MAC NEILLE COMPLETION OF CENTERS AND CENTERS OF MAC NEILLE COMPLETIONS OF LATTICE EFFECT ALGEBRAS

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If element $z$ of a lattice effect algebra $(E, \oplus, 0, 1)$ is central, then the interval $[0, z]$ is a lattice effect algebra with the new top element $z$ and with inherited partial binary operation $\oplus$. It is a known fact that if the set $C(E)$ of central elements of $E$ is an atomic Boolean algebra and the supremum of all atoms of $C(E)$ in $E$ equals to the top element of $E$, then $E$ is isomorphic to a subdirect product of irreducible effect algebras ([18]). This means that if there exists a MacNeille completion $\hat{E}$ of $E$ which is its extension (i.e. $E$ is densely embeddable into $\hat{E}$) then it is possible to embed $E$ into a direct product of irreducible effect algebras. Thus $E$ inherits some of the properties of $\hat{E}$. For example, the existence of a state in $\hat{E}$ implies the existence of a state in $E$. In this context, a natural question arises if the MacNeille completion of the center of $E$ (denoted as $\mathcal{MC}(C(E))$) is necessarily the same as the center of $\hat{E}$, i.e., if $\mathcal{MC}(C(E)) = C(\hat{E})$ is necessarily true. We show that the equality is not necessarily fulfilled. We find a necessary condition under which the equality may hold. Moreover, we show also that even the completeness of $C(E)$ and its bifullness in $E$ is not sufficient to guarantee the mentioned equality.

Keywords: lattice effect algebra, center, atom, MacNeille completion

Classification: 03G12, 03G27, 06B99

1. INTRODUCTION AND PRELIMINARIES

Effect algebras, introduced by D.J. Foulis and M.K. Bennett [3], have their importance in the investigation of uncertainty. Lattice ordered effect algebras generalize orthomodular lattices and MV-algebras. Thus they may include non-compatible pairs of elements as well as unsharp elements.

**Definition 1.1.** (Foulis and Bennett [3]) An effect algebra is a system $(E; \oplus, 0, 1)$ consisting of a set $E$ with two different elements $0$ and $1$, called zero and unit, respectively and $\oplus$ is a partially defined binary operation satisfying the following conditions for all $p, q, r \in E$:

(E1) If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.

(E2) If $q \oplus r$ is defined and $p \oplus (q \oplus r)$ is defined, then $p \oplus q$ and $(p \oplus q) \oplus r$ are defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$. 
(E3) For every \( p \in E \) there exists a unique \( q \in E \) such that \( p \oplus q \) is defined and \( p \oplus q = 1 \).

(E4) If \( p \oplus 1 \) is defined then \( p = 0 \).

The element \( q \) in (E3) will be called the \textit{supplement} of \( p \), and will be denoted as \( p' \).

In the whole paper, for an effect algebra \((E, \oplus, 0, 1)\), writing \( a \oplus b \) for arbitrary \( a, b \in E \) will mean that \( a \oplus b \) exists. On an effect algebra \( E \) we may define another partial binary operation \( \ominus \) by

\[
a \ominus b = c \iff b \oplus c = a.
\]

The operation \( \ominus \) induces a partial order on \( E \). Namely, for \( a, b \in E \) \( b \leq a \) if there exists a \( c \in E \) such that \( a \ominus b = c \). If \( E \) with respect to \( \leq \) is lattice ordered, we say that \( E \) is a \textit{lattice effect algebra}. For the sake of brevity we will write just LEA. Further, in this article we often briefly write ‘an effect algebra \( E \)’ skipping the operations.

If every pair \( x, y \) of elements of a LEA \( E \) is \textit{compatible}, meaning that \( x \lor y = (x \ominus y) \oplus (x \land y) \) then \( E \) is called an \textit{MV-effect algebra} \cite{11, 12}.

S. P. Gudder \cite{5, 6} introduced the notion of sharp elements and sharply dominating lattice effect algebras. Recall that an element \( x \) of the LEA \( E \) is called \textit{sharp} if \( x \land x' = 0 \). Jenča and Riečanová in \cite{7} proved that in every lattice effect algebra \( E \) the set \( S(E) = \{x \in E; x \land x' = 0\} \) of sharp elements is an orthomodular lattice which is a \textit{sub-effect algebra} of \( E \), meaning that if among \( x, y, z \in E \) with \( x \ominus y = z \) at least two elements are in \( S(E) \) then \( x, y, z \in S(E) \). Moreover \( S(E) \) is a \textit{full sublattice} of \( E \), hence supremum of any set of sharp elements, which exists in \( E \), is again a sharp element. Further, each maximal subset \( M \) of pairwise compatible elements of \( E \), called \textit{block} of \( E \), is a sub-effect algebra and a full sublattice of \( E \) and \( E = \bigcup \{M \subseteq E; M \text{ is a block of } E\} \) (see \cite{15, 16}). \textit{Central elements} and centers of effect algebras were defined in \cite{11}. In \cite{13, 14} it was proved that in every lattice effect algebra \( E \) the center

