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REDUCTION AND TRANSFER EQUIVALENCE OF NONLINEAR CONTROL SYSTEMS: UNIFICATION AND EXTENSION VIA PSEUDO-LINEAR ALGEBRA

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The paper applies the pseudo-linear algebra to unify the results on reducibility, reduction and transfer equivalence for continuous- and discrete-time nonlinear control systems. The necessary and sufficient condition for reducibility of nonlinear input-output equation is presented in terms of the greatest common left factor of two polynomials describing the behaviour of the 'tangent linearized system' equation. The procedure is given to find the reduced (irreducible) system equation that is transfer equivalent to the original system equation. Besides unification, the tools of pseudo-linear algebra allow to extend the results also for systems defined in terms of difference, q-shift and q-difference operators.

Keywords: nonlinear control systems, input-output models, reduction, pseudo-linear algebra, transfer equivalence

Classification: 93C10, 93B20, 93B25

1. INTRODUCTION

In every research field a moment arrives when there is a need to make the existing knowledge base more compact to facilitate the growth of the field. Nonlinear control theory that has been developed actively since mid 80-s is not an exception. There are many results in nonlinear control theory showing significant similarities in continuous- and discrete-time cases. The paper shows how to apply the pseudo-linear algebra to unify the study of continuous- and discrete-time nonlinear control systems to the general solutions of the problems from which the results for continuous- and discrete-time systems follow as special cases. This will be demonstrated on the example of the system reduction problem though we argue that the pseudo-linear algebra can be used to unify the study that relies on the 'tangent linearized system description', formulated in terms of differential one-forms [10], like accessibility, feedback linearization, inversion etc. We apply the pseudo-linear algebra to unify the results on reducibility, reduction and transfer equivalence for single-input single-output (SISO) continuous- and different discrete-time control system descriptions. In [20] for the same purpose the calculus on homogeneous time scales has been applied, and this motivates our choice since it allows to compare the two approaches providing tools for unification and extension. The main tool of the time scale calculus is a delta-derivative but that of the pseudo-linear algebra is a pseudo-linear map, including the delta-derivative as a special case but accommodating additionally a shift operator that is more conventional in control theory (unlike the difference operator being a special case of delta-derivative) and also a q-shift operator.

Pseudo-linear algebra [1, 5], alternatively called Ore algebra, provides the tools to study the common properties of linear differential, difference, shift and other type of operators, such as q-difference and q-shift operators, expressed in terms of the skew polynomials. These tools have been applied earlier for study of controllability of linear time-varying control systems, see [4, 7] and the references therein. For nonlinear control systems, pseudo-linear algebra was first used in the authors' conference paper [15], though its possibility was implicitly suggested already in [28]. Note that in order to make the tools of pseudo-linear algebra applicable for nonlinear systems, one has to find first the 'tangent linearized system' equations in terms of differential one-forms by applying the (Kähler) differential operator to the system equations. That way one can associate with the control system a pseudo-linear map, defined over the differential (or difference – in the discrete-time case) field of meromorphic functions in system variables (see [18, 31, 32]), pretty much in the similar manner it has been done in the linear time-varying case. Then, in the practical terms, checking irreducibility and finding the reduced system requires to find the greatest common left factor of two polynomials from the skew (Ore) polynomial ring, with the polynomial indeterminate being the pseudo-linear operator. The resulting time-varying linear models can be used for checking the solvability of various problems of interest but for finding the solutions one has to address the integrability aspects. However, even for the analysis, the application of the respective results from the linear time-varying theory requires special attention and is not, in most cases, directly applicable. The main source of the trouble comes from the fact that the polynomial coefficients as the time-varying functions are not free parameters, but the composite functions depending on time via the system's state, input and output variables, see also [27, 30].

To conclude, the main novelty of this paper, compared to [20] is that it allows to unify more cases into a single framework from which the continuous- and discrete-time results follow as special cases. Of course, [20] and this paper have many overlappings since both work with 'tangent linearized systems' and with polynomials from the skew (Ore) polynomial ring that act as operators on the differential one-forms, describing the 'tangent linearized system'[1]. However, these skew rings are different since the one in [20] does not accommodate the case when delta-derivative equals to zero, corresponding to the pure shift case. The latter implies that in this paper the multiplication rule in the polynomial ring may take a different form when compared to [20], and then of course, the computations with polynomials differ. Finally, the main results look like very similar, but otherwise, of course, one could not speak about unification. Note that there exist results in nonlinear control theory, relying on the 'tangent linearized system description', and not carrying over from continuous

[1]The 'tangent linearized system' approach together with the Ore rings has been used earlier in a number of papers devoted to the study of nonlinear control systems like [16, 18, 29, 31, 32]. However, these papers focus on the special cases of the Ore rings and do not address the unification power of the Ore ring.

to the discrete-time case, see for example [19]. Application of pseudo-linear algebra will help to reveal and explain such discrepancies between the results concerning the differential equations and their discrete-time analogues. A practical advantage of the unification of the solutions via the tools of pseudo-linear algebra is in the reduction of the implementation load since one may now write a single Maple function that covers all the special cases. The extra bonus is the existing software in Maple, in the form of built-in-packages like OreTools [1] and OreModules [8] that can handle a number of issues, related to Ore polynomials.

The paper is organized as follows. In Section 2 we give a brief exposition of the pseudo-linear algebra. Section 3 describes the control system in terms of two polynomials from the skew polynomial ring, being the pseudo-linear operators that act on differential one-forms. The results of this section may be also considered as an original contribution since up to now there does not exist a journal paper demonstrating *whether* and *how* the pseudo-linear algebra is applicable in the study of nonlinear control systems. In Section 4 the necessary and sufficient reducibility condition is given and in Section 5 the reduction procedure is described. Moreover, in this section the new definition of transfer equivalence of two systems is related explicitly to the equality of their (recently introduced) transfer functions, and the reduced system is shown to be transfer equivalent to the original system. In Section 6 three examples are given, and finally, Section 7 concludes the paper.

2. PSEUDO-LINEAR ALGEBRA

Definition 2.1. Let $\mathcal{K}$ be a field and $\sigma : \mathcal{K} \to \mathcal{K}$ an automorphism of $\mathcal{K}$. A map $\delta : \mathcal{K} \to \mathcal{K}$ which satisfies

$$\begin{aligned} \delta(\alpha+\beta) &= \delta(\alpha)+\delta(\beta) \\ \delta(\alpha\beta) &= \sigma(\alpha)\delta(\beta)+\delta(\alpha)\beta \end{aligned} \tag{1}$$

is called a pseudo- or σ-derivation[2].

Obviously, $\sigma(\alpha/\beta) = \sigma(\alpha)/\sigma(\beta)$ and $\delta(\alpha/\beta) = (\beta\delta(\alpha) - \alpha\delta(\beta))/(\sigma(\beta)\beta)$ for $\alpha, \beta \in \mathcal{K}$ with $\beta \neq 0$.

Definition 2.2. A σ-differential field is a triple $(\mathcal{K}, \sigma, \delta)$ where $\mathcal{K}$ is a field, σ is an automorphism of $\mathcal{K}$ and δ is a σ-derivation.

The σ-differential field $\mathcal{K}$ is the starting point for constructions used in characterizing reducibility property of different nonlinear control systems.

