be a function of type (1). It is trivial to check that F satisfies (C1), (C2) and (C3). Moreover, for every $x \in \mathbb{I} \setminus \{0,1\}$, $D^+F_y(x)$ is finite for every $y \in \mathbb{I}$, and we have

$$D^+F_y(x) = \begin{cases} \dfrac{y}{y+1-x} D^+\delta\left(\dfrac{y}{y+1-x}\right) - \delta\left(\dfrac{y}{y+1-x}\right), & y \le x, \\ \left(1 - \dfrac{x}{x+1-y}\right) D^+\delta\left(\dfrac{x}{x+1-y}\right) + \delta\left(\dfrac{x}{x+1-y}\right), & y > x. \end{cases}$$

Now, the proof follows from the fact that condition (C4)(ii) is equivalent to the fact that, for all $x \in \mathbb{I} \setminus \mathcal{Z}$, where $\mathcal{Z}$ is a countable set, h_x is increasing on $\mathbb{I}$. $\square$

Remark 3.4. If, for all $x \in \mathbb{I} \setminus \mathcal{Z}$, h_x given by (2) is increasing on $\mathbb{I}$, then it follows that

$$f(t) = (1-t)\, D^+\delta(t) + \delta(t), \tag{3}$$

$$g(t) = tD^+\delta(t) - \delta(t), \tag{4}$$

are increasing on $\mathbb{I}$.

Lemma 3.5. Let δ be a convex diagonal. Then:

(a) f and g given by (3) and (4), respectively, are increasing on $\mathbb{I}$;

(b) $m(t) = (1-2t)\delta'(t) + 2\delta(t) \ge 0$ for all $t \in \mathbb{I}$, with the possible exception of a countable set of points.

Proof. If δ is convex, then $t \mapsto D^+\delta(t)$ is increasing (see [26, Theorem 14.10]). It follows that, for all $t_1, t_2 \in \mathbb{I}$, $t_1 < t_2$,

$$\begin{aligned} f(t_2) - f(t_1) &\ge (1-t_2)D^+\delta(t_2) - (1-t_1)D^+\delta(t_1) + (t_2 - t_1)D^+\delta(t_1) \\ &= (1-t_2)\left(D^+\delta(t_2) - D^+\delta(t_1)\right) \ge 0, \end{aligned}$$

which implies that f is increasing.

Analogously, for all $t_1, t_2 \in \mathbb{I}$, $t_1 < t_2$,

$$\begin{aligned} g(t_2) - g(t_1) &= t_2 D^+\delta(t_2) - \delta(t_2) - t_1 D^+\delta(t_1) + \delta(t_1) \\ &\ge t_2 D^+\delta(t_2) - t_1 D^+\delta(t_1) - (t_2 - t_1)\, D^+\delta(t_2) \ge 0. \end{aligned}$$

Finally, we have to show that $m(t) \geq 0$ for all $t \in \mathbb{I}$ such that $\delta'(t)$ is well-defined. Let $t \in \mathbb{I}$, $t \neq \frac{1}{2}$. Then the function $t \mapsto \frac{\delta(t)}{1-2t}$ is increasing since it is the product of two increasing functions. Therefore, $D^+\left(\frac{\delta(t)}{1-2t}\right) \geq 0$, which is equivalent to $m(t) \geq 0$ on $[0, 1/2)$. Analogously, we can prove that $m(t) \geq 0$ on $(1/2, 1]$. □

Finally, we need the following result, which is related to [26, Theorem 14.10].

Lemma 3.6. Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$. If f is absolutely continuous and D^+f is increasing, then f is convex.

Thanks to all the previous lemmata, we can now prove our characterization.

Proof. [Proof of Theorem 2.1] Let δ is convex. By Lemma 3.3 we have to verify that h_x is increasing. We distinguish three different cases. If $0 \leq t_1 < t_2 \leq x$, then

$$h_x(t_1) = g(t_1) \leq g(t_2) = h_x(t_2).$$

If $x < t_1 < t_2 \leq 1$, then

$$h_x(t_1) = f(t_1) \leq f(t_2) = h_x(t_2).$$

Finally, since $m(t) \geq 0$ it follows that, for $t_1 \leq x < t_2$,

$$h_x(t_1) = g(t_1) \leq f(t_2) = h_x(t_2).$$

By Lemma 3.3 we can conclude that F_δ is a copula.

On the other hand, if F_δ is a copula, Lemma 3.3 ensures that f and g are increasing and, hence, the sum $f + g = D^+\delta$ is increasing too. Since δ is absolutely continuous, then it is convex (see Lemma 3.6). □

4. CONCLUDING REMARKS

We have studied the class of biconic copulas, which depend on their diagonal section and are linear on some specific segments of the unit square.

The copulas F_δ of type (1) are characterized by the fact that their diagonal section is convex. Actually, the convexity of δ characterizes several constructions of copulas with given diagonal section. For example, we can consider quasi–homogeneous copulas [19], and the copulas of type

$$D_\delta(x, y) = \min\left\{x, y, \delta\left(\frac{x+y}{2}\right)\right\}$$

introduced in [4].

Furthermore, by using diagonal patchwork constructions [6, 8, 22], all these copulas can be glued together in order to obtain new copulas with a prescribed diagonal section.

Inspired by the idea of biconic copulas [11], Sempi suggested the following problem (see [20, Problem 2.4]).

Given a fixed $x_0 \in]0,1[$ characterize all $\delta\colon \mathbb{I} \to \mathbb{I}$ such that there exists a copula C with $C(x,x) = \delta(x)$ for all $x \in \mathbb{I}$, and C is linear on the segments connecting the diagonal point (x_0, x_0) with the corner points $(1,0)$ and $(0,1)$. Also, try to construct pointwise maximal and minimal elements of this class.

It should be mentioned that, while the copulas we have considered obviously satisfy the Sempi's requirements, the solution to the previous problem gives a larger class of copula, as the following example shows.

Example 4.1. Let $x_0 \in]0,1[$. Let C be the ordinal sum of (Π,Π) with respect to the partition $([0,x_0],[x_0,1])$. Then C is a possible solution to Sempi's problem; but C does not belong to $\mathcal{C}_{\mathrm{BC}}$.

Finally, notice that pointwise maximal and minimal elements of the copulas from previous problem coincide with M and W, respectively, which are copulas in $\mathcal{C}_{\mathrm{BC}}$.

ACKNOWLEDGEMENT

We wish to thank Prof. Bernard De Baets who generously sent us his ongoing research papers before the publication. We also thank the Reviewers for careful reading and useful suggestions.

The first author acknowledges the support of Free University of Bozen–Bolzano (Italy), School of Economics and Management, via the project "Stochastic models for lifetimes".

(Received December 6, 2010)

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