\[
C(E) = \{x \in E; (\forall y \in E)y = (y \land x) \lor (y \land x')\} = S(E) \cap B(E),
\]

where \( B(E) = \bigcap \{M \subseteq E; M \text{ is a block of } E\} \). Since \( S(E) \) is an orthomodular lattice and \( B(E) \) is an MV-effect algebra, we obtain that \( C(E) \) is a Boolean algebra. Note that \( E \) is an orthomodular lattice if and only if \( E = S(E) \) and \( E \) is an MV-effect algebra if and only if \( E = B(E) \). Thus \( E \) is a Boolean algebra if and only if \( E = S(E) = B(E) = C(E) \).

Recall that an element \( p \) of an effect algebra \( E \) is called an \textit{atom} if and only if \( p \) is a minimal non-zero element of \( E \) and \( E \) is \textit{atomic} if for each \( x \in E \), \( x \neq 0 \), there exists an atom \( p \leq x \).

**Definition 1.2.** Let \((E, \oplus, 0)\) be an effect algebra. To each \( a \in E \) we define its \textit{isotropic index}, notation \( \text{ord}(a) \), as the maximal positive integer \( n \) such that

\[
na := a \oplus \cdots \oplus a
\]

\text{\textit{n-times}}.
exists. We set \( \text{ord}(a) = \infty \) if \( na \) exists for each positive integer \( n \). We say that \( E \) is Archimedean, if for each \( a \in E, a \neq 0, \text{ord}(a) \) is finite.

An element \( u \in E \) is called finite, if there exists a finite system of atoms \( a_1, \ldots, a_n \) (which are not necessarily distinct) such that \( u = a_1 \oplus \cdots \oplus a_n \). An element \( v \in E \) is called cofinite, if there exists a finite element \( u \in E \) such that \( v = u' \).

We say that for a finite system \( F = (x_j)_{j=1}^k \) of not necessarily different elements of an effect algebra \( (E, \oplus, 0, 1) \) is \( \oplus \)-orthogonal if \( x_1 \oplus x_2 \oplus \cdots \oplus x_n = (x_1 \oplus x_2 \oplus \cdots \oplus x_{n-1}) \oplus x_n \) exists in \( E \) (briefly we will write \( \bigoplus_{j=1}^n x_j \)). We define also \( \oplus \emptyset = 0 \).

**Definition 1.3.** For a lattice \( (L, \land, \lor) \) and a subset \( D \subseteq L \) we say that \( D \) is a bifull sublattice of \( L \), if and only if for any \( X \subseteq D, \bigvee_L X \) exists if and only if \( \bigvee_D X \) exists and \( \bigwedge_L X \) exists if and only if \( \bigwedge_D X \) exists, in which case \( \bigvee_L X = \bigvee_D X \) and \( \bigwedge_L X = \bigwedge_D X \).

Recall that an element \( a \in L \), where \( (L, \land, \lor) \) is a lattice, is called a compact element if for arbitrary \( D \subseteq L \) with \( \bigvee D \in L \), if \( a \leq \bigvee D \) then \( a \leq \bigvee F \) for some finite set \( F \subseteq D \). The lattice \( L \) is called compactly generated if every element of \( L \) is a join of compact elements.

**Lemma 1.4.** Let \( (E, \oplus, \lor, \land, 0, 1) \) be an atomic Archimedean lattice effect algebra. Then

(i) (see \[10\]) a block \( M \) of \( E \) is atomic if there exists a maximal pairwise compatible set \( A \) of atoms of \( E \) such that \( A \subseteq M \) and if \( M_1 \) is a block of \( E \) with \( A \subseteq M_1 \), then \( M_1 = M \). Moreover for all \( x \in E \) and all \( a \in A \) the following holds

\[
x \in M \iff x \leftrightarrow a,
\]

(ii) (see \[17\]) to every nonzero element \( x \in E \) there exist mutually distinct atoms \( a_\alpha \in E \) and positive integers \( k_\alpha \) for \( \alpha \in I \) such that

\[
x = \bigoplus_{\alpha \in I} (k_\alpha a_\alpha) = \bigvee_{\alpha \in I} (k_\alpha a_\alpha).
\]