A *left* polynomial is an element which can be uniquely written in the form

$$a = \sum_{i=0}^{n} \alpha_i \partial^{n-i}, \quad \alpha_i \in \mathcal{K} \tag{2}$$

where ∂ is a formal variable (polynomial indeterminate) and $a \neq 0$ iff at least one of the functions α_i, $i = 0, \ldots, n$ is nonzero. If $\alpha_0 \neq 0$, then the positive integer n is

[2]Note that σ-derivation may be interpreted as the delta-derivative from the *homogeneous* time scale calculus.

called the *degree* of the left polynomial a and denoted by $\mathrm{d}^0(a)$. In addition, we set $\mathrm{d}^0(0) = -\infty$.

Any automorphism σ and σ-derivation δ induce a (left) non-commutative skew polynomial ring. The term skew means that the derivative (shift, difference) operator does not commute with every element in the coefficient field.

Definition 2.3. The left skew polynomial ring given by σ and δ is the ring $\mathcal{K}[\partial; \sigma, \delta]$ of polynomials in ∂ over $\mathcal{K}$ with the usual addition, and the (non-commutative) multiplication defined, for any $\alpha \in \mathcal{K}$, by the commutation rule

$$\partial \cdot \alpha = \sigma(\alpha) \cdot \partial + \delta(\alpha). \tag{3}$$

This rule can be uniquely extended to multiplication on monomials by

$$(\alpha\partial^n)(\beta\partial^m) = (\alpha\partial^{n-1})(\sigma(\beta)\partial^{m+1} + \delta(\beta)\partial^m),$$

and then to arbitrary polynomials by

$$\left(\sum_{i=0}^{n} \alpha_i \partial^{n-i}\right)\left(\sum_{j=0}^{m} \beta_j \partial^{m-j}\right).$$

A ring is called an *integral domain*, if it does not contain any zero divisors. This means that if a and b are two elements of ring such that $ab = 0$, then $a = 0$ or $b = 0$.

Proposition 2.4. (McConnell and Robson [24])

(i) The ring $\mathcal{K}[\partial; \sigma, \delta]$ is an integral domain.

(ii) If a and b are nonzero skew polynomials, then $\mathrm{d}^0(ab) = \mathrm{d}^0(a) + \mathrm{d}^0(b)$.

Moreover, the ring $\mathcal{K}[\partial; \sigma, \delta]$ satisfies the so-called (left) Ore condition.

Proposition 2.5. (Left Ore condition). (Ore [25]) For all non-zero $a, b \in \mathcal{K}[\partial; \sigma, \delta]$, there exist non-zero $a_1, b_1 \in \mathcal{K}[\partial; \sigma, \delta]$ such that $a_1 b = b_1 a$.

Elements of such a ring are called skew polynomials or non-commutative polynomials or Ore polynomials.

For any differential field $\mathcal{K}$ with a time-derivation $\delta = \frac{\mathrm{d}}{\mathrm{d}t}$, $\mathcal{K}[D; 1_{\mathcal{K}}, \delta]$ is the ring of linear ordinary differential operators. If σ is the automorphism over $\mathcal{K}$ which takes t to $t+1$, then $\mathcal{K}[E; \sigma, 0]$ is the ring of linear ordinary shift (recurrence) operators, while $\mathcal{K}[E; \sigma, \Delta]$, where $\Delta = \frac{1}{\mu}(\sigma - 1_{\mathcal{K}})$, $\mu \in I\!\!R$ is the ring of linear ordinary difference operators. If σ is the automorphism over $\mathcal{K}$ which takes t to qt, then $\mathcal{K}[Q; \sigma, \Delta]$ with $\Delta = \frac{1}{\mu}(\sigma - 1_{\mathcal{K}})$ is the ring of linear ordinary q-difference operators, see [1, 5].

Definition 2.6. Let V be a vector space over a field $\mathcal{K}$. A map $\theta : V \to V$ is called pseudo-linear if

$$\begin{aligned} \theta(w+v) &= \theta(w) + \theta(v) \\ \theta(\alpha w) &= \sigma(\alpha)\theta(w) + \delta(\alpha)w \end{aligned} \tag{4}$$

for any $\alpha \in \mathcal{K}$, $w, v \in V$.

Table 1. Basic types of operators.

Case	σ	δ	θ	$f(t)$
differential	$1_{\mathcal{K}}$	$\frac{\mathrm{d}}{\mathrm{d}t}$	δ	$\frac{\mathrm{d}f(t)}{\mathrm{d}t}$
shift	$t \to t+1$	0	σ	$f(t+1)$
difference	$t \to t+1$	Δ	δ	$\frac{1}{\mu}[f(t+1)-f(t)]$
q-shift	$t \to qt$	0	σ	$f(qt)$
q-difference	$t \to qt$	Δ	δ	$\frac{1}{\mu}[f(qt)-f(t)]$

Note that any field $\mathcal{K}$ is a vector space itself. Hence, we can consider pseudo-linear maps over $\mathcal{K}$ assuming that (4) holds for an $\alpha, w, v \in \mathcal{K}$. Obviously, any pseudo-derivation δ over $\mathcal{K}$ is a pseudo-linear map, *simply by letting* $\theta = \delta$. If $\delta = 0$ then $\theta = \sigma$ and (4) is clearly satisfied. Also a difference operator $\Delta = \frac{1}{\mu}(\sigma - 1_{\mathcal{K}})$, is a pseudo-linear map *by letting* $\theta = \Delta$. Note that (4) is again satisfied, since $\Delta(\alpha w) = \sigma(\alpha)\Delta(w) + \Delta(\alpha)w = \frac{1}{\mu}\sigma(\alpha)(\sigma(w)-w) + \frac{1}{\mu}(\sigma(\alpha)-\alpha)w = \frac{1}{\mu}[\sigma(\alpha w)-\alpha w]$. Thus, pseudo-linear maps allow to handle differential, shift and difference structures from a unified viewpoint.

The basic types of operators that can be addressed within the pseudo-linear algebra are listed in Table 1. Note that only the 1st, 3rd and 5th operator can be handled within the time scale formalism, whereas the 2nd and 4th cannot.

Any pseudo-linear map $\theta : V \to V$ *induces an action* denoted by $*$

$$* : \mathcal{K}[\partial, \sigma, \delta] \times V \to V; \left(\sum_{i=0}^{n} \alpha_i \partial^i\right) * w = \sum_{i=0}^{n} \alpha_i \theta^i(w)$$

for any $w \in V$. For the sake of simplicity, below the symbol $*$ is dropped. Multiplication in $\mathcal{K}[\partial; \sigma, \delta]$ corresponds to the composition of operators and $(rs)w = r(sw)$ for any $r, s \in \mathcal{K}[\partial; \sigma, \delta]$ and $w \in V$. So the elements of $\mathcal{K}[\partial; \sigma, \delta]$ can be viewed as operators acting on V.

An algebraic setting for dealing and studying theoretic properties of nonlinear control systems is often built up by introducing the notion of differential form. From that point of view it will be important to satisfy commutativity of differential operator d with a pseudo-linear operator θ, i.e. $\mathrm{d} \cdot \theta = \theta \cdot \mathrm{d}$. Note that for this purpose in case of $\Delta = \frac{1}{\mu}(\sigma - 1_{\mathcal{K}})$, the parameter μ is assumed to be a real number and hence the q-differential case, where $\sigma : t \to qt$ and $\delta = \frac{1}{qt-t}(\sigma - 1_{\mathcal{K}})$, does not accommodate into our approach.

