KYBERNETIKA — VOLUME 46 (2010), NUMBER 5, PAGES 890–906

NULL CONTROLLABILITY OF A NONLINEAR DIFFUSION SYSTEM IN REACTOR DYNAMICS

Kumarasamy Sakthivel, Krishnan Balachandran,
Jong-Yeoul Park and Ganeshan Devipriya

In this paper, we prove the exact null controllability of certain diffusion system by rewriting it as an equivalent nonlinear parabolic integrodifferential equation with variable coefficients in a bounded interval of $\mathbb{R}$ with a distributed control acting on a subinterval. We first prove a global null controllability result of an associated linearized integrodifferential equation by establishing a suitable observability estimate for adjoint system with appropriate assumptions on the coefficients. Then this result is successfully used with some estimates for parabolic equation in L^k spaces together with classical fixed point theorem, to prove the null controllability of the nonlinear model.

Keywords: controllability, observability, parabolic integrodifferential equation

Classification: 93B05, 93B07

1. INTRODUCTION

It is interesting to note that in various fields of physics and engineering, many of the applications begin with a partial (or ordinary) differential equation and, through simplifying assumptions, we arrive at an integral or integrodifferential equation. Consider for example, in the analysis of space time dependent nuclear reactor dynamics, if the effect of a linear temperature feedback is taken into consideration and the reactor model is considered as an infinite rod, then the one group neutron flux $y(t,x)$ and the temperature $v(t,x)$ in the reactor are given by the following coupled equation (see, [20, 21, 27])

$$\begin{aligned} y_t - (a(t,x)y_x)_x &= (b(t,x)v + c_1(t,x) - 1)\Sigma_f y \qquad (t>0, -\infty < x < \infty) \quad (1)\\ \widetilde{\rho} c_2 v_t &= c_3 \Sigma_g y, \qquad (2) \end{aligned}$$

where a is the diffusion coefficient and $\Sigma_f, \Sigma_g, b, \widetilde{\rho}, c_1, c_2$ and c_3 are the physical quantities. By integrating (2) in the interval $(0,t)$ and substituting it in (1), we obtain the following nonlinear integrodifferential diffusion equation:

$$y_t - (a(t,x)y_x)_x = \beta b(t,x) y \int_0^t y(r,x)\,\mathrm{d}r + c(t,x)y \qquad (t>0, -\infty < x < \infty)$$

where the constant β and the coefficient c are the quantities associated with the initial temperature and various physical parameters.

Therefore, in this paper we consider the following nonlinear integrodifferential control problem(taking $\beta = 1$)

$$\begin{cases} z_t - (a(t,x)z_x)_x + b(t,x)z \displaystyle\int_0^t z(r,x)\,\mathrm{d}r + c(t,x)z = 1_\omega u(t,x), & \text{in } Q \\ z = 0, \quad \text{on } \Sigma \\ z(0,x) = z_0(x), \quad \text{in } \Omega, \end{cases} \tag{3}$$

where $\Omega = (i_1, i_2)$ is a bounded interval in $\mathbb{R}$, $T > 0$ is a fixed time. The notations $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \{i_1\} \cup (0,T) \times \{i_2\}$ is the boundary and 1_ω is the characteristic function of an open set $\omega \subset \Omega$. The state $z = z(t,x)$ and the control $u = u(t,x)$ (which acts on the system through ω) are unknowns to be determined for any arbitrary but fixed initial data $z_0 \in H_0^1(\Omega)$.

Moreover, we assume that the coefficients $a \in C^{1,2}(\bar{Q}), c \in L^\infty(\bar{Q})$ and b is sufficiently smooth satisfying the following conditions:

$$a(t,x) \geq a_{\min} > 0 \quad \text{and} \quad b(t,\cdot)|_{t=0,T} = 0, \quad b(t,x) \geq 0. \tag{4}$$

The system (3) is said to be null controllable at time T if, for each $z_0 \in H_0^1(\Omega)$, there exists $u \in L^2((0,T) \times \omega)$ such that the associated solution satisfies $z(T,x) \equiv 0$ in Ω.

The existence, uniqueness and asymptotic behavior of the solutions of the nonlinear integrodifferential equation of the form (3) have been studied by several authors (see, for example, [20, 21]). The problem of exact null controllability of related models without the integral term as a nonlinearity has been studied by several authors in the last two decades. We will explain briefly some results from the literature. For the first time, Lebeau and Robbiano [18] studied the null controllability of the linear heat equation in a bounded domain $\Omega \subset \mathbb{R}^n$ by a localized control force which acts on a subdomain $\omega \subset \Omega$ by using the spectral decomposition of the solutions. Fursikov and Imanuvilov [14] proved these results for semilinear equation

$$y_t - \Delta y + f(y) = 1_\omega u, \text{ in } (0,T) \times \Omega$$

when the function f is sublinear and also they proved the exact controllability of more general semilinear parabolic equations with variable coefficients by establishing a global Carleman estimate for linearized problem. A related result has been proved by Imanuvilov and Yamamoto [15] for parabolic equations in Sobolev spaces of negative order. Barbu [4] (see, also [5] for a related result) and Fernandez-Cara and Zuazua [13] generalized those results of [14] for some superlinear nonlinearities using a classical fixed point method and a Carleman type estimate for associated linear equation. Anita and Barbu [2] studied some interesting results on null controllability of nonlinear convective heat equations for $n = 1, 2, 3$. The null controllability of the diffusion equation

$$y_t - (a(y))_{xx} = 1_\omega u, \text{ in } (0,T) \times \Omega,$$

where Ω is a bounded interval in $\mathbb{R}$, has been discussed by Beceanu [8] by establishing a Carleman estimate for appropriate linearized equation.

Besides, approximate controllability of the linear heat equations with memory kernels has been analyzed by Barbu and Iannelli [6] under some technical conditions on the memory kernels by means of Laplace transform and Yong and Zhang [28] studied the controllability of the heat equation with hyperbolic memory kernel. The paper by Sakthivel et al. [22] discusses the exact controllability of some nonlinear parabolic integrodifferential equations with periodic boundary conditions when the nonlinearity is of globally Lipschitz and the control $u \in L^2((0,T) \times \omega)$. Whereas in this work we deal with a particular nonlinear term consisting of an integral in which the associated linear model is discussed when the linearized coefficient is in $L^\infty(Q)$ which leads us to proving some regularity on the control and this result is then applied to establish some new regularity result for the nonlinear system. Apart from these results, very recently Sakthivel et al. [24] obtained the local exact null controllability of nonlinear parabolic integrodifferential equation with nonlinearities under the integral sign is of globally Lipschitz with homogeneous Dirichlet boundary conditions.

Remark 1.1. The paper by Sakthivel et al. [23] studies the null controllability of certain parabolic equations with zeroth order memory integral of the form

$$\int_0^t k(t,r)z(r,x)\,\mathrm{d}r$$

with the assumption that the smooth kernel $k(t,\cdot)$ has support with respect to time, that is, $\operatorname{supp}(k(\cdot,t)) \subset (t_0,t_1)$ for some arbitrary but fixed t_0 and t_1 with $0 < t_0 < t_1 < T$. It is worth noting that in the duality process, the coefficient $b(t,x)$ in (1) turns into a kind of kernel function in which we have not imposed such assumption and it forces us to prove a new estimate to complete the proof of Theorem 2.2 which makes this paper substantially different from [23]. Further, we remark that the model we have considered here is completely different from the above noted literature due to the peculiar nonlinearity arises when we combine the system (1) – (2) to get an equivalent model.