It is known that if \( E \) is a distributive effect algebra (i.e., the effect algebra \( E \) is a distributive lattice – e.g., if \( E \) is an MV-effect algebra) then \( C(E) = S(E) \). If moreover \( E \) is Archimedean and atomic then the set of atoms of \( C(E) = S(E) \) is the set \( \{ n_\alpha a; a \in E \text{ is an atom of } E \} \), where \( n_\alpha = \text{ord}(a) \) (see \[19\]). Since \( S(E) \) is a bifull sublattice of \( E \) if \( E \) is an Archimedean atomic LEA (see \[12\]), we obtain that

\[
1 = \bigvee_{C(E)} \{ p \in C(E); p \text{ is an atom of } C(E) \} = \bigvee_{E} \{ p \in C(E); p \text{ is an atom of } C(E) \}
\]

for every Archimedean atomic distributive lattice effect algebra \( E \). In \[8\] it was shown that there exists a LEA \( E \) for which this property fails to be true. Important properties of Archimedean atomic lattice effect algebras with atomic center were proven by Riečanová in \[20\].
Theorem 1.5. (Riečanová [20]) Let $E$ be an Archimedean atomic lattice effect algebras with atomic center $C(E)$. Denote by $A_E$ the set of all atoms of $E$ and by $A_{C(E)}$ the set of all atoms of $C(E)$. The following conditions are equivalent:

1. $\bigvee_E A_{C(E)} = 1$.
2. For every atom $a \in A_E$ there exists an atom $p_a \in A_{C(E)}$ such that $a \leq p_a$.
3. For every $z \in C(E)$ it holds
   \[ z = \bigvee_{C(E)} \{p \in A_{C(E)}; p \leq z\} = \bigvee_{E} \{p \in A_{C(E)}; p \leq z\} . \]
4. $C(E)$ is a bifull sub-lattice of $E$.

In this case $E$ is isomorphic to a subdirect product of Archimedean atomic irreducible lattice effect algebras.

2. MACNEILLE COMPLETION OF A LEA $E$ WHOSE CENTER IS NOT BIFULL IN $E$

This section is based on an example published by the author in [8]. For reader’s comfort in Section 2.1 we repeat the substantial parts of this paper where the LEA $E$ whose center is not bifull in $E$, is constructed. In Section 2.2 we make the completion of $E$.

2.1. Construction of a LEA $E$ whose center is not bifull in $E$

Let us have the following sequences of elements (sets):

- $a_0 = \{(x, y) \in \mathbb{R}^2; 0 \leq x \leq 1, y \in \mathbb{R}\}$
- $a_l = \{(x, y) \in \mathbb{R}^2; l < x \leq l + 1, y \in \mathbb{R}\}$, for $l = 1, 2, \ldots$
- $b_0 = \{(x, y) \in \mathbb{R}^2; -1 \leq x < 0, y \in \mathbb{R}\}$
- $b_l = \{(x, y) \in \mathbb{R}^2; -l - 1 \leq x < -l, y \in \mathbb{R}\}$, for $l = 1, 2, \ldots$ (2)
- $c_j = \{(x, y) \in \mathbb{R}^2; -j \leq x \leq j, y \leq j \cdot x\}$, for $j = 1, 2, \ldots$
- $d_j = \{(x, y) \in \mathbb{R}^2; -j \leq x \leq j, y > j \cdot x\}$, for $j = 1, 2, \ldots$
- $p_j = \{j\}$, for $j = 1, 2, \ldots$

For such a choice of elements, the elements $q_1 \neq q_2$ are compatible if and only if $q_1 \cap q_2 = \emptyset$.

Denote $\hat{B}_0$, $\hat{B}_j$ (for $j = 1, 2, \ldots$) complete atomic Boolean algebras with the corresponding sets of atoms $A_0$, $A_j$ ($j = 1, 2, \ldots$), given by

\begin{align*}
A_0 & = \bigcup_{i=0}^{\infty} \{a_i\} \cup \bigcup_{i=0}^{\infty} \{b_i\} \cup \bigcup_{j=1}^{\infty} \{p_j\}, \\
A_j & = \bigcup_{i=j}^{\infty} \{a_i\} \cup \bigcup_{i=j}^{\infty} \{b_i\} \cup \bigcup_{j=1}^{\infty} \{p_j\} \cup \{c_j, d_j\}.
\end{align*}
Disjointness among some elements of the system (2) is equivalent with the fact that $A_0$ and $A_j$ ($j = 1, 2, \ldots$) are unique maximal sets of pairwise compatible atoms.