3. CONTROL SYSTEMS

In the polynomial approach, SISO nonlinear control system is described by two skew polynomials that act as differential, difference or shift operators on input and output differentials [18, 20, 31, 32]. In this subsection we consider a wide class of nonlinear control systems and unify the polynomial formalism, replacing the differential, difference or shift operators in polynomials by a more general pseudo-linear map that accommodates all three cases and many more.

For the sake of simplicity, for $y(t)$ we write just y, and the symbol $y^{\langle 1\rangle}$ stands for a pseudo-linear operator: $y^{\langle 1\rangle} = \theta(y)$. It can be a derivation, $y^{\langle 1\rangle} = \dot{y}$, that corresponds to the continuous-time case, a shift $y^{\langle 1\rangle} = \sigma(y)$, or a difference, $y^{\langle 1\rangle} = \frac{1}{\mu}(\sigma(y) - y)$ with $\mu \in I\!R$, that correspond to two alternative discrete-time cases. Consider a nonlinear SISO control system, described by the i/o equation

$$y^{\langle n\rangle} = \phi\left(y, \dots, y^{\langle n-1\rangle}, u, \dots, u^{\langle s\rangle}\right) \tag{5}$$

where $u \in I\!R$ and $y \in I\!R$ denote the input and the output of the system and ϕ is a real analytic function, n and s are nonnegative integers such that $s < n$. Let $\varphi(y, \dots, y^{\langle n\rangle}, u, \dots, u^{\langle s\rangle}) := y^{\langle n\rangle} - \phi(y, \dots, y^{\langle n-1\rangle}, u, \dots, u^{\langle s\rangle})$. Then equation (5) can be rewritten as

$$\varphi(y, \dots, y^{\langle n\rangle}, u, \dots, u^{\langle s\rangle}) = 0. \tag{6}$$

Associate with system (5) the field $\mathcal{K}$ of meromorphic functions of the independent system variables $\{y^{\langle j\rangle}, 0 \le j \le n-1, u^{\langle k\rangle}, k \ge 0\}$. Assume that system (5) is generically submersive, i. e.

$$\partial\sigma^n(y)/\partial(y, u) \not\equiv 0 \tag{7}$$

is satisfied generically, i. e. almost everywhere except on a set of measure zero. Assumption (7) reduces to the well known condition in the case of discrete-time nonlinear systems when $y^{\langle 1\rangle} = \sigma(y)$ [23] and is trivially satisfied in the case of continuous-time nonlinear systems when $\sigma(y) = y$. Under (7), σ is an automorphism of $\mathcal{K}$. Let δ be a pseudo-derivation defined on $\mathcal{K}$. The field $\mathcal{K}$ can be endowed with σ-differential structure, determined by system equation (5) and the triple $(\mathcal{K}, \sigma, \delta)$ forms a σ-differential field. $\mathcal{K}$ is not inversive in general, i. e. some elements of $\mathcal{K}$ may not have preimages with respect to σ. Note that under assumption (7) there exists, up to an isomorphism, a unique σ-differential field $\mathcal{K}^*$ called the inversive closure of $\mathcal{K}$ [9]. Here we assume that the inversive closure is given and by abuse of notation we use the same symbol $\mathcal{K}$ for both. A construction of the inversive closure follows the same line as in [2, 21]; for the case ϕ in (5) being a rational function, a more detailed construction is given in [16].

The nonlinear system (5) can be represented in terms of two skew polynomials in the ring $\mathcal{K}[\partial; \sigma, \delta]$. By differentiating (5) we obtain,

$$\mathrm{d}y^{\langle n\rangle} - \sum_{i=0}^{n-1} \frac{\partial\phi}{\partial y^{\langle i\rangle}} \mathrm{d}y^{\langle i\rangle} = \sum_{i=0}^{s} \frac{\partial\phi}{\partial u^{\langle i\rangle}} \mathrm{d}u^{\langle i\rangle}$$

or, alternatively

$$p\mathrm{d}y = q\mathrm{d}u \tag{8}$$

where $p = \partial^n - \sum_{i=0}^{n-1} \frac{\partial\varphi}{\partial y^{\langle i\rangle}} \partial^i$ and $q = \sum_{i=0}^{s} \frac{\partial\varphi}{\partial u^{\langle i\rangle}} \partial^i$ and $p, q \in \mathcal{K}[\partial; \sigma, \delta]$, i. e. are polynomials over the σ-differential field $\mathcal{K}$.

Let $\sigma^n := \underbrace{\sigma \circ \sigma \circ \ldots \circ \sigma}_{n \text{ times}}$, and $\delta^n = \underbrace{\delta \circ \delta \circ \ldots \circ \delta}_{n \text{ times}}$.

We define a pseudo-linear operator θ, associated with system (5) and acting on $\mathcal{K}$, separately for derivation, shift and difference operators.

First, if $\sigma = 1_{\mathcal{K}}$ and $\delta = \mathrm{d}/\mathrm{d}t$, a pseudo-linear operator $\theta = \delta$ and

$$\delta\varphi(\{y^{\langle j\rangle}, u^{\langle k\rangle}\}) = \frac{\partial\varphi}{\partial y^{\langle j\rangle}}\delta y^{\langle j\rangle} + \frac{\partial\varphi}{\partial u^{\langle k\rangle}}\delta u^{\langle k\rangle} \tag{9}$$

where $\delta y^{\langle j\rangle} = y^{\langle j+1\rangle}$, $j = 0, \ldots, n-2$, $\delta u^{\langle k\rangle} = u^{\langle k+1\rangle}$, but $\delta y^{\langle n-1\rangle} = \phi(y, \ldots, y^{\langle n-1\rangle}, u, \ldots, u^{\langle s\rangle})$.

Second, if $\delta = 0$, a pseudo-linear operator $\theta = \sigma$ and

$$\sigma\varphi(\{y^{\langle j\rangle}, u^{\langle k\rangle}\}) = \varphi(\{\sigma y^{\langle j\rangle}, \sigma u^{\langle k\rangle}\}) \tag{10}$$

where $\sigma y^{\langle j\rangle} = y^{\langle j+1\rangle}$, $\sigma u^{\langle k\rangle} = u^{\langle k+1\rangle}$, $\sigma y^{\langle n-1\rangle} = \phi(y, \ldots, y^{\langle n-1\rangle}, u, \ldots, u^{\langle s\rangle})$.

Finally, if $\delta = \frac{1}{\mu}(\sigma - 1_{\mathcal{K}}) := \Delta$ with $\mu \in I\!R$, then a pseudo-linear operator $\theta = \Delta$ and

$$\Delta\varphi(\{y^{\langle j\rangle}, u^{\langle k\rangle}\}) = \frac{1}{\mu}\left[\varphi(\{\sigma y^{\langle j\rangle}, \sigma u^{\langle k\rangle}\}) - \varphi(\{y^{\langle j\rangle}, u^{\langle k\rangle}\})\right] \tag{11}$$

where $\sigma = \mu\Delta + 1_{\mathcal{K}}$, $\Delta y^{\langle j\rangle} = y^{\langle j+1\rangle}$, $\Delta u^{\langle k\rangle} = u^{\langle k+1\rangle}$, and $\Delta y^{\langle n-1\rangle} = \phi(y, \ldots, y^{\langle n-1\rangle}, u, \ldots, u^{\langle s\rangle})$.