Remark 1.2. Though for simplicity, we restrict this paper for one dimensional case, in the actual reactor systems, the temperature is a function of position x, which may be one, two or three dimensional. So the discussion we have carried out in this paper can also be extended for higher dimensions ($n = 2,3$) of the model

$$z_t - \nabla\cdot(a(t,x)\nabla z) + b(t,x)z\int_0^t z(r,x)\,\mathrm{d}r + c(t,x)z = 1_\omega u(t,x) \quad \text{in} \quad (0,T)\times\Omega,$$

with the same assumptions on the coefficients a, b and c and with suitable regularity on the initial data. More precisely, from the regularity result we have proved in Lemma 3.2, it is clear that for $z_0 \in H^2(\Omega) \subset W_3^{2-\frac{2}{3}}(\Omega)$ and $u \in L^3(Q)$, one can conclude by the Sobolev imbeddings $W_3^{2,1}(Q) \subset L^\infty(Q)$, for $n = 1,2,3$ (see, [19]) so that Theorem 3.4 can be validated for $n = 1,2,3$. Besides, the result can further be extended to a larger class of initial data by means of smoothing effect of the heat equation.

Now we describe some spaces which will be used throughout the paper to formulate our results. For each positive integer m and $p > 1$, or $p = \infty$, we denote as usual by $W^{m,p}(\Omega)$, the Sobolev space of functions in $L^p(\Omega)$ whose weak derivatives of order less than or equal to m are also in $L^p(\Omega)$. When $p = 2$ instead of $W^{m,2}(\Omega)$, we shall write $H^m(\Omega)$. Besides, we need the space $L^2(0,T;H^1(\Omega))$ of all equivalence classes of square integrable functions from $(0,T)$ to $H^1(\Omega)$. The space $L^2(0,T;L^2(\Omega))$ is analogously defined. Moreover, we set

$$\begin{aligned}
&C^{1,2}(\bar{Q}) = \big\{w(t,x) \in C(\bar{Q}) : w_t, w_x, w_{xx} \in C(\bar{Q})\big\},\\
&W_k^{2,1}(Q) = \big\{w(t,x) \in L^k(Q) : D_t^r D_x^m w \in L^k(Q),\ 2r + m \le 2,\ 2 \le k \le +\infty\big\},\\
&H^1(0,T;L^2(\Omega)) = \big\{w(t,x) \in L^2(0,T;L^2(\Omega)) : \mathrm{d}w/\mathrm{d}t \in L^2(0,T;L^2(\Omega))\big\}.
\end{aligned}$$

For more details about these spaces, one can refer to Adams and Fournier [1] and the classical monograph by Ladyzenskaya et al. [17].

This paper is arranged as follows: In Section 2 we establish a Carleman estimate for linear adjoint diffusion equation and deduce an observability estimate. In Section 3, we first prove the global null controllability of the linear diffusion equation and then we establish the null controllability of the nonlinear model (3) making use of Lemma 3.2 and classical fixed point arguments.

2. OBSERVABILITY ESTIMATES

It is well known that the exact controllability of a linear system can be reduced to the observability estimate of its dual system. In the same way, controllability of a semilinear system can be reduced to an estimate, provided the observability constant depends on the coefficients of the "linearized" systems. Thus, one of the main problems in the theory of exact controllability is how to construct the observability estimates for the linear system. There are several methods to prove the observability estimates, for example, multiplier techniques [16, 19], Carleman estimates [14, 26] and microlocal analysis [7]. The very recent survey article by Zuazua [29] explains most of the available methods developed by various authors during the last few decades to study the controllability of partial differential equations. However, to the best of our knowledge, the most effective method in proving the observability estimate is the method of Carleman estimates.

In this direction, we first consider the linearized system

$$\begin{cases} z_t - (a(t,x)z_x)_x + g(t,x)\displaystyle\int_0^t z(r,x)\,\mathrm{d}r + c(t,x)z = 1_\omega u(t,x), & \text{in } Q\\ z = 0, \quad \text{on } \Sigma\\ z(0,x) = z_0(x), \quad \text{in } \Omega, \end{cases} \tag{5}$$

where $g(t,x) = w(t,x)b(t,x)$, for some function $w \in L^\infty(Q)$. In order to establish the controllability of (5) it is sufficient to derive a Carleman type estimate for dual problem

$$\begin{cases} -y_t - \left(a(t,x)y_x\right)_x + G_t^T * y + c(t,x)y = 0, & \text{in } Q \\ y = 0, & \text{on } \Sigma \\ y(T,x) = y_T(x), & \text{in } \Omega, \end{cases} \tag{6}$$

where $y_T \in L^2(\Omega)$ and the notation $G_t^T * y$ stands for

$$G_t^T * y = \int_t^T g(r,x)y(r,x)\,\mathrm{d}r. \tag{7}$$

The following lemma is the most fundamental tool in proving the Carleman type estimates. The proof of this lemma can be found in [14].

Lemma 2.1. Let $\overline{\omega}_0 \subset \omega$ be an arbitrary fixed subset of Ω. Then there exists a function $\psi \in C^2(\bar{\Omega})$ such that

$$\psi > 0 \;\forall x \in \Omega, \;\; \psi = 0 \text{ on } \partial\Omega \;\text{ and }\; |\psi_x| > 0 \;\forall x \in \overline{\Omega \setminus \omega_0}.$$

Further, we are in need of the following weight functions

$$\phi(t,x) = e^{\lambda\psi(x)}/\pi(t) \quad \text{ and } \quad \alpha(t,x) = \left(e^{\lambda\psi(x)} - e^{2\lambda\Psi}\right)/\pi(t), \tag{8}$$

where $\pi(t) = t(T-t)$ and $\Psi = \|\psi(x)\|_{C(\bar{\Omega})}$, the parameter $\lambda > 1$ and the function ψ is defined in the above lemma. From the definitions of ϕ and α, we note that

$$\phi_t = e^{\lambda\psi(x)}(2t-T)/\pi^2(t), \quad \alpha_t = (e^{\lambda\psi(x)} - e^{2\lambda\Psi})(2t-T)/\pi^2(t),$$

and so

$$|\phi_t| \le \tilde{C}_1\phi^2, \quad |\alpha_t| = |\alpha(\ln(\pi^{-1}(t))_t)| \le \tilde{C}_1\phi^2, \quad |\alpha_{tt}| \le \tilde{C}_1\phi^3, \tag{9}$$

where the constant $\tilde{C}_1 > 0$ is independent of $(t,x) \in Q$ and $\lambda > 1$. It is easy to see that $\phi_x = \alpha_x = \lambda\phi\psi_x$. Further, ψ is a continuous function with compact support in Ω; then there exist constants $\tilde{C}_2, \tilde{C}_3$ and $\tilde{C}_4$ such that

$$\tilde{C}_2 = \sup_{x\in\Omega}|\psi_x|, \;\; \tilde{C}_3 = \sup_{x\in\Omega}|\psi_{xx}| \;\text{ and }\; \tilde{C}_4 = \sup_{x\in\Omega}|\psi_x|^2$$

hold. But, for simplicity, throughout the proof we shall use the generic constant C alone. Here we see that the weight function α approaches $-\infty$ at $t = 0$ and $t = T$, and this helps us to get the desired observability estimate. Also the additional parameter λ is essential in order to obtain the control of the constant which enable us to handle arbitrarily large coefficients in the coupling terms.