For elements $u_1, u_2 \in \hat{B}_l$, $l = 0, 1, 2, \ldots$, such that $u_1 \cap u_2 = \emptyset$ we introduce the partial operation $\oplus_l$ by

$$u_1 \oplus_l u_2 = u_1 \cup u_2.$$  \hfill (5)

Observe that if $u_1, u_2 \in \hat{B}_i \cap \hat{B}_j$, then

$$u_1 \oplus_i u_2 = u_1 \oplus_j u_2.$$  \hfill (6)

This is the reason why we will omit the index denoting operation $\oplus$ in the whole paper. Moreover we have the following equality

$$c_j \oplus d_j = \bigoplus_{i=0}^{j-1} (a_i \oplus b_i) = \{(x, y) \in \mathbb{R}^2; -j \leq x \leq j\}, \quad \text{for all } j = 1, 2, \ldots. \hfill (7)$$

The complete Boolean algebras $\hat{B}_0, \hat{B}_j$, $j = 1, 2, \ldots$, have the following top elements:

$$\mathbb{R}^2 \cup \mathbb{N} = 1 = 1_0 = a_0 \oplus b_0 \oplus \bigoplus_{i=1}^{\infty} (a_i \oplus b_i \oplus p_i)$$  \hfill (8)

$$\mathbb{R}^2 \cup \mathbb{N} = 1 = 1_1 = (c_1 \oplus d_1) \oplus \bigoplus_{i=1}^{\infty} (a_i \oplus b_i \oplus p_i)$$  \hfill (9)

$$\mathbb{R}^2 \cup \mathbb{N} = 1 = 1_j = (c_j \oplus d_j) \oplus \bigoplus_{i=1}^{\infty} (a_i \oplus b_i \oplus p_i) \oplus \bigoplus_{i=1}^{j-1} p_i,$$  \hfill (10)

for all $j = 2, 3, \ldots$.

An element $u \in \hat{B}_l$ is finite if and only if $u = q_1 \oplus q_2 \oplus \cdots \oplus q_n$ for an $n \in \mathbb{N}$ and $q_1, q_2, \ldots, q_n \in A_l$. Set $Q_l = \{u \in B_l; u \text{ is finite}\}$, $l = 0, 1, 2, \ldots$. Then $Q_l$ is a generalized Boolean algebra, since $\hat{B}_l = Q_l \cup \hat{B}_l^\dag$ is a Boolean algebra, where
Fig. 2. Illustration of the element $a_3 \oplus b_3 \oplus c_3 \oplus d_3$.

$Q_i^* = \{ u^*; u^* = 1_l \ominus u \text{ and } u \in Q_l \}$ (see [21], or [2, pp. 18-19]). This means that $B_l$ is a Boolean subalgebra of finite and cofinite elements of $\hat{B}_l$ ($l = 0, 1, 2, \ldots$).

**Theorem 2.1.** (Kalina [8]) Denote $E = \bigcup_{l=0}^{\infty} B_l$. Then $(E, \oplus, \vee, \wedge, 0, 1)$ is a compactly generated LEA with the family $(B_l)_{l=0}^{\infty}$ of atomic blocks of $E$. The center of $E$, $C(E)$, is not a bifull sublattice of $E$.

### 2.2. MacNeille completion of $E$

Let us denote

$$\hat{E} = \bigcup_{l=0}^{\infty} \hat{B}_l. \quad (11)$$

First we show the following lemma.

**Lemma 2.2.** $(\hat{E}, \oplus, \wedge, \vee, 0, 1)$ is a lattice effect algebra.

**Proof.** Equation (11) shows that $\oplus$ is well defined. We show that this operation is commutative and associative. Let $q_1, q_2, q_3 \in \hat{E}$ are elements such that $q_1 \oplus q_2$ is defined and $(q_1 \oplus q_2) \oplus q_3$ is also defined. Then $q_1, q_2$ are disjoint sets and $(q_1 \oplus q_2)$ and $q_3$ are also disjoint sets. These imply that $q_1, q_2, q_3$ is a triple of pairwise disjoint sets and hence the commutativity and associativity follows immediately. Following $(\hat{E}, \oplus)$ is an effect algebra.

We show now that $(\hat{E}, \wedge, \vee, 0, 1)$ is a bounded lattice.