Over the σ-differential field $\mathcal{K}$ one can define the vector space $\mathcal{E}$ of differential one-forms spanned by the differentials of elements $\mathcal{K}$, that is $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\xi; \xi \in \mathcal{K}\}$. Any element $v \in \mathcal{E}$ is a vector of the form $v = \sum_{i=1}^{s}\gamma_i\mathrm{d}\xi_i$ where all $\gamma_i \in \mathcal{K}$. We say that $v \in \mathcal{E}$ is exact if $v = \mathrm{d}\varphi$ for some $\varphi \in \mathcal{K}$.

For $F \in \mathcal{K}$ we define $\mathrm{d} : \mathcal{K} \to \mathcal{E}$ (called the differential of F) as follows[3]

$$\mathrm{d}F := \sum_{i=0}^{n-1}\frac{\partial F}{\partial y^{\langle i\rangle}}\mathrm{d}y^{\langle i\rangle} + \sum_{l=0}^{k}\frac{\partial F}{\partial u^{\langle l\rangle}}\mathrm{d}u^{\langle l\rangle}.$$

The vector space $\mathcal{E}$ can also be endowed with the σ-differential structure. However, this time there is no need to define actions separately. Each pseudo-linear operator $\theta : \mathcal{K} \to \mathcal{K}$ induces pseudo-linear operator $\theta : \mathcal{E} \to \mathcal{E}$ as follows

$$\theta(v) = v^{\langle 1\rangle} = \sum_i[\sigma(\gamma_i)\mathrm{d}(\theta(\xi_i)) + \delta(\gamma_i)\mathrm{d}\xi_i]. \tag{12}$$

The operator θ commutes with operator d, $\theta(\mathrm{d}\varphi) = \mathrm{d}(\theta(\varphi))$, and reduces to the well-known rules $\delta v = \sum_i[\gamma_i\mathrm{d}(\delta\xi_i) + \delta(\gamma_i)\mathrm{d}\xi_i]$ and $\sigma v = \sum_i\sigma(\gamma_i)\mathrm{d}(\sigma\xi_i)$, for the special cases of continuous-time systems ($\sigma = 1_{\mathcal{K}}$, $\theta = \delta = \mathrm{d}/\mathrm{d}t$) and discrete-time systems ($\delta = 0, \theta = \sigma$), respectively. Now it is clear why μ in (11) has to be a real constant, since otherwise $\Delta(\mathrm{d}\varphi) \neq \mathrm{d}(\Delta\varphi)$, or going into details $\frac{1}{\mu}(\mathrm{d}(\sigma\varphi) - \mathrm{d}\varphi) \neq \mathrm{d}(\frac{1}{\mu}(\sigma\varphi - \varphi))$. Using the induction principle one can prove the following lemma.

[3]Note that in case of rational functions $F(\cdot)$, $\mathrm{d}F$ may be understood as Kähler differential [17], see [16]. The latter applies sometimes in more general cases too, see Remark C1 in [11].

Lemma 3.1. Let $F \in \mathcal{K}$ and $i \in I\!N$. Then $\theta^i(\mathrm{d}F) = \mathrm{d}F^{\langle i \rangle}$.

We briefly demonstrate some basic computations in $(\mathcal{K}, \sigma, \delta)$ and $\mathcal{E}$ by the following example.

Example 3.2. Consider the nonlinear q-difference system with $q = 2$,

$$y(2t) - y(t) = y(t)u(t),$$

rewritten in terms of the pseudo-linear operator as

$$y^{\langle 1 \rangle} = \Delta y = yu. \tag{13}$$

The σ-differential field associated with (13) is $(\mathcal{K}, \sigma, \Delta)$, where σ takes t to $2t$, $\Delta = \sigma - 1_{\mathcal{K}}$, and the pseudo-linear operator $\theta = \Delta$. Note that $(\mathcal{K}, \sigma, \Delta)$ has σ-differential structure, defined by the system equation. If for instance $\varphi = y^2$, then $\varphi^{\langle 1 \rangle} = \Delta\varphi = (\sigma y)^2 - y^2 = (\Delta y + y)^2 - y^2 = (yu + y)^2 - y^2$.

Also σ-differential structure of $\mathcal{E}$ is defined by the system equations. If for instance $v = 2u\mathrm{d}y$, then $v^{\langle 1 \rangle} = \Delta v = 2\sigma(u)\mathrm{d}\sigma(y) - 2u\mathrm{d}y = 2(\Delta u + u)\mathrm{d}(\Delta y + y) - 2u\mathrm{d}y = 2(u^{\langle 1 \rangle} + u)\mathrm{d}(yu + y) - 2u\mathrm{d}y$, or if computed directly by (12), $v^{\langle 1 \rangle} = 2\sigma(u)\mathrm{d}(\Delta y) + 2\Delta(u)\mathrm{d}y = 2(u^{\langle 1 \rangle} + u)\mathrm{d}(yu) + 2u^{\langle 1 \rangle}\mathrm{d}y$ which yields the same result.

Since $\mathcal{K}[\partial; \sigma, \delta]$ is an Ore ring, one can construct the division ring of fractions. If $p = p_1 p_2$ and $\mathrm{d}^0(p_1) > 0$, then p_1 is called a *left divisor* of p and p is called *left divisible* by p_1. If for $p_1, p_2 \in \mathcal{K}[\partial; \sigma, \delta]$, p_c is a left divisor of $p_1 - p_2$, then p_c is called a *common left divisor* of p_1 and p_2. If the degree of p_c is the greatest of all common left divisors of $p_1 - p_2$, then p_c is called the *greatest common left divisor* (gcld).

To find the gcld one can use the *left Euclidean division algorithm* [5]. To perform the left Euclidean division algorithm it is sufficient that σ be an automorphism (that holds under Assumption (7)). For given two polynomials p_1 and p_2 with $\mathrm{d}^0(p_1) > \mathrm{d}^0(p_2)$ there exist a unique polynomial γ_1 and a unique left remainder polynomial p_3 such that

$$p_1 = p_2\gamma_1 + p_3, \quad \mathrm{d}^0(p_3) < \mathrm{d}^0(p_2).$$

Using the (left) Euclidean division algorithm, after $k - 1$ steps, we obtain

$$\begin{array}{rcl} p_2 & = & p_3\gamma_2 + p_4 \\ & \vdots & \\ p_{k-2} & = & p_{k-1}\gamma_{k-2} + p_k \\ p_{k-1} & = & p_k\gamma_{k-1}. \end{array}$$

Hence the gcld of p_1 and p_2 is p_k. Moreover, eliminating polynomials $p_{k-1}, \ldots, p_3$ we get the *Bézout identity*, i. e. there exist polynomials $r, s \in \mathcal{K}[\partial; \sigma, \delta]$ such that $p_1 r + p_2 s = p_k$. The gcld is only unique up to multiplication of it by a functions from $\mathcal{K}$, but it can be made unique by requiring it to be monic. Thus for two polynomials $\tilde{p}_1$ and $\tilde{p}_2$ we have

$$p_1 = p_k\tilde{p}_1, \qquad p_2 = p_k\tilde{p}_2.$$

If $\mathrm{d}^0(p_k) = 0$, then the polynomials p_1 and p_2 are called *left coprime* (or alternatively, *relatively left prime*).