First we establish a Carleman estimate for the variant of the dual problem (6), namely,

$$\begin{cases} y_t + \left(ay_x\right)_x - G_t^T * y = f(t,x), & \text{in } Q \\ y = 0, & \text{on } \Sigma \\ y(T,x) = y_T(x), & \text{in } \Omega, \end{cases} \tag{10}$$

where the integral $G_t^T * y$ is defined in (7). Now we are ready to state and prove the main result of this section.

Theorem 2.2. (Carleman Type Estimate) Let the functions y and f satisfy (10) and ϕ, α be defined as in (8). Suppose assumptions (4) on the coefficients a, b and c hold true and also assume that $\|w\|_{L^\infty(Q)} \leq \rho$. Then for any $\lambda \geq \lambda_0(\Omega, T)$ and $s \geq s_0(\Omega, T, \lambda, \rho, a, b)$, the following inequality holds:

$$\int_Q e^{2s\alpha}\big((s\phi)^3 y^2 + s\phi y_x^2\big)\,\mathrm{d}x\mathrm{d}t \leq C(\lambda)\Big(\int_Q e^{2s\alpha} f^2\,\mathrm{d}x\mathrm{d}t + \int_{Q_\omega} e^{2s\alpha} s^3\phi^3 y^2\,\mathrm{d}x\mathrm{d}t\Big), \quad (11)$$

where $Q_\omega = (0,T)\times\omega$ and the constant $C(\lambda) > 0$ is independent of y and s.

Proof. Let us make the change of variable for the unknown function $p(t,x) = e^{s\alpha}y(t,x)$ in (10). Then it becomes

$$\begin{cases} p_t(t,x) + L_1 p(t,x) - L_2 p(t,x) = f_s(t,x), \ \text{ in } \ Q \\ p = 0, \ \text{ on } \Sigma \\ p(0,x) = p(T,x) = 0, \ \text{ in } \Omega, \end{cases} \quad (12)$$

where

$$\begin{aligned} L_1 p &= -2sa\lambda\phi\psi_x p_x - 2sa\lambda^2\phi\psi_x^2 p, & (13)\\ L_2 p &= -s^2 a\lambda^2\phi^2\psi_x^2 p + s\alpha_t p - a p_{xx} - sa\lambda^2\phi\psi_x^2 p + sa\lambda\phi\psi_{xx} p, & (14)\\ f_s &= e^{s\alpha} f + e^{s\alpha} G_t^T * e^{-s\alpha} p + a_x s\lambda\phi\psi_x p - a_x p_x. & (15) \end{aligned}$$

In virtue of (12), we obtain

$$\|p_t + L_1 p\|^2_{L^2(Q)} + \|L_2 p\|^2_{L^2(Q)} - 2\langle p_t, L_2 p\rangle_{L^2(Q)} - 2\langle L_1 p, L_2 p\rangle_{L^2(Q)} = \|f_s\|^2_{L^2(Q)}.$$

Clearly, this would imply the following inequality

$$-\langle L_1 p, L_2 p\rangle_{L^2(Q)} \leq \langle p_t, L_2 p\rangle_{L^2(Q)} + \frac{1}{2}\|f_s\|^2_{L^2(Q)}. \quad (16)$$

Now we have to obtain the lower bound for the left hand side L^2 integrals and the upper bound for the right hand side integrals as well. Proceeding with the computations similar to [9, 14, 23], one indeed obtain the following lemma.

Lemma 2.3. Suppose all the assumptions of Theorem 2.2 are satisfied. Then there exists a $\hat\lambda_0(\Omega, T) > 0$ such that for an arbitrary $\lambda \geq \hat\lambda_0$, there exists a $\hat s_0(\Omega, T, \lambda, a)$ such that for every $s \geq \hat s_0$, the solution of the problem (12) satisfies the inequality:

$$\begin{aligned} &\int_Q s^3\lambda^4\phi^3 p^2\,\mathrm{d}x\mathrm{d}t + \int_Q s\lambda^2\phi p_x^2\,\mathrm{d}x\mathrm{d}t \\ &\leq \ C\Big(\int_{Q_{\omega_0}} s^3\lambda^4\phi^3 p^2\,\mathrm{d}x\mathrm{d}t + \int_{Q_{\omega_0}} s\lambda^2\phi p_x^2\,\mathrm{d}x\mathrm{d}t + \|f_s\|^2_{L^2(Q)}\Big), \quad (17) \end{aligned}$$

where the constant $C > 0$ depends only on Ω, ω, T and $Q_{\omega_0} = (0,T)\times\omega_0$.

We now continue the proof of Theorem 2.2. Estimating the last term in (17), we obtain

$$\begin{aligned}\|f_s\|^2_{L^2(Q)} \;\leq\; & 4\int_Q e^{2s\alpha} f^2 \,\mathrm{d}x\mathrm{d}t + 4\int_Q e^{2s\alpha}(G_t^T * e^{-s\alpha}p)^2 \,\mathrm{d}x\mathrm{d}t \\ & + C\int_Q s^2\lambda^2\phi^2 p^2 \,\mathrm{d}x\mathrm{d}t + C\int_Q p_x^2 \,\mathrm{d}x\mathrm{d}t. \qquad (18)\end{aligned}$$

Note that for any sufficiently large $\lambda \geq \lambda_0$ and $s \geq s_0$ (see, for example [23] for precise values of λ_0, s_0), the last two integrals in (18) can go with the similar integrals on the left hand side of (17). Thus we have

$$\begin{aligned}&\int_Q s^3\lambda^4\phi^3 p^2 \,\mathrm{d}x\mathrm{d}t + \int_Q s\lambda^2\phi p_x^2 \,\mathrm{d}x\mathrm{d}t \leq C\int_Q e^{2s\alpha} f^2 \,\mathrm{d}x\mathrm{d}t \\ &+ C\Big(\int_{Q_{\omega_0}} s^3\lambda^4\phi^3 p^2 \,\mathrm{d}x\mathrm{d}t + \int_{Q_{\omega_0}} s\lambda^2\phi p_x^2 \,\mathrm{d}x\mathrm{d}t + \int_Q e^{2s\alpha}(G_t^T * e^{-s\alpha}p)^2 \,\mathrm{d}x\mathrm{d}t\Big). \qquad (19)\end{aligned}$$

Next, we come back to the original variable y by substituting $p = e^{s\alpha}y$ into the above inequality as follows

$$\begin{aligned}&\int_Q e^{2s\alpha}(s^3\phi^3 y^2 + s\phi y_x^2) \,\mathrm{d}x\mathrm{d}t \leq C\int_Q e^{2s\alpha} f^2 \,\mathrm{d}x\mathrm{d}t \\ &\quad + C(\lambda)\Big(\int_{Q_{\omega_0}} e^{2s\alpha}(s^3\phi^3 y^2 + s\phi y_x^2) \,\mathrm{d}x\mathrm{d}t + \int_Q e^{2s\alpha}(G_t^T * y)^2 \,\mathrm{d}x\mathrm{d}t\Big) \qquad (20)\end{aligned}$$

for $\lambda \geq \lambda_0$ and $s \geq s_0$, where the constant C is somehow greater than the constant defined in the preceding estimate. To estimate the last term in (20), we are in need of the following lemma.