Let $h_1, h_2 \in \hat{E}$ be arbitrary elements. First assume that $h_1 \leftrightarrow h_2$. Then there is an $i \in \{0, 1, 2, \ldots\}$ such that $h_1 \in \hat{B}_i$, $h_2 \in \hat{B}_i$. Since $\hat{B}_i$ is a complete Boolean algebra, $h_1 \vee h_2$ and $h_1 \wedge h_2$ are well defined.
Assume that \( h_1 \not\rightarrow h_2 \). Then there are some \( 0 \leq i < s \) such that \( h_1 \in \hat{B}_i \) and \( h_2 \in \hat{B}_s \). This means that for \( h_1 \) and \( h_2 \) we have

\[
    h_1 = \bigoplus_{i=0}^{\infty} (\alpha_i a_i + \beta_i b_i) + \bigoplus_{j=1}^{\infty} \pi_j p_j, \quad \text{if } i = 0,
\]

\[
    h_2 = \bigoplus_{i=0}^{\infty} (\alpha'_i a'_i + \beta'_i b'_i) + \bigoplus_{j=1}^{\infty} \pi'_j p'_j, \quad \text{if } i = 0,
\]

where \( \alpha_i, \beta_i, \gamma_i, \delta_i, \pi_j \in \{0, 1\} \) for \( l = 0, 1, 2, \ldots, i = 1, 2, \ldots \) and \( j = 1, 2, \ldots, \alpha'_i, \beta'_i, \gamma'_i, \delta'_i, \pi'_j \in \{0, 1\} \) for \( l = 1, 2, \ldots, s = 1, 2, \ldots \) and \( j = 1, 2, \ldots \). Because of formula (7) and the non-compatibility of \( h_1 \) and \( h_2 \), if we denote by \( \Gamma_i \) all atoms of \( A_i \) which are non-compatible with \( c_s \) (or equivalently, which are non-compatible with \( d_s \)), for \( h_1 \) we get that there exists a \( q \in \Gamma_i \) such that \( q \leq h_1 \) and at the same time

\[
    \bigoplus_{l=0}^{s-1} (a_l \oplus b_l) \not\leq h_1, \quad \text{if } i = 0,
\]

\[
    c_i \oplus d_i \oplus \bigoplus_{l=1}^{s-1} (a_l \oplus b_l) \not\leq h_1, \quad \text{if } i \neq 0.
\]

For \( h_2 \) we get that either \( c_s \leq h_2 \) or \( d_s \leq h_2 \), and \( c_s \oplus d_s \not\leq h_2 \).

In all other cases we would get the compatibility of \( h_1 \) and \( h_2 \). Hence we have

\[
    h_1 \wedge h_2 = \bigoplus_{l=s}^{\infty} (\hat{\alpha}_l a_l \oplus \hat{\beta}_l b_l) \oplus \bigoplus_{m=1}^{\infty} \hat{\pi}_m p_m,
\]

\[
    h_1 \vee h_2 = c_s \oplus d_s \oplus \bigoplus_{l=s}^{\infty} (\hat{\alpha}_l a_l \oplus \hat{\beta}_l b_l) \oplus \bigoplus_{m=1}^{\infty} \hat{\pi}_m p_m
\]

\[
= \bigoplus_{l=0}^{s-1} (a_l \oplus b_l) \oplus \bigoplus_{l=s}^{\infty} (\hat{\alpha}_l a_l \oplus \hat{\beta}_l b_l) \oplus \bigoplus_{m=1}^{\infty} \hat{\pi}_m p_m,
\]

where \( \hat{\alpha}_l = \min\{\alpha_i, \alpha'_i\} \), \( \hat{\beta}_l = \min\{\beta_i, \beta'_i\} \), \( \hat{\alpha}_l = \max\{\alpha_i, \alpha'_i\} \), \( \hat{\beta}_l = \max\{\beta_i, \beta'_i\} \) for \( l \in \{s, 2s + 1, \ldots\} \), and \( \hat{\pi}_m = \min\{\pi_m, \pi'_m\} \), \( \hat{\pi}_m = \max\{\pi_m, \pi'_m\} \) for \( m \in \{1, 2, \ldots\} \).

The fact that \( (\hat{E}, \oplus, \wedge, \vee, 0, 1) \) is a LEA is due to formulas (14) and (15).

In what follows we will denote the LEA \( (\hat{E}, \oplus, \wedge, \vee, 0, 1) \) just briefly as \( \hat{E} \).