4. REDUCIBILITY CONDITION

The reduction of an input-output (i/o) differential (or difference in the discrete-time case) equation is an important subtask in finding the minimal, i.e. accessible and observable realization of the nonlinear control system. Denote $f^{\langle i \ldots k\rangle} := (f^{\langle i\rangle}, \ldots, f^{\langle k\rangle})$.

Definition 4.1. A function $\varphi_r \not\equiv$ constant in $\mathcal{K}$, such that $\varphi_r(0,\ldots,0) = 0$, is said to be an *autonomous variable* for control system (5) if there exist an integer $\nu \geq 1$ and a non-zero analytic function F so that

$$F\left(\varphi_r^{\langle 0 \ldots \nu\rangle}\right) = 0. \tag{14}$$

Definition 4.2. Control system (5) is said to be *irreducible* if there does not exist any non-zero autonomous variable in $\mathcal{K}$. Otherwise system (5) is called *reducible*.

If system (5) is reducible, then there exist an autonomous variable $\varphi_r = \varphi_r\left(y^{\langle 0..r\rangle}, u^{\langle 0..l\rangle}\right)$ with $r < n$ and a non-zero analytic function F such that (6) can be rewritten as

$$\varphi = kF\left(\varphi_r^{\langle 0..\nu\rangle}\right) = 0,$$

where $\nu \geq 1$ and k is a non-zero element of $\mathcal{K}$. Since $\varphi_r \not\equiv$ const and $\nu \geq 1$, then $r \geq 1$ and $\nu + r \geq n$.

The necessary and sufficient condition for reducibility of system (5) is given below and expressed in terms of the subspace $\mathcal{H}_\infty$ of $\mathcal{H}_1 := \operatorname{span}_{\mathcal{K}}\{\mathrm{d}y^{\langle i\rangle}, \mathrm{d}u^{\langle j\rangle}, 0 \leq i \leq n-1, 0 \leq j \leq s\} \subset \mathcal{E}$ defined by

$$\mathcal{H}_\infty := \operatorname{span}_{\mathcal{K}}\{\omega : \omega^{\langle k\rangle} \in \mathcal{H}_1, \forall k \geq 0\}. \tag{15}$$

Theorem 4.3. The system (5) is reducible iff $\mathcal{H}_\infty \neq \{0\}$.

Proof. Follows directly from the proofs of Lemma 9 and Proposition 4.4 in [6] and [2], respectively. □

The next theorem gives the alternative necessary and sufficient condition for reducibility of (5) in terms of differential polynomials from the ring $\mathcal{K}[\partial; \sigma, \delta]$.

Theorem 4.4. The control system (5) is reducible in the sense of Definition 4.2 if and only if the polynomials p and q in (8), associated with system (5) are not left coprime[4].

[4]Of course, one may take a step further like in Proposition 6.5 of [16] and prove that the module $\mathcal{E}$ is torsion-free iff the (monic) greatest common left divisor of p and q is 1, getting that way an intrinsic (and not equation-based as in this paper) reducibility condition as suggested by [12, 13] for linear systems. We have not done this since the focus of our paper is on unification, including unification of the computations and in order to verify whether the module is torsion-free, one may check whether p and q have a trivial greatest common left divisor.

P r o o f. *Necessity.* Suppose the system (5) is reducible, meaning that there exist functions φ_r and F such that (14) holds. Note that

$$\mathrm{d}\varphi_r = \sum_{i=0}^{r} \frac{\partial \varphi_r}{\partial y^{\langle i \rangle}} \mathrm{d}y^{\langle i \rangle} + \sum_{j=0}^{l} \frac{\partial \varphi_r}{\partial u^{\langle j \rangle}} \mathrm{d}u^{\langle j \rangle}. \tag{16}$$

Let $\tilde{\varphi} := F\left(\varphi_r^{\langle 0..\nu \rangle}\right)$. Then $\tilde{\varphi} \in \mathcal{K}$ and

$$\mathrm{d}\tilde{\varphi} = \sum_{k=0}^{\nu} \frac{\partial F}{\partial \varphi_r^{\langle k \rangle}} \mathrm{d}\varphi_r^{\langle k \rangle}. \tag{17}$$

By Lemma 3.1, $\partial^i \mathrm{d}y = \mathrm{d}y^{\langle i \rangle}$, $\partial^j \mathrm{d}u = \mathrm{d}u^{\langle j \rangle}$ and $\partial^k \mathrm{d}\varphi_r = \mathrm{d}\varphi_r^{\langle k \rangle}$ allowing to rewrite (16) and (17) in terms of differential polynomials in operator ∂ over $\mathcal{K}$:

$$\mathrm{d}\varphi_r = \left(\sum_{i=0}^{r} \frac{\partial \varphi_r}{\partial y^{\langle i \rangle}} \partial^i \right) \mathrm{d}y + \left(\sum_{j=0}^{l} \frac{\partial \varphi_r}{\partial u^{\langle j \rangle}} \partial^j \right) \mathrm{d}u := \tilde{p}\mathrm{d}y - \tilde{q}\mathrm{d}u \tag{18}$$

and

$$\mathrm{d}\tilde{\varphi} = \left(\sum_{k=0}^{\nu} \frac{\partial F}{\partial \varphi_r^{\langle k \rangle}} \partial^k \right) \mathrm{d}\varphi_r := \varsigma \mathrm{d}\varphi_r \tag{19}$$

Then the equation $\mathrm{d}\tilde{\varphi} = 0$ can be rewritten as

$$\varsigma \left[\tilde{p}\mathrm{d}y - \tilde{q}\mathrm{d}u \right] = 0, \tag{20}$$

where $\mathrm{d}^0(\varsigma) = \nu \geq 1$ and $\mathrm{d}^0(\tilde{p}) = r < n$. Moreover, $n = \mathrm{d}^0(p) \leq \mathrm{d}^0(\varsigma \cdot \tilde{p}) = \nu + r$, so in the similar manner as in the proof of Lemma 6.5 in [16] the right-hand division yields $\varsigma\tilde{p} = \alpha p + r$ with $\mathrm{d}^0(r) < n$. Since $p\mathrm{d}y = q\mathrm{d}u$ and $\varsigma\tilde{p}\mathrm{d}y = \varsigma\tilde{q}\mathrm{d}u$, $r\mathrm{d}y = [\varsigma\tilde{q} - \alpha q]\mathrm{d}u$. This gives rise to a linear relation among the vectors $\{\mathrm{d}y, \ldots, \mathrm{d}y^{\langle n-1 \rangle}, u^{\langle k \rangle}, k \geq 0\}$ because $\mathrm{d}^0(r) < n$. Hence $r = 0$ and $\varsigma\tilde{q} = \alpha q$. Consequently, $\varsigma\tilde{p} = \alpha p$. The remaining part of the proof is by contradiction. Suppose that polynomials p and q in (8) are left coprime. Then α is the greatest common left divisor of polynomials αp, αq and from Bézout identity there exist polynomials $u, v \in \mathcal{K}[\partial; \sigma, \delta]$ such that $\alpha p u + \alpha q v = \alpha$. Then we get $\varsigma\tilde{p}u + \varsigma\tilde{q}v = \alpha$. Therefore, there exist $\chi = \tilde{p}u + \tilde{q}v \in \mathcal{K}[\partial; \sigma, \delta]$ such that $\alpha = \xi\chi$. Hence $\xi\tilde{p} = \xi\chi p$ and consequently, $\tilde{p} = \chi p$ yielding $\mathrm{d}^0(\tilde{p}) \geq \mathrm{d}^0(p)$, a contradiction to $\mathrm{d}^0(\tilde{p}) = r < n = \mathrm{d}^0(p)$. Hence the polynomials p and q have a common left divisor.