Lemma 2.4. Suppose all the assumptions of Theorem 2.2 hold true. Then, there exists a constant $C > 0$ depending only on Ω, T, ρ and b satisfying the estimate:

$$\int_Q e^{2s\alpha}(G_t^T * y)^2 \,\mathrm{d}x\mathrm{d}t \leq C\int_Q e^{2s\alpha} y^2 \,\mathrm{d}x\mathrm{d}t. \qquad (21)$$

Proof. We shall prove this lemma, with the help of the assumptions on b and certain interesting properties of the weight functions. Let us first set

$$\tilde{b}(\tau, x) = -b_\tau(\tau, x) \;\text{ and }\; y^t(\zeta, x) = \int_{\frac{T}{2}}^{\zeta} |y(r, x)| \,\mathrm{d}r, \;\; \zeta \leq t.$$

Then the formal integration by parts in time yields

$$\int_0^T \tilde{b}(\tau, x) y^t(\tau, x) \,\mathrm{d}\tau = \int_0^T b(\tau, x)|y(\tau, x)| \,\mathrm{d}\tau - (b(\tau, x) y^t(\tau, x))\Big|_{\tau=0}^{\tau=T},$$

and therefore

$$\begin{aligned}\int_Q e^{2s\alpha}(G_t^T * y)^2\,\mathrm{d}x\mathrm{d}t &\le \rho\int_Q e^{2s\alpha}\Big(\int_0^T b(\zeta,x)|y(\zeta,x)|\,\mathrm{d}\zeta\Big)^2\,\mathrm{d}x\mathrm{d}t\\ &= \rho\int_Q e^{2s\alpha}\Big(\int_0^T \tilde{b}(\zeta,x)y^t(\zeta,x)\,\mathrm{d}\zeta\Big)^2\,\mathrm{d}x\mathrm{d}t\\ &\le \rho C_b T\int_Q e^{2s\alpha}\Big(\int_0^T \mathrm{d}\zeta\Big|\int_{\frac{T}{2}}^t |y(r,x)|\,\mathrm{d}r\Big|^2\Big)\,\mathrm{d}x\mathrm{d}t\\ &= \rho C_b T^2\int_Q e^{2s\alpha}\Big|\int_{\frac{T}{2}}^t |y(r,x)|\,\mathrm{d}r\Big|^2\,\mathrm{d}x\mathrm{d}t. \end{aligned}\tag{22}$$

Now, let us observe that

$$\begin{aligned}&\int_Q e^{2s\alpha}\Big|\int_{\frac{T}{2}}^t |y(r,x)|\,\mathrm{d}r\Big|^2\,\mathrm{d}x\mathrm{d}t\\ &\quad= \int_0^{\frac{T}{2}}\int_\Omega e^{2s\alpha}\Big|\int_{\frac{T}{2}}^t |y(r,x)|\,\mathrm{d}r\Big|^2\,\mathrm{d}x\mathrm{d}t + \int_{\frac{T}{2}}^T\int_\Omega e^{2s\alpha}\Big|\int_{\frac{T}{2}}^t |y(r,x)|\,\mathrm{d}r\Big|^2\,\mathrm{d}x\mathrm{d}t \equiv I_1 + I_2.\end{aligned}$$

From the definition of the weight function α, it is clear that

$$\alpha_t = -2\frac{(e^{2\lambda\Psi} - e^{\lambda\psi})\big(t-\frac{T}{2}\big)}{\pi^2(t)} \ge 0 \quad\text{for}\quad 0\le t\le \frac{T}{2},$$

which leads to

$$\begin{aligned} I_1 &\le \int_0^{\frac{T}{2}}\int_\Omega e^{2s\alpha}\big(\frac{T}{2}-t\big)\Big(\int_t^{\frac{T}{2}} |y(r,x)|^2\,\mathrm{d}r\Big)\,\mathrm{d}x\mathrm{d}t\\ &\le \frac{C(\Omega,T)}{s}\int_0^{\frac{T}{2}}\int_\Omega (e^{2s\alpha})_t\Big(\int_t^{\frac{T}{2}} |y(r,x)|^2\,\mathrm{d}r\Big)\,\mathrm{d}x\mathrm{d}t\\ &\le C(\Omega,T)\int_0^{\frac{T}{2}}\int_\Omega e^{2s\alpha}|y(t,x)|^2\,\mathrm{d}x\mathrm{d}t \quad\text{for } 0\le t\le\frac{T}{2} \quad\text{and}\quad s\ge s_0,\end{aligned}$$

since the boundary terms disappear due to the fact that $\alpha(0,x) = -\infty$ and inner integral vanishes at $t = T/2$. For any $t \in (T/2, T)$, applying Fubini's theorem together with the fact that $e^{2s\alpha}$ is a decreasing function in $(T/2, T)$, we also have

$$\begin{aligned} I_2 &= \int_{\frac{T}{2}}^T\int_\Omega\Big(\int_r^T e^{2s\alpha(t,x)}\,\mathrm{d}t\Big)|y(r,x)|^2\,\mathrm{d}x\mathrm{d}r\\ &\le C(T)\int_{\frac{T}{2}}^T\int_\Omega e^{2s\alpha(r,x)}|y(r,x)|^2\,\mathrm{d}x\mathrm{d}t \quad\text{for}\quad \frac{T}{2}\le r\le t\le T.\end{aligned}$$

Hence, by combining the preceding estimates, one can complete the proof. □

In order to complete the proof of Theorem 2.2, it is sufficient to express the integral of $e^{2s\alpha}s\phi y_x^2$ over Q_{ω_0} in the right hand side of (20), in terms of $e^{2s\alpha}s^3\phi^3y^2$ over a larger domain Q_ω. To this end, we multiply the first equation in (10) by $\theta(y) := e^{2s\alpha}s\phi\chi y$ and integrate over Q_ω, where the function $\chi \in C_0^2(\omega)$ satisfies $\chi = 1$ in ω_0, $0 \le \chi \le 1$, to get

$$\begin{aligned}-\int_{Q_\omega}\theta(y)(ay_x)_x\,\mathrm{d}x\mathrm{d}t \quad &\le \quad \frac{\eta}{2}\int_{Q_\omega}e^{2s\alpha}f^2\,\mathrm{d}x\mathrm{d}t+\frac{1}{\eta}\int_{Q_\omega}e^{2s\alpha}s^2\phi^2y^2\,\mathrm{d}x\mathrm{d}t \qquad (23)\\ &\quad+\int_{Q_\omega}\theta(y)y_t\,\mathrm{d}x\mathrm{d}t+\frac{\eta}{2}\int_{Q_\omega}e^{2s\alpha}(G_t^T*y)^2\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Consequently following the similar estimates in [14, 23] and using Lemma 2.4, we obtain

$$\int_{Q_{\omega_0}}e^{2s\alpha}s\phi y_x^2\,\mathrm{d}x\mathrm{d}t \quad \le \quad C(\lambda)\Big(\int_Q e^{2s\alpha}f^2\,\mathrm{d}x\mathrm{d}t+\int_{Q_\omega}e^{2s\alpha}s^3\phi^3y^2\,\mathrm{d}x\mathrm{d}t\Big), \qquad (24)$$

where we have used the assumption on the coefficient a. Thus substituting the estimation (24) into the inequality (20) and choosing sufficiently large enough $\lambda \ge \lambda_0$, $s \ge s_0$, one can conclude the proof of Theorem 2.2. □

Now we reduce the following observability estimate from Theorem 2.2 and it will be the main ingredient for the proof of the controllability of the linear problem. This inequality enables us to estimate the solutions in the entire domain by observing them in small subdomain only.