**Theorem 2.3.** \( \hat{E} \) is a complete lattice.

**Proof.** Since \( \hat{E} \) is the union of countably many blocks \( \hat{B}_i \) and each block \( \hat{B}_i \) is a complete Boolean algebra, it is enough to show that \( \hat{E} \) is a \( \sigma \)-complete lattice. Each element \( q \in \hat{E} \) has its supplement, hence we show just the \( \sigma \)-completeness with respect to \( \vee \). Assume that \( (h_{k_i})_{i=1} \) be a sequence of pairwise non-compatible
elements of $\hat{E}$, where $h_{k_i} \in \hat{B}_{k_i}$ and $(k_i)_{i=1}^\infty$ is an increasing sequence of non-negative integers. Then the element $h_{k_1}$ can be expressed by formula (12) replacing $i$ by $k_1$, and $h_{k_i}$ (for $i > 1$) can be expressed by formula (13) replacing $s$ by $k_i$. Then by formula (15) we have that

$$\bigvee_{i=1}^t h_{k_i} = c_{k_t} \oplus d_{k_t} \oplus \bigoplus_{j=k_t}^\infty (\hat{\alpha}_j a_j \oplus \hat{\beta}_j b_j) \oplus \bigoplus_{m=1}^\infty \hat{\pi}_m p_m,$$

where

$$\hat{\alpha}_j = \begin{cases} 1, & \text{if } a_j \leq h_{k_i} \text{ for an } 1 \leq i \leq t, \\ 0, & \text{otherwise}, \end{cases}$$

$$\hat{\beta}_j = \begin{cases} 1, & \text{if } b_j \leq h_{k_i} \text{ for an } 1 \leq i \leq t, \\ 0, & \text{otherwise}, \end{cases}$$

$$\hat{\pi}_j = \begin{cases} 1, & \text{if } p_j \leq h_{k_i} \text{ for an } 1 \leq i \leq t, \\ 0, & \text{otherwise}. \end{cases}$$

Formulas (2) imply

$$\bigvee_{i=1}^t (c_{k_i} \oplus d_{k_i}) = \mathbb{R}^2$$

which gives

$$\bigvee_{i=1}^\infty h_{k_i} = \mathbb{R}^2 \oplus \bigoplus_{m=1}^\infty \hat{\pi}_m p_m, \quad \text{where} \quad \hat{\pi}_j = \begin{cases} 1, & \text{if } p_j \leq h_{k_i} \text{ for an } 1 \leq i \leq t, \\ 0, & \text{otherwise}. \end{cases}$$

This completes the proof that $\hat{E}$ is a complete lattice. \hfill \Box

**Theorem 2.4.** The atomic Archimedean LEA $E = \bigcup_{l=0}^\infty B_l$ can be densely embedded into $\hat{E} = \bigcup_{l=0}^\infty \hat{B}_l$.

**Proof.** Since each of the atomic complete Boolean algebras $\hat{B}_l$, for $l = 0, 1, 2, \ldots$, is generated by countably many atoms, the completeness of each particular $\hat{B}_l$ is equivalent with its $\sigma$-completeness. Further, the atomic Boolean algebras $B_l$ contain all finite elements of $\hat{B}_l$. This implies that each $B_l$ can be densely embedded into $\hat{B}_l$. Hence we have that $E = \bigcup_{l=0}^\infty B_l$ can be densely embedded into $\hat{E} = \bigcup_{l=0}^\infty \hat{B}_l$, and the proof is finished. \hfill \Box

Let us denote by $\tilde{B}_0, \tilde{B}_j$ (for $j = 1, 2, \ldots$) the following complete Boolean algebras generated by corresponding sets of atoms $\tilde{A}_0, \tilde{A}_j$:

$$\tilde{A}_0 = \bigcup_{i=0}^\infty \{a_i\} \cup \bigcup_{l=0}^\infty \{b_l\},$$

$$\tilde{A}_j = \bigcup_{i=j}^\infty \{a_i\} \cup \bigcup_{i=j}^\infty \{b_i\} \cup \{c_j, d_j\}.$$
Further we denote
\[ \hat{E}_1 = \bigcup_{i=0}^{\infty} \hat{B}_i. \]

We can embed \( \hat{E}_1 \) into \( \hat{E} \). In this sense \( \hat{E}_1 \) is equipped with the partial operation \( \oplus \) inherited from \( \hat{E} \).

**Lemma 2.5.** \( \hat{E}_1 \) is a complete atomic Archimedean LEA with its center equal to \( C(\hat{E}_1) = \{0_{E_1}, 1_{E_1}\} \) and \( 1_{E_1} \) is an infinite element.

**Proof.** To show that \( \hat{E}_1 \) is a complete atomic Archimedean LEA we could repeat the proofs of Lemma 2.2 and of Theorem 2.3 just skipping the atoms \( \{p_1, p_2, \ldots\} \) from all formulas.

We show now that \( C(\hat{E}_1) = \{0_{E_1}, 1_{E_1}\} \). Formulas (2) imply that \( 1_{E_1} = \mathbb{R}^2 \).