Sufficiency. Suppose the polynomials p, q with $\mathrm{d}^0(p) = n$ and $\mathrm{d}^0(q) = s$ have a common left factor ς with $\mathrm{d}^0(\varsigma) = \nu > 0$. Then the equation (8) can be rewritten as

$$p\mathrm{d}y - q\mathrm{d}u = \varsigma \left[\tilde{p}\mathrm{d}y - \tilde{q}\mathrm{d}u \right] = 0 \tag{21}$$

where $\mathrm{d}^0(\tilde{p}) = n - \nu$, $\mathrm{d}^0(\tilde{q}) = s - \nu$, $\varsigma = \varsigma_\nu \partial^\nu + \ldots + \varsigma_1 \partial + \varsigma_0$ with $\varsigma_\nu \neq 0$ and $\varsigma_i \in \mathcal{K}$ for $0 \leq i \leq \nu$.

We will show that the one-form $\omega := \tilde{p}\mathrm{d}y - \tilde{q}\mathrm{d}u$ belongs to the subspace $\mathcal{H}_\infty$. From (21),

$$\varsigma\omega = \sum_{i=0}^{\nu} \varsigma_i \omega^{\langle i \rangle} = 0,$$

showing that the one-forms $\{\omega^{\langle i \rangle}, 0 \leq i \leq \nu\}$ are linearly dependent over $\mathcal{K}$ and

$$\omega^{\langle \nu \rangle} = \sum_{i=0}^{\nu-1} \frac{\varsigma_i}{\varsigma_\nu} \omega^{\langle i \rangle}.$$

Using the induction principle one can prove that there exist $\beta_{ik} \in \mathcal{K}$, $i = 0, \ldots, \nu-1$ such that

$$\omega^{\langle k \rangle} = \sum_{i=0}^{\nu-1} \beta_{ik} \omega^{\langle i \rangle} \quad \forall k \geq 0. \tag{22}$$

Since $\mathrm{d}^0(\tilde{p}) = n - \nu$ and $\mathrm{d}^0(\tilde{q}) = s - \nu$, the one-forms $\omega^{\langle k \rangle} \in \mathcal{H}_1$, for $k = 0, \ldots, \nu$ yielding $\omega^{\langle k \rangle} \in \mathcal{H}_1$, for all $k \geq 0$. Thus, $\omega \in \mathcal{H}_\infty$ by the definition of $\mathcal{H}_\infty$, see (15). Hence, by (22) $\mathcal{H}_\infty \neq \{0\}$, and from Theorem 4.3 system (5) is reducible. □

5. REDUCTION AND TRANSFER EQUIVALENCE

If the polynomials p and q have the gcld χ, then the differential one-form $\omega = \tilde{p}\mathrm{d}y - \tilde{q}\mathrm{d}u$ (see (18)) is either exact or can be made exact by multiplying it with an integrating factor, see [10, 20, 23] for the continuous-time and two discrete-time cases, shift and difference operator based cases, respectively. Then there exist functions $\alpha \in \mathcal{K}$ and $\varphi_{ir} \in \mathcal{K}$ such that $\alpha\omega = \mathrm{d}\varphi_{ir}$. Because χ is the gcld of p and q, polynomials $\tilde{p}$ and $\tilde{q}$ are left coprime. In analogy with the continuous-time case [10] we call $\mathrm{d}\varphi_{ir}$ an *irreducible differential form* and $\varphi_{ir}(\cdot) = c$ an *irreducible i/o equation* of system (5).

Proposition 5.1. In the irreducible equation the highest order of the pseudo-linear operator acting on input u is strictly lower than that acting on output y.

Proof. The proof is by contradiction, yielding $s \geq n$. □

Since the mathematical tools we employ require that instead of working with the equations themselves we work with their (Kähler) differentials, the systems $\varphi(\cdot) = 0$ and $\varphi(\cdot) + \text{constant} = 0$ are indistinguishable for an arbitrary constant. Of course, at the last step when the irreducible one-form has to be integrated to obtain the irreducible system equation, one has to specify the integrating constant. In general, this constant can be defined by the solution (equilibrium point) of the system, assuming that the reduced system is not allowed to have an equilibrium point not shared with the original equation. Then the one-forms will be integrated around the shared equilibrium point to get the reduced system equations. Typically, e. g. in [10], one takes this constant equal to zero. This choice may yield a situation, described in [10] as 'the system of the form (5) under study *does not admit an irreducible form*' meaning that one cannot decide whether the system under consideration is irreducible or not (see Example 6.2 below). The proposition below shows that generically the reduced equation satisfies also the submersivity assumption and can be expressed in the form (5).

Proposition 5.2. The system (5) in the neighbourhood of a generic solution admits an irreducible i/o equation $\varphi_{ir} = c$, that can be locally uniquely expressed (up to an integrating constant c) in the explicit form (5), satisfying assumption (7).

Proof. The proof relies on the fact that, according to the proof of Theorem 4.4, $\mathrm{d}\varphi(\cdot) = 0$ can be rewritten in the form $\mathrm{d}\varphi = p\mathrm{d}y - q\mathrm{d}u = \chi\mathrm{d}\varphi_{ir}$, where χ (unique, if required to be monic) is the gcld of two polynomials p and q. To obtain the irreducible i/o equation, we take $\varphi_{ir} = c$. In the neighbourhood of a generic point $\varphi_{ir} - c = 0$ can be expressed in the form (5), since otherwise the original equation would not have a form (5), which implies uniqueness. Now assume that (7) is not satisfied for the irreducible equation. Then the original equation does not satisfy assumption (7) either. $\square$

The system reduction, based on the notion of the autonomous element, is closely related to the concept of transfer equivalence of nonlinear control systems.

Definition 5.3. (Transfer equivalence) (Conte et al. [10], Kotta et al. [23])[5] Two systems Σ_1 and Σ_2 (that admit an irreducible i/o equation) are said to be transfer equivalent if they have the same irreducible i/o equation.

In the linear case all systems admit an irreducible i/o equation and the definition above coincides with the classical definition, saying that two systems are transfer equivalent if their respective transfer functions are equal. However, at the time when Definition 5.3 was introduced, the notion of transfer function has not yet been extended for nonlinear control systems. Since now this extension exists [14], our goal is to reformulate Definition 5.3 in such a way that its dependance on transfer function becomes explicit. For that, we first recall the definition of the transfer function [15] for nonlinear control systems, defined in terms of the pseudo-linear operator. Note that for continuous-time nonlinear systems the new definition was already suggested in [26], but again, called the input-output equivalence.