Corollary 2.5. Suppose all the assumptions of Theorem 2.2 are satisfied. Then there exists a constant $C > 0$, such that for any sufficiently large $\lambda \ge \lambda_0$, $s \ge s_1$ and for any $r \in (0, 2)$, the following estimate holds:

$$\int_\Omega y^2(0,x)\,\mathrm{d}x \le C\int_{Q_\omega}e^{rs\alpha}y^2\,\mathrm{d}x\mathrm{d}t, \qquad (25)$$

where y is the solution to the adjoint problem (6).

P r o o f. Multiplying (6) by y and integrating on Ω and applying Young's inequality, we get

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega y^2\,\mathrm{d}x+\int_\Omega ay_x^2\,\mathrm{d}x \quad \le \quad \frac{1}{2}(1+2\|c\|_{L^\infty(\bar{Q})})\int_\Omega y^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega(G_t^T*y)^2\,\mathrm{d}x,$$

whence it follows that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{C_1t}\int_\Omega y^2(t,x)\mathrm{d}x\Big) \le e^{C_1t}\int_\Omega(G_t^T*y)^2\mathrm{d}x, \qquad (26)$$

where we have used the assumption on a and $C_1 > 0$ is a constant. Integrating over the time interval $0 \le t \le T/4$, we get

$$\int_\Omega y^2(0,x)\,\mathrm{d}x \le e^{C_1T}\Big(\int_\Omega y^2(T/4,x)\,\mathrm{d}x+\int_0^{\frac{T}{4}}\int_\Omega(G_t^T*y)^2\,\mathrm{d}x\mathrm{d}t\Big).$$

Again integrating (26) from $T/4$ to t with $t \in [T/4, 3T/4]$ and then combining with the preceding estimate, one can obtain that

$$\int_\Omega y^2(0,x)\,\mathrm{d}x \le e^{C_1 T}\Big(\int_\Omega y^2(t,x)\,\mathrm{d}x + \int_{Q^{3T/4}} (G_t^T * y)^2\,\mathrm{d}x\mathrm{d}t\Big) \tag{27}$$

where $Q^{3T/4} = (0, 3T/4) \times \Omega$. But recalling the assumption on the coefficient b and proceeding with the computation similar to the estimate (22), one indeed get

$$\int_{Q^{3T/4}} (G_t^T * y)^2\,\mathrm{d}x\mathrm{d}t \quad \le \quad \rho C_b T^3 \int_{Q^{3T/4}} \Big|\int_{\frac{T}{2}}^{t} y^2\,\mathrm{d}r\Big|\,\mathrm{d}x\mathrm{d}t \le C_2 \int_{\frac{T}{2}}^{\frac{3T}{4}}\int_\Omega y^2\,\mathrm{d}x\mathrm{d}t.$$

Integrating with time over the interval $(T/4, 3T/4)$, the inequality (27), now reduces to

$$\int_\Omega y^2(0,x)\,\mathrm{d}x \le C_3 \int_{\frac{T}{4}}^{\frac{3T}{4}}\int_\Omega y^2\,\mathrm{d}x\mathrm{d}t, \tag{28}$$

where the constant $C_3 > 0$ depends on Ω, T, ρ and the coefficients b, c. Now by density, applying the Carleman estimate to the system (6), we get

$$\int_Q e^{2s\alpha} s^3\phi^3 y^2\,\mathrm{d}x\mathrm{d}t + \int_Q e^{2s\alpha} s\phi y_x^2\,\mathrm{d}x\mathrm{d}t \le C(\lambda)\int_{Q_\omega} e^{2s\alpha} s^3\phi^3 y^2\,\mathrm{d}x\mathrm{d}t, \tag{29}$$

for any $s \ge s_1 = \max\{s_0, C\|c\|_{L^\infty(\bar{Q})}^{2/3}\}$.

It is easy to see that $(e^{2s\alpha}\phi^3) \ge C_4 > 0$ for all $(t,x) \in [T/4, 3T/4] \times \bar{\Omega}$, for sufficiently large $s \ge s_1$ (see, [12] for a sharp estimate). For any $0 < r < 2$ and $s \ge s_1$, we also have $(e^{(2-r)s\alpha}\phi^3) \le C_5 < \infty$ for all $(t,x) \in (0,T) \times \bar{\Omega}$. With this remark, combining the estimate (28) with (29), one can conclude the proof. □

3. CONTROLLABILITY RESULTS

3.1. Controllability of the Linear Diffusion Equation

In this section, we shall obtain a solution to the global exact null controllability problem for the linear model (5) as the limit of an approximation process, constructed with the aid of a family of appropriate optimal control problems for system (5). To derive the needed estimates for the solutions of the optimal control problems we shall use Pontryagin's maximum principle and the Carleman estimate (11) derived for the backward adjoint equation of (5). Throughout this sequel we use the functions α and ϕ as defined in (8).

Theorem 3.1. Let Ω be an open bounded set in $\mathbb{R}$. Suppose assumptions (4) on the coefficients a, b and c are satisfied and also assume that $\|w\|_{L^\infty(Q)} \le \rho$. Then there exist $\lambda \ge \lambda_0$ and $s \ge s_1$ as defined in Corollary 2.5, such that for any $z_0 \in H_0^1(\Omega)$, there exist (u, z) satisfying (5) and the terminal condition $z(T,x) \equiv 0$ a. e. $x \in \Omega$.

P r o o f. Let $T > 0$ be fixed and $z_0 \in H_0^1(\Omega)$. For any $\epsilon > 0$ and $r \in (0, 2)$, consider the optimal control problem subject to (5) with

$$J(u, z) = \int_Q e^{-rs\alpha} u^2 \,\mathrm{d}x\mathrm{d}t + \frac{1}{\epsilon} \int_\Omega z^2(T, x) \,\mathrm{d}x \to \inf . \tag{30}$$

It is well known that by the classical arguments, this minimization problem has a unique solution (u_ϵ, z_ϵ) for every $\epsilon > 0$. Next we shall show that (u_ϵ, z_ϵ) converges (on a subsequence of $\{\epsilon\}$) to (u, z) and this will be proved to be a solution of the control problem (5). By Pontryagin's maximum principle (see, [10, 13]), the control u_ϵ is characterized as $u_\epsilon = 1_\omega e^{rs\alpha} y_\epsilon$ a. e. in Q, where $r \in (0, 2)$ and y_ϵ is a solution of the following backward adjoint equation

$$\begin{cases} -(y_\epsilon)_t - \left(a(y_\epsilon)_x\right)_x + G_t^T * y_\epsilon + c y_\epsilon = 0, & \text{in } Q \\ y_\epsilon = 0, & \text{on } \Sigma \\ y_\epsilon(T, x) = -\frac{1}{\epsilon} z_\epsilon(T, x), & \text{in } \Omega. \end{cases} \tag{31}$$

Next, we obtain an *a priori* estimate for the solution z of (5). First, we multiply (5) by z_ϵ and integrate over Ω. Then applying Young's inequality and Hölder's inequality, we get