Assume that there is yet another element of \( C(\hat{E}_1) \). Let us denote this element by \( z \). Assume that no atom from the set of atoms \( \{c_1, d_1, c_2, d_2, \ldots, c_j, d_j, \ldots\} \) is below \( z \). Since \( z \neq 0_{E_1} \), there exists an atom \( a_i \leq z \) (or \( b_i \leq z \)). Then we get that \( c_i+1 \cap z \neq \emptyset \) and \( c_i+1 \not\leq z \) and hence \( c_i+1 \not\in z \). We may conclude that \( z \) is not a central element in this case. Assume that \( c_j \leq z \) (or \( d_j \leq z \)) for some \( j = 1, 2, \ldots \) and there is a \( k \) such that \( (c_k \oplus d_k) \not\leq z \). Then formulas (2) imply that either \( c_k \) or \( d_k \) in non-compatible with \( z \) and followingly \( z \) is not a central element. This consideration gives that if \( z \) is a central element then all atoms from the set of atoms \( \{c_1, d_1, c_2, d_2, \ldots, c_j, d_j, \ldots\} \) are below \( z \). Since
\[ \bigvee_{j=1}^{\infty} (c_j \oplus d_j) = \mathbb{R}^2, \]
we get that \( C(\hat{E}_1) = \{0_{E_1}, 1_{E_1}\} \).

To conclude the proof we have to show that \( 1_{\hat{E}_1} \) is an infinite element of \( \hat{E}_1 \). This is due to the fact that \( 1_{\hat{E}_1} \) is an infinite element of each of the blocks \( \hat{B}_i \).

**Lemma 2.6.** Let us denote by \( \hat{B} \) the complete Boolean algebra generated by the set of atoms \( \{p_1, p_2, \ldots, p_j, \ldots\} \). Then \( \hat{E} \) is isomorphic to the direct product \( \hat{B} \times \hat{E}_1 \).

**Proof.** The isomorphism between \( \hat{E} = \bigcup_{i=0}^{\infty} \hat{B}_i \) and the direct product \( \hat{B} \times \hat{E}_1 \) follows from the fact that each of the blocks \( \hat{B}_i \) is isomorphic to the direct product \( \hat{B} \times \hat{B}_i \).

**Theorem 2.7.** Let \( E = \bigcup_{i=0}^{\infty} B_i \) and \( \hat{E} = \bigcup_{i=0}^{\infty} \hat{B}_i \). Denote \( \mathcal{MC}(C(E)) \) the MacNeille completion of \( C(E) \). Then the following holds
\[ \mathcal{MC}(C(E)) \subseteq C(\hat{E}). \]

**Proof.** Set \( 1_{\hat{E}_1} \) the top element of \( \hat{E}_1 \). Then \( 1_{\hat{E}_1} \in C(\hat{E}) \). Since there is no non-zero central element of \( \hat{E} \) below \( 1_{\hat{E}_1} \), we may conclude that \( 1_{\hat{E}_1} \) is an atom of \( C(\hat{E}) \).
On the other hand, $1_{\hat{E}_1}$ is neither a finite nor a cofinite element of $\hat{E}$ and hence $1_{\hat{E}_1} \notin C(\hat{E})$. Since $1_{\hat{E}_1}$ is an atom of $C(\hat{E})$, we get immediately $1_{\hat{E}_1} \notin MC(C(\hat{E}))$ and the proof of the theorem is finished. \hfill \Box

Theorem 2.7 can be generalized into the following

**Theorem 2.8.** Let $\mathcal{E}$ be an atomic Archimedean LEA with atomic center $C(\mathcal{E})$ that is not a bifull sublattice of $\mathcal{E}$. Let $MC(C(\mathcal{E}))$ be the MacNeille completion of $C(\mathcal{E})$ and $\hat{\mathcal{E}}$ the MacNeille completion of $\mathcal{E}$. Then the following holds

$$MC(C(\mathcal{E})) \subseteq C\left(\hat{\mathcal{E}}\right).$$

**Proof.** Because $C(\mathcal{E})$ is not a bifull sublattice of $\mathcal{E}$, due to Theorem 1.5 we have that

$$\bigvee_{\mathcal{E}} \{q \in C(\mathcal{E}); q \text{ is an atom of } C(\mathcal{E})\}$$

does not exist in $\mathcal{E}$ but

$$\bigvee_{C(\mathcal{E})} \{q \in C(\mathcal{E}); q \text{ is an atom of } C(\mathcal{E})\} = 1$$

Set $z = \left(\bigvee_{\mathcal{E}} \{q \in C(\mathcal{E}); q \text{ is an atom of } C(\mathcal{E})\}\right)'$. Then obviously

$$z \in \hat{\mathcal{E}}$$

holds and at the same time, since there is no non-zero element of $C(\mathcal{E})$ that is below $z$, $z \notin MC(C(\mathcal{E}))$. \hfill \Box