By Proposition 2.5 each two elements of $\mathcal{K}[\partial;\sigma,\delta]$ have a common left multiple, and $\mathcal{K}[\partial;\sigma,\delta]$ can be embedded into a non-commutative quotient field [25] by defining quotients as $b^{-1}\cdot a$ where $a, b \in \mathcal{K}[\partial;\sigma,\delta]$ and $b \neq 0$. Addition is defined by reducing two quotients to the same denominator $b_1^{-1}a_1 + b_2^{-1}a_2 = (d_2b_1)^{-1}(d_2a_1 + d_1a_2)$ where $d_2b_1 = d_1b_2$ by Ore condition. Multiplication is defined by $(b_1^{-1}a_1)(b_2^{-1}a_2) = (d_2b_1)^{-1}c_1a_2$, where $d_2a_1 = c_1b_2$ again by Ore condition. The resulting quotient field of skew polynomials is denoted by $\mathcal{K}(\partial;\sigma,\delta)$. Once a fraction of two skew

[5]Note, however that in [10] the equivalence in the sense of Definition 5.3 is called input-output (i/o) equivalence. We prefer to call it transfer equivalence since the latter is consistent with the linear case.

polynomials is defined, one can introduce a transfer function of nonlinear control system.

Definition 5.4. (Halás and Kotta [15]) An element $F(\partial) \in \mathcal{K}(\partial; \sigma, \delta)$ such that $\mathrm{d}y = F(\partial)\mathrm{d}u$ is said to be a transfer function of the SISO nonlinear system (5).

So, for polynomial system description (8), the transfer function $F(\partial) = p^{-1}q$.

Though one can always associate to a proper rational function $F(\partial) = p^{-1}q$ a corresponding input-output differential form, $\omega = p\mathrm{d}y - q\mathrm{d}u$, this one-form is not necessarily integrable. If the i/o differential form is integrable or can be made integrable multiplying it by an integrating factor, then there exists an input-output equation of the form (5) such that the transfer function of this i/o equation is $F(\partial)$. In other words, every control system can be expressed as a quotient of the skew polynomials, but not every quotient of skew polynomials necessarily represents a control system.

Observe that in case the systems Σ_1 and Σ_2 have the same irreducible differential form, their transfer functions are equal. Really, consider two systems, Σ_1 being reducible and Σ_2 irreducible but having the same irreducible differential form $\omega = \tilde{p}\mathrm{d}y - \tilde{q}\mathrm{d}u$. Then $F_{\Sigma_2}(\partial) = \tilde{p}^{-1}\tilde{q}$ and $F_{\Sigma_1}(\partial) = p^{-1}q = (\varsigma\tilde{p})^{-1}(\varsigma\tilde{q}) = \tilde{p}^{-1}\tilde{q} = F_{\Sigma_2}(\partial)$. This leads us to the alternative definition of transfer equivalence directly in terms of the transfer functions.

Definition 5.5. (Transfer equivalence). Two systems Σ_1 and Σ_2 are transfer equivalent if they have the same irreducible differential form, or said alternatively, have the same transfer functions.

6. EXAMPLES

Example 6.1. Consider a system described by the i/o equation

$$y^{\langle 2\rangle} - (yu)^{\langle 1\rangle} + y^{\langle 1\rangle} - uy = 0. \tag{23}$$

By applying the differential operator d for (23) and taking into account that by (4) $(yu)^{\langle 1\rangle} = \sigma(u)y^{\langle 1\rangle} + \delta(u)y$, the equation (8) takes the form

$$[\partial^2 + (1 - \sigma(u))\partial - (\delta(u) + u)]\mathrm{d}y = [\sigma(y)\partial + \delta(y) + y]\mathrm{d}u. \tag{24}$$

Applying the Euclidean division algorithms for p and q, we get

$$p = q\left(\frac{1}{y}\partial - \frac{u}{y}\right)$$

iff $y \neq 0$. So q is the gcld of p and q. Hence, system (23) is reducible and $\tilde{p} = \frac{1}{y}\partial - \frac{u}{y}$, $\tilde{q} = 1$. Then

$$\mathrm{d}\varphi_r = \left(\frac{1}{y}\partial - \frac{u}{y}\right)\mathrm{d}y - \mathrm{d}u$$

or alternatively $\mathrm{d}[y^{\langle 1\rangle} - yu] = 0$. The autonomous variable for system (23) is the function $\varphi_r(y, y^{\langle 1\rangle}, u) = y^{\langle 1\rangle} - yu$ and (23) may be rewritten as $\varphi_r^{\langle 1\rangle} + \varphi_r = 0$. Finally, note that the reduced system is $y^{\langle 1\rangle} = yu$.

The general solution $y^{\langle 1\rangle} = yu$ yields $\dot{y} = yu$ or $\sigma(y) = yu$ in the continuous- and shift-operator-based discrete-time cases, respectively. Below we will demonstrate that the computations, when done separately for these cases, are consistent with the general solution. Assume first that $\sigma(y) = y$ and $\partial := \theta = \delta$, that corresponds to the continuous-time case and the field $\mathcal{K}[\partial, 1_{\mathcal{K}}, \delta]$. Then by applying the d operator for

$$\ddot{y} - \dot{y}u - \dot{u}y + \dot{y} - yu = 0 \tag{25}$$

being the special case of (23), (8) takes now the form

$$[\partial^2 + (1-u)\partial - (\dot{u}+u)]\mathrm{d}y = [y\partial + \dot{y} + y]\mathrm{d}u. \tag{26}$$

Applying the left Euclidean division algorithm, we get

$$p = q\left(\frac{1}{y}\partial - \frac{u}{y}\right),$$

if only $y \neq 0$. So $q = y\partial + (y + \dot{y})$ is the gcld of $p = \partial^2 + (1-u)\partial - (\dot{u}+u)$ and q. Then $\left(\frac{1}{y}\partial - \frac{u}{y}\right)\mathrm{d}y = \mathrm{d}u$, or alternatively, $\mathrm{d}[\dot{y} - yu] = 0$. Hence, system (25) is reducible. The autonomous variable is the function $\varphi_r(y, \dot{y}, u) = \dot{y} - yu$ and (25) can be rewritten as $\dot{\varphi}_r + \varphi_r = 0$.

Assume now that $\delta = 0$ and $\partial := \theta = \sigma$ that corresponds to the discrete-time shift-operator-based case, and the field $\mathcal{K}[\partial; \sigma, 0]$. By applying the d operator for $\sigma^2(y) - \sigma(y)\sigma(u) + \sigma(y) - uy = 0$, being the special case of (23), equation (8) takes now the form

$$[\partial^2 + (1-\sigma(u))\partial - u]\mathrm{d}y = [\sigma(y)\partial + y]\mathrm{d}u. \tag{27}$$

Applying the left Euclidean division algorithm, we get that the gcld of p and q is $\sigma(y)\partial + y$ and $\tilde{p} = \frac{1}{y}\partial - \frac{u}{y}$, $\tilde{q} = 1$. The reduced system reads as $\sigma(y) = yu$.

The generically reducible system (5) may not admit irreducible i/o equation for every constant c.