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega z_\epsilon^2 \,\mathrm{d}x + \int_\Omega a(t, x)(z_\epsilon)_x^2 \,\mathrm{d}x \\ &\qquad \le \left(1 + \|c\|_{L^\infty(\bar{Q})}\right) \int_\Omega z_\epsilon^2 \,\mathrm{d}x + \frac{\rho^2 C_b T}{2} \int_0^t \int_\Omega z_\epsilon^2 \,\mathrm{d}x\mathrm{d}r + \frac{1}{2} \int_\Omega 1_\omega^2 u_\epsilon^2 \,\mathrm{d}x. \end{aligned}$$

Integrating from 0 to t, for $t \in (0, T)$, we get

$$\begin{aligned} &\int_\Omega z_\epsilon^2(t, x) \,\mathrm{d}x + 2 \int_0^t \int_\Omega a(t, x)(z_\epsilon)_x^2 \,\mathrm{d}x\mathrm{d}r \\ &\qquad \le C_1 \left(\int_0^t \|z_\epsilon\|_{L^2((0,r)\times\Omega)}^2 \,\mathrm{d}r + \|z_0\|_{L^2(\Omega)}^2 + \|e^{-rs\alpha/2} u_\epsilon\|_{L^2(Q_\omega)}^2 \right), \end{aligned}$$

where $C_1 > 0$ is a constant. We note that $e^{rs\alpha} \le \bar{C} < +\infty$, $\forall\ (t, x) \in Q$ and by duality argument, the last control integral can be bounded by the L^2 norm of z_0 (see, [23]). Now making use of the assumption on the coefficient a and the Poincaré inequality, it is clear that

$$\int_0^t \int_\Omega (z_\epsilon)_x^2 \,\mathrm{d}x\mathrm{d}r \le C_2 \|z_0\|_{L^2(\Omega)}^2 \ \forall t \in (0, T), \tag{32}$$

where the constant $C_2 > 0$ depends on Ω, T and the coefficients $a_{\min}, b, c$ and ρ. Moreover, the system (5) can equivalently be written as

$$\begin{cases} (z_\epsilon)_t - a(z_\epsilon)_{xx} = 1_\omega u_\epsilon + a_x(z_\epsilon)_x - g \displaystyle\int_0^t z_\epsilon(r, x) \,\mathrm{d}r - c z_\epsilon, & \text{in } Q \\ z_\epsilon = 0, & \text{on } \Sigma \\ z_\epsilon(0, x) = z_0(x), & \text{in } \Omega. \end{cases} \tag{33}$$

Now squaring both sides of the equation (33) and integrating on $(0,t)\times\Omega$, one can get

$$\begin{aligned}&\int_\Omega a(t,x)(z_\epsilon)_x^2\,\mathrm{d}x+\frac{1}{2}\int_0^t\int_\Omega\Big((z_\epsilon)_t^2+2a^2(t,x)(z_\epsilon)_{xx}^2\Big)\,\mathrm{d}x\mathrm{d}r\\ \leq\ &\int_\Omega a(0,x)(z_0(x))_x^2\,\mathrm{d}x+4\int_0^t\int_\Omega 1_\omega^2u_\epsilon^2\,\mathrm{d}x\mathrm{d}r+C_a\int_0^t\int_\Omega(z_\epsilon)_x^2\,\mathrm{d}x\mathrm{d}r\\ &+4\big(\|c\|^2_{L^\infty(\bar{Q})}+\rho^2C_bT^2\big)\int_0^T\int_\Omega z_\epsilon^2\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Once again with the assumption on a together with the Poincaré inequality and the estimate (32), we arrive at

$$\int_\Omega(z_\epsilon)_x^2\,\mathrm{d}x+\int_0^T\int_\Omega((z_\epsilon)_t^2+(z_\epsilon)_{xx}^2)\,\mathrm{d}x\mathrm{d}t\leq C_3\big(\|z_0\|^2_{H_0^1(\Omega)}+\|z_0\|^2_{L^2(\Omega)}\big),\qquad(34)$$

where the constant $C_3>0$ depends only on Ω,T and the coefficients $a_{\min},b,c$ and ρ. Making use of the above *a priori* estimates and following the similar arguments in [23, 24], one can conclude the proof. □

For the proof of controllability of the nonlinear system, we are further in need of some regularity on the control.

Lemma 3.2. Suppose all the assumptions of Theorem 3.1 are satisfied. Then for any $z_0\in H_0^1(\Omega)$, there exist (u,z) satisfying (5) such that $z(T,x)\equiv0$ a. e. $x\in\Omega$ and

$$\|u\|^2_{L^k(Q)}\leq C_k\|z_0\|^2_{L^2(\Omega)},\quad\text{for any}\quad k\in(2,+\infty).\qquad(35)$$

P r o o f. Let us start by setting $\tilde{y}_\epsilon=e^{rs\alpha}y_\epsilon$, to have from (6) that

$$\begin{cases}(\tilde{y}_\epsilon)_t+a(\tilde{y}_\epsilon)_{xx}=g_{y_\epsilon} & \text{in } Q\\ \tilde{y}_\epsilon=0 & \text{on } \Sigma\\ \tilde{y}_\epsilon(T,x)=0 & \text{in } \Omega,\end{cases}\qquad(36)$$

where

$$\begin{aligned}g_{y_\epsilon}&=e^{rs\alpha}G_t^T*y_\epsilon+\big[2a(e^{rs\alpha})_x-a_xe^{rs\alpha}\big](y_\epsilon)_x+\big[ce^{rs\alpha}+(e^{rs\alpha})_t+a(e^{rs\alpha})_{xx}\big]y_\epsilon\\ &=I_1+I_2+I_3.\end{aligned}$$

Now from the parabolic regularity (see, [17]) and the estimate (29), we obtain the following estimates: First, we note that

$$\|\tilde{y}_\epsilon\|_{W_2^{2,1}(Q)}\leq C\|g_{y_\epsilon}\|_{L^2(Q)}.$$

Next we proceed estimating the integrals $I_i, i=1,2,3$ one by one. By Lemma 2.4, it is clear that

$$\|I_1\|^2_{L^2(Q)}\leq C(\Omega,T)\rho C_b\|e^{2(r-1)s\alpha}\phi^{-3}\|_{L^\infty(Q)}\int_Q e^{2s\alpha}\phi^3y_\epsilon^2\,\mathrm{d}x\mathrm{d}t\leq C\int_{Q_\omega}e^{rs\alpha}y_\epsilon^2\,\mathrm{d}x\mathrm{d}t,$$

for the proper choice of $r \geq 1$. Simple calculation shows that

$$\|I_2\|^2_{L^2(Q)} \leq C(\Omega)C_a\|e^{2(r-1)s\alpha}(\phi^{-1}+\phi)\|_{L^\infty(Q)} \int_Q e^{2s\alpha}\phi(y_\epsilon)^2_x \,\mathrm{d}x\mathrm{d}t \leq C\int_{Q_\omega} e^{rs\alpha}y_\epsilon^2 \,\mathrm{d}x\mathrm{d}t.$$

Moreover, we have

$$\begin{aligned} \|I_3\|^2_{L^2(Q)} &\leq C(\Omega,T)C_a\|e^{2(r-1)s\alpha}(\phi^{-1}+\phi^{-3}+\phi)\|_{L^\infty(Q)} \int_Q e^{2s\alpha}\phi^3 y_\epsilon^2 \,\mathrm{d}x\mathrm{d}t \\ &\leq C\int_{Q_\omega} e^{rs\alpha}y_\epsilon^2 \,\mathrm{d}x\mathrm{d}t. \end{aligned}$$