3. SEARCHING FOR A SUFFICIENT CONDITION UNDER WHICH $MC(C(\mathcal{E})) = C\left(\hat{\mathcal{E}}\right)$ HOLDS

Theorem 2.8 gives us a necessary condition under which, for an atomic Archimedean lattice effect algebra $\mathcal{E}$ the equality

$$MC(C(\mathcal{E})) = C\left(\hat{\mathcal{E}}\right)$$

(17)

is valid. Once we have find a necessary condition, it is natural to look for a sufficient condition. We are going to present an example helping us to solve this problem.

Let us take the complete atomic Archimedean LEA $\hat{E}_1$ given by formula 16 and its isomorphic copy denoted by $\hat{E}_2$. Since all atoms of $\hat{E}_1$ are compact elements, the following assertion is straightforward

**Lemma 3.1.** The Archimedean atomic LEA $\hat{E}_1 \times \hat{E}_2$ is compactly generated. Further, its center $C(\hat{E}_1 \times \hat{E}_2)$ has the following elements

$$C\left(\hat{E}_1 \times \hat{E}_2\right) = \{0, 1, 1_{\hat{E}_1}, 1_{\hat{E}_2}\},$$

where $1_{\hat{E}_1}$ and $1_{\hat{E}_2}$ are the top elements of $\hat{E}_1$ and $\hat{E}_2$, respectively.
Let us denote $E_f$ the set of all finite and cofinite elements of $\hat{E}_1 \times \hat{E}_2$.

**Theorem 3.2.** $E_f$ is an atomic Archimedean LEA which is densely embeddable into $\hat{E}_1 \times \hat{E}_2$. The center of $E_f$ is the following

$C(E_f) = \{0, 1\}$.

**Proof.** The fact that $E_f$ is an atomic Archimedean LEA which is densely embeddable into $\hat{E}_1 \times \hat{E}_2$, follows from Lemma 3.1. Since $1_{\hat{E}_1}$ and $1_{\hat{E}_2}$ are neither finite nor cofinite elements of $\hat{E}_1 \times \hat{E}_2$, we have that $C(E_f) = \{0, 1\}$.

Let $\hat{B}$ be an arbitrary atomic Boolean algebra and $q_i$, for $i$ running through an appropriate index set $I$, be atoms of $\hat{B}$. Then, due to Theorem 1.5 $\hat{B}$ is isomorphic with a subdirect product of $\\{0_{\hat{B}}, z_i\}_{i \in I}$.

**Theorem 3.3.** There exists an atomic Archimedean LEA $E_{\hat{B}}$ whose center is isomorphic with $\hat{B}$ and for which equality (17) does not hold.

**Proof.** $\hat{B}$ is a subdirect product of $\\{0_{\hat{B}}, z_i\}$ for $i \in I$. Instead of $\\{0_{\hat{B}}, z_1\}$ we take the atomic Archimedean LEA $E_f$. Then the center of the corresponding subdirect product of $E_f$ and of the system $\\{0_{\hat{B}}, z_i\}$ for $i \in I \setminus \{1\}$ is isomorphic to $\hat{B}$, but due to Lemma 3.1 we have

$\mathcal{MC}(C(E_{\hat{B}})) = \mathcal{MC}(\hat{B}) \subsetneq \mathcal{MC}(E_{\hat{B}})$.

□

4. CONCLUSIONS

In this paper we studied the equality

$\mathcal{MC}(C(E)) = C(\hat{E})$,

where $E$ is an atomic Archimedean LEA and $\hat{E}$ its MacNeille completion. Particularly, we were interested in finding conditions expressible by means of properties of $C(E)$, under which the equality holds. We proved that there exists an atomic Archimedean LEA $E$ for which equality is violated. Further, we proved that the bifullness of the center $C(E)$ in $E$ is necessary for the equality to be true. Moreover we showed that even the completeness of the center and the bifullness of $C(E)$ in $E$ is not sufficient to guarantee the above equality and for an arbitrary atomic Boolean algebra $B$ there exists an atomic Archimedean LEA whose center is equal to $B$ and for which the above equality is not fulfilled.
ACKNOWLEDGEMENT

The support of Science and Technology Assistance Agency under the contract No. APVV-0375-06, and of the VEGA grant agency, grant number 1/0373/08, is kindly acknowledged. The author is grateful to the anonymous reviewers for their valuable comments helping to improve the manuscript.

(Received June 30, 2010)

REFERENCES


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