Example 6.2. Consider the nonlinear system, described by the i/o equation

$$\varphi = uy^{\langle 2\rangle} - u^{\langle 1\rangle}y^{\langle 1\rangle} = 0.$$

Note that φ has been obtained, for $u \neq 0$ and $\sigma(u) \neq 0$, as $u\sigma(u)[(\partial - 1^{\langle 1\rangle})]\varphi_{ir}$, where $\varphi_{ir} = y^{\langle 1\rangle}/u$. If we make the computations, we get the reduced differential form

$$\mathrm{d}\varphi_{ir} = \mathrm{d}\left(\frac{y^{\langle 1\rangle}}{u}\right).$$

The typical choice $c = 0$ does not work here. In [10, 21] in such a situation it is said that the system 'does not admit the irreducible form', meaning that one can not decide whether the system under consideration is irreducible or not. The choice $c = 0$ comes from the assumption $\varphi(0, \ldots, 0) = 0$ which is natural in the linear setting but not, in general, in the nonlinear case. For example, assuming $\varphi(1,1,1,1)=0$ yields[6] $c=1$, and the reduced system equation $y^{\langle 1\rangle}=u$.

Note that, in general, the system may be reducible in one ring of polynomials but not in the other(s). The next example servers as a demonstration.

Example 6.3. Consider the nonlinear system, described by the i/o equation

$$y^{\langle 2\rangle}-y^{\langle 1\rangle}u^{\langle 1\rangle}+y^{\langle 1\rangle}-yu=0. \tag{28}$$

By applying the differential operator d for (28) we obtain equation (8) with

$$p=\partial^2+(1-u^{\langle 1\rangle})\partial-u,\quad q=y^{\langle 1\rangle}\partial+y.$$

Applying the left Euclidean division algorithm, we obtain $p=q\gamma+r$ with $\gamma=\gamma_1\partial+\gamma_2$ and $r=r_1$. Next, note that

$$\begin{aligned}q\gamma+r &= (y^{\langle 1\rangle}\partial+y)(\gamma_1\partial+\gamma_2)+r_1= \\ &= y^{\langle 1\rangle}\partial(\gamma_1)\partial+y^{\langle 1\rangle}\partial(\gamma_2)+y\gamma_1\partial+y\gamma_2+r_1.\end{aligned}$$

Finally, using the relationship (3), we get

$$q\gamma+r=y^{\langle 1\rangle}\sigma(\gamma_1)\partial^2+(y\gamma_1+y^{\langle 1\rangle}\sigma(\gamma_2)+y^{\langle 1\rangle}\delta(\gamma_1))\partial+y\gamma_2+y^{\langle 1\rangle}\delta(\gamma_2)+r_1.$$

Matching the terms of the same power of the polynomial indeterminate ∂, we get a system of equations for γ_1, γ_2 and r_1

$$\begin{aligned}y^{\langle 1\rangle}\sigma(\gamma_1) &= 1\\ y\gamma_1+y^{\langle 1\rangle}\sigma(\gamma_2)+y^{\langle 1\rangle}\delta(\gamma_1) &= 1-u^{\langle 1\rangle}\\ y\gamma_2+y^{\langle 1\rangle}\delta(\gamma_2)+r_1 &= -u.\end{aligned} \tag{29}$$

Now, if we consider the shift operator based discrete-time case when one interprets ∂ as $\theta=\sigma$, $\delta=0$, the equations (29) take the form

$$\begin{aligned}\sigma(y)\sigma(\gamma_1) &= 1\\ y\gamma_1+\sigma(y)\sigma(\gamma_2) &= 1-\sigma(u)\\ y\gamma_2+r_1 &= -u\end{aligned}$$

yielding a solution $\gamma_1=1/y$, $\gamma_2=-u/y$ and $r_1=0$. The latter means that the gcld of p and q is γ, the system (28) is reducible and the irreducible system equation is $\sigma(y)=uy$.

However, if we consider the continuous-time case when one interprets ∂ as $\theta=\delta=\frac{\mathrm{d}}{\mathrm{d}t}$ and $\sigma=1_{\mathcal{K}}$, the equations (29) take the form

$$\begin{aligned}\dot y\gamma_1 &= 1\\ y\gamma_1+\dot y\gamma_2+\dot y\dot\gamma_1 &= 1-\dot u\\ y\gamma_2+\dot y\dot\gamma_2+r_1 &= -u\end{aligned}$$

yielding $r_1\neq 0$.

[6] One assumes here that the equilibrium point of the reduced system is also an equilibrium point of the original system.

To conclude, since the shift and derivation operators have remarkably different properties, when the computations are made by hand it is easier to make the computations separately in the rings $\mathcal{K}[\partial;\sigma,0]$ and $\mathcal{K}[\partial;1_{\mathcal{K}},\delta]$. However, the widespread use of computers promotes the idea that modern mathematical tools are those that can be easily transformed into a computer program. The important point to be mentioned here is that there is a big difference in the concept of simplicity for human being and for computer algebra system. Moreover, it is important to stress that behind the separate computations exists a general mathematical abstraction that accommodates all the cases. So, one can always apply the general algorithms to get the general solutions to the general problems, and later just specify the given σ and δ in order to get from the general solution the one of the interest. This has an advantage from the point of view of possible implementation of the procedures, for example, in the computer algebra system Maple, since in our setting, one may now write a single general Maple function for a general Ore ring without the advance knowledge on σ and δ and then simply specify the meaning of the operators σ and δ when calling this Maple function.

7. CONCLUSIONS

In this paper, the algebraic formalism of differential one-forms and the related polynomial approach was unified and extended for a wide class of nonlinear control systems. In the unification and extension pseudo-linear algebra played a key role. Though differential, shift, difference, q-shift and q-difference operators have remarkably different properties, they all accommodate into this mathematical abstraction as the special cases. However, note that our approach does not accommodate q-differential operator since it does not commute with the differential operator d. The tools introduced were applied to study the problems of reducibility, reduction and transfer equivalence. Our results include the earlier results for nonlinear discrete- and continuous-time cases [18, 31, 32] respectively and are similar to those of linear theory.

Note that another approach was suggested in the literature for unification and extension of the study of irreducibility and reduction problems [20]. The latter approach relies on system description on homogeneous time scale that accommodates the continuous-time systems and the discrete-time systems described in terms of difference operator. Actually, time scale formalism allows to unify only the study of systems described in terms of pseudo-derivation, therefore leaving out the more conventional system descriptions in terms of the different shift operators. To conclude, in comparison with the homogeneous time-scale formalism, pseudo-linear algebra is favorable since

- it accommodates more cases into a single framework, allowing to reduce the implementation load
- there exists a built-in-package, OreTools [1] that addresses the computations with Ore polynomials and can be a good starting point for control application related implementations

- it is a purely algebraic approach in nature.

However, pseudo-linear algebra, at least in its present version, is unable to tackle the systems defined on non-homogeneous but regular time scales [3].

An important partly open problem is how to lift the non-trivial irreducible differential form $\mathrm{d}\varphi_{ir}$ (or alternatively, the autonomous element of the 'tangent linearized system') to autonomous element $\varphi_{ir} + c$ of the respective nonlinear system. Though it has been proved that the irreducible differential form is closed (perhaps after multiplication it by a integrating factor), the question remains how to define the constant c. In majority of cases, including the book [10], the problem is left unexplored by assuming that $\varphi(0, \dots, 0) = 0$ yielding the only choice $c = 0$. However, to achieve a complete foundation of the approach, based on the 'study of tangent linearized systems', such aspect must be worked out as well in a full rigor.

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