In view of the preceding estimates, one indeed get

$$\|\tilde{y}_\epsilon\|^2_{W_2^{2,1}(Q)} \leq C_4\int_{Q_\omega} e^{rs\alpha}y_\epsilon^2 \,\mathrm{d}x\mathrm{d}t, \tag{37}$$

where the constant $C_4 > 0$ depends on Ω, T, ρ and the coefficients a, b, c. Recalling the regularity result $W_2^{2,1}(Q) \subset L^k(Q)$, for all $k \in (2, +\infty)$ and going back to the definition of the control in Theorem 3.1, we then obtain that

$$\|u_\epsilon\|^2_{L^k(Q)} = \|1_\omega\tilde{y}_\epsilon\|^2_{L^k(Q)} \leq C_k\int_{Q_\omega} e^{rs\alpha}y_\epsilon^2 \,\mathrm{d}x\mathrm{d}t \leq C_k\|z_0\|^2_{L^2(\Omega)}. \tag{38}$$

Thus, from the existence theory of parabolic boundary value problems in $L^k(Q)$ (see, [17]), it follows at least for a subsequence of ϵ, that for $\epsilon \to 0$

$$\begin{aligned} u_\epsilon &\to u \text{ weakly in } L^k(Q), \\ z_\epsilon &\to z \text{ weakly in } W_k^{2,1}(Q) \cap L^2(0,T;H_0^1(\Omega)) \end{aligned}$$

and (u, z) satisfy the system (5) with $z(T, x) \equiv 0$, a. e. in Ω. The estimate for the control follows from (38). The proof is thus completed. □

3.2. Controllability of the Nonlinear Diffusion Equation

For the study of controllability of the nonlinear problem, it is well known that the fixed point method is the most effective one and it has been effectively used by several authors (see, for instance, [2, 3, 14] and the recent survey paper [11]), in which the controllability problem is transferred to a fixed point problem for an appropriate nonlinear operator in a suitable function space. In this context, we apply Kakutani's fixed point theorem along with the exact null controllability results of the associated linear problem to discuss the null controllability of the nonlinear model (3).

Moreover, as far as the exact controllability of the parabolic equation is concerned, the fixed point approach can be applied whenever, among other things, the way the constants arising in Carleman estimates depend on the coefficients of the linearized systems is known in detail. We remark that the existence of u is again equivalent to deriving a suitable observability inequality for the adjoint system. Thus, the idea is to find a fixed point of the mapping $\tilde{z} \to z$, where z is, with the initial data $z_0 \in H_0^1(\Omega)$ together with some control u, a solution to the linearized system

$$\begin{cases} z_t - (az_x)_x + b\tilde{z}\displaystyle\int_0^t z(r,x)\,\mathrm{d}r + cz = 1_\omega u, & \text{in } Q \\ z = 0, \quad \text{on } \Sigma \\ z(0,x) = z_0(x), \quad \text{in } \Omega, \end{cases} \tag{39}$$

satisfying the final condition $z(T,x) \equiv 0$, a. e. in Ω, for each $\tilde{z} \in N$, which is the set defined by

$$N = \{w \in L^\infty(Q) : \|w\|_{L^\infty(Q)} \le \rho\},$$

where ρ is an arbitrary but fixed positive constant. It can be seen that the set is closed and the fact that $\|w\|_{L^\infty(Q)}$ is bounded ensures its precompactness. Indeed there is a classical result (see, [25]), which states the following:

Theorem 3.3. Let X, Y and B be Banach spaces such that $X \subset B \subset Y$ with compact imbedding $X \hookrightarrow B$. Let $1 \le p \le \infty$. If $\mathcal{F}$ is a bounded subset of $L^p(0,T;X)$ and

$$\|\tau_h f - f\|_{L^p(0,T-h;Y)} \to 0, \text{ as } h \to 0, \text{ uniformly for } f \in \mathcal{F},$$

where $\tau_h f$ is the translated function of f with $(\tau_h)f(t) = f(t+h)$ for $h > 0$, then $\mathcal{F}$ is relatively compact in $L^p(0,T;B)$ (and in $C([0,T];B)$ if $p = \infty$).

Now we are ready to state and prove the main result of this section.

Theorem 3.4. Let Ω be an open bounded interval in $\mathbb{R}$. Suppose assumptions (4) on the coefficients a, b and c are satisfied. Then for each $z_0 \in H_0^1(\Omega)$, there exist $(u,z) \in L^2(Q) \times W_2^{2,1}(Q) \cap L^2(0,T;H_0^1(\Omega))$ satisfying (3) such that $z(T,x) \equiv 0$ a. e. in Ω.

Proof. For each $\tilde{z} \in N$, let us define the set valued mapping $\Phi : N \to 2^N$ such that

$$\begin{aligned} \Phi(\tilde{z}) \quad = \quad & \Big\{ z \in W_k^{2,1}(Q) \cap L^2(0,T;H_0^1(\Omega)), \text{ for any } k \in (2,+\infty) \\ & \text{and } z(T,x) \equiv 0 \text{ a. e. } x \in \Omega \text{ with } \|u\|^2_{L^k(Q)} \le C_k \|z_0\|^2_{L^2(\Omega)} \Big\}. \end{aligned}$$

Let us observe that, if the solution z to (3) lies in N, then it also solves (39). Hence in order to prove this theorem, it suffices to show that Φ has a fixed point in N. In this context, we now prove that all the conditions to apply the Kakutani fixed point theorem in L^2 topology are satisfied.

Let us first look at for any $z_0 \in H_0^1(\Omega)$, the solution z to (39) satisfies the estimate

$$\int_\Omega z_x^2\,\mathrm{d}x + \int_Q (z_t^2 + z_{xx}^2)\,\mathrm{d}x\mathrm{d}t \le C\big(\|z_0\|^2_{H_0^1(\Omega)} + \|z_0\|^2_{L^2(\Omega)}\big),$$

and therefore by the Sobolev imbeddings, we have

$$\int_{\Omega} z_x^2 \,\mathrm{d}x + \int_Q (z_t^2 + z_{xx}^2)\,\mathrm{d}x\mathrm{d}t \le C\|z_0\|^2_{H_0^1(\Omega)}. \tag{40}$$

For each $\tilde{z} \in N$, by Lemma 3.2 and the estimate (40), it is clear that $\Phi(\tilde{z})$ is a closed nonempty and convex subset of $L^2(Q)$. Further from the existence theory of parabolic boundary value problems (see, [17]), we also get

$$\|z\|^2_{L^\infty(Q)} \le \tilde{C}\|z_0\|^2_{H_0^1(\Omega)}. \tag{41}$$

Thus if the initial data is sufficiently small (for instance $\|z_0\|_{H_0^1(\Omega)} \le \rho/\tilde{C}$), one can obtain from the definition of N that, $\Phi(N) \subset N$. Besides, it follows from the estimate (40) that, there exists a compact set $\mathcal{N} \subset L^2(Q)$, such that for each $\tilde{z} \in N$, $\Phi(\tilde{z}) \subset \mathcal{N}$ and by the boundedness of Φ in $W_2^{2,1}(Q) \cap L^2(0,T;H_0^1(\Omega))$, which injects compactly in $L^2(Q)$.

It remains to show that Φ is upper semicontinuous and that can be shown from the fact that it has a closed graph. Indeed, let $\tilde{z}_n \in N$, $\tilde{z}_n \to \tilde{z}$ in $L^2(Q)$ and $z_n \in \Phi(\tilde{z}_n) \to z$ in $L^2(Q)$ and let u_n be the corresponding controls. Then by Lemma 3.2, we infer that the following convergence holds on a subsequence:

$$\begin{aligned} u_n &\to u \text{ weakly in } L^2(Q),\\ z_n &\to z \text{ weakly in } W_2^{2,1}(Q) \cap L^2(0,T;H_0^1(\Omega)). \end{aligned}$$

Since (u_n, z_n) is a solution of the linearized system

$$\begin{cases} (z_n)_t - (a(z_n)_x)_x + b\tilde{z}_n \int_0^t z_n(r,x)\,\mathrm{d}r + cz_n = 1_\omega u_n, & \text{in } Q\\ z_n = 0, \quad \text{on } \Sigma\\ z_n(0,x) = z_0(x), \quad \text{in } \Omega, \end{cases} \tag{42}$$

and therefore passing to weak limit, we can conclude that $z \in \Phi(\tilde{z})$. Eventually using the Kakutani fixed point theorem in the $L^2(Q)$ topology for the mapping Φ, we obtain that there is at least one $z \in N$, such that $z \in \Phi(z)$. It is clear that such z is a solution to the system (3) satisfying $z(T,x) \equiv 0$, a. e. in Ω. □

ACKNOWLEDGEMENT

The work of second author is supported by NBHM, Department of Atomic Energy, India (Grant No: 48/3/2006/R&D-II/8336) and Brain Pool Program of Korea Research Foundation Grant(2008). The authors wish to thank the referees for fruitful comments and suggestions which lead to this improved version.

(Received March 20, 2009)

REFERENCES

[1] R. A. Adams and J. F. Fournier: Sobolev Spaces. Second edition. Academic Press, New York 2003.

[2] S. Anita and V. Barbu: Null controllability of nonlinear convective heat equation. ESAIM: Control, Optimization and Calculus of Variations *5* (2000), 157–173.

[3] K. Balachandran and J. P. Dauer: Controllability of nonlinear systems in Banach spaces; A survey. J. Optim. Theory Appl. *115* (2002), 7–28.

[4] V. Barbu: Exact controllability of superlinear heat equation. Appl. Math. Optim. *42* (2000), 73–89.

[5] V. Barbu: Controllability of parabolic and Navier–Stokes equations. Sci. Math. Japon. *56* (2002), 143–211.

[6] V. Barbu and M. Iannelli: Controllability of the heat equation with memory. Differential and Integral Equations *13* (2000), 1393–1412.

[7] C. Bardos, G. Lebeau, and J. Rauch: Controle et stabilisation de l'equation des ondes. SIAM J. Control Optim. *30* (1992), 1024–1065.

[8] M. Beceanu: Local exact controllability of the diffusion equation in one dimension. Abstract Appl. Anal. *14* (2003), 793–811.

[9] D. Chae, O. Yu. Imanuvilov, and S. M. Kim: Exact controllability for semilinear parabolic equations with Neumann boundary conditions. J. Dynamical and Control Systems *2* (1996), 449–483.

[10] H. O. Fattorini and S. S. Sritharan: Necessary and sufficient conditions for optimal controls in viscous flow problems. Proc. Royal Soc. London A *124* (1994), 211–251.

[11] E. Fernández-Cara and S. Guerrero: Global Carleman inequalities for parabolic systems and applications to controllability. SIAM J. Control Optim. *45* (2006), 1395–1446.

[12] E. Fernández-Cara and E. Zuazua: The cost of approximate controllability for heat equations: The linear case. Adv. in Differential Equations *5* (2000), 465–514.

[13] E. Fernández-Cara and E. Zuazua: Null and approximate controllability for weakly blowing up semilinear heat equations. Ann. Inst. H. Poincaré Anal. Non Linéaire *17* (2000), 583–616.

[14] A. V. Fursikov and O. Yu. Imanuvilov: Controllability of Evolution Equations. Lecture Notes Ser., Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul 1996.

[15] O. Yu. Imanuvilov and M. Yamamoto: Carleman inequalities for parabolic equations in Sobolev spaces of negative order and exact controllability for semilinear parabolic equations. Publications of RIMS, Kyoto University *39* (2003), 227–274.

[16] V. Komorník: Exact Controllability and Stabilization: The Multiplier Method. John Wiley, Paris 1995.

[17] O. A. Ladyzenskaya, V. A. Solonikov, and N. Uralceva: Linear and Quasilinear Equations of Parabolic Type. Transl. Math. Monographs, Providence 1968.

[18] G. Lebeau and L. Robbiano: Contròle exact de l'équation de la chaleur. Comm. Partial Differential Equations *20* (1995), 335–356.

[19] J. L. Lions: Controlabilité exacte, stabilisation et perturbations the systémes distribués. Tome 1 and Tome 2. Masson 1988.

[20] B. G. Pachpatte: On a nonlinear diffusion system arising in reactor dynamics. J. Math. Anal. Appl. *94* (1983), 501–508.

[21] C. V. Pao: Bifurcation analysis of a nonlinear diffusion system in reactor dynamics. Appl. Anal. *9* (1979), 107–119.

[22] K. Sakthivel, K. Balachandran, and S. S. Sritharan: Controllability and observability theory of certain parabolic integrodifferential equations. Comp. and Math. Appl. *52* (2006), 1299–1316.

[23] K. Sakthivel, K. Balachandran, and S. S. Sritharan: Exact controllability of nonlinear diffusion equations arising in reactor dynamics. Nonlinear Anal.: Real World Appl. *9* (2008) 2029–2054.

[24] K. Sakthivel, K. Balachandran, and B. R. Nagaraj: On a class of nonlinear parabolic control systems with memory effects. Internat. J. Control *81* (2008), 764–777.

[25] J. Simon: Compact sets in the space $L^p(0;T;B)$. Ann. Mat. Pura Appl. *146* (1986), 65–96.

[26] D. Tataru: Boundary controllability for conservative partial differential equations. Appl. Math. Optim. *31* (1995), 257–295.

[27] E. G. Yanik and G. Fairweather: Finite element methods for parabolic and hyperbolic partial integrodifferential equation. Nonlinear Anal. *12* (1988), 785–809.

[28] J. Yong and X. Zhang: Exact controllability of the heat equation with hyperbolic memory kernel. Control Theory of Partial Differential Equations. Chapman & Hall/CRC, Boca Raton 2005, pp. 387–401.

[29] E. Zuazua: Controllability and observability of partial differential equations: some results and open problem. In: Handbook of Differential Equations: Evolutionary Equations *3* (2007), 527–621.

Kumarasamy Sakthivel, Department of Mathematics, Bharathiar University, Coimbatore - 641 046. India.
e-mail: pktsakthi@gmail.com

Krishnan Balachandran, Department of Mathematics, Pusan National University, Busan - 609-735. Korea.
e-mail: kbkb1956@yahoo.com

Jong-Yeoul Park, Department of Mathematics, Pusan National University, Busan - 609-735. Korea.
e-mail: jyepark@pusan.ac.kr

Ganeshan Devipriya, Department of Mathematics, Bharathiar University, Coimbatore - 641 046. India.
e-mail: devipriyaganeshan@gmail.com