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THE SINGLE (AND MULTI) ITEM PROFIT MAXIMIZING CAPACITATED LOT–SIZE (PCLSP) PROBLEM WITH FIXED PRICES AND NO SET–UP

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This paper proposes a specialized LP-algorithm for a sub problem arising in simple Profit maximising Lot-sizing. The setting involves a single (and multi) item production system with negligible set-up costs/times and limited production capacity. The producer faces a monopolistic market with given time-varying linear demand curves.

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1. INTRODUCTION

Lot-size problems has drawn a great deal of attention in OR/Operations Management research literature. In their purest form, lot-size problems focus on the trade-off between set-up and inventory costs in generalized production environments. Single item lot-size problems are successfully treated through specialized Dynamic Programming algorithms [10, 12], while multi item problems prove harder to solve. Many authors have proposed various algorithmic approaches in order to handle the NP-hardness of these problems – see e. g. [3]. Recently, several authors have analyzed extensions to classical lot-sizing through pricing. Examples of this line of research may be found in [5, 6] and [8]

Following the notation of Haugen et al. in [5], we look at a version of the PCLSP-problem with a single product and negligible set-up costs. Such a problem may be formulated as:

$$\text{Max } Z = \sum_{t=1}^{T} [(\alpha_t - \beta_t \cdot p_t)p_t - h_t I_t - c_t x_t] \tag{1}$$

s.t.

$$a_t x_t \le R_t \quad \forall\, t \tag{2}$$
$$x_t + I_{t-1} - I_t = \alpha_t - \beta_t \cdot p_t \quad \forall\, t \tag{3}$$
$$x_t \ge 0 \quad \forall\, t \tag{4}$$
$$I_t \ge 0, \quad \forall\, t \tag{5}$$
$$\frac{\alpha_t}{\beta_t} \ge p_t \ge 0 \quad \forall\, t \tag{6}$$

with decision variables (x_t, I_t, p_t) and parameters $(T, \alpha_t, \beta_t, h_t, c_t, a_t, R_t)$:

x_t = the amount produced (of the given product) in period t
I_t = amount of product held in inventory between periods t and $t+1$
p_t = price of item in period t
T = number of time periods
α_t = constant in linear demand function for item in period t
β_t = slope of linear demand function in period t
h_t = unit storage cost between periods t and $t+1$
c_t = unit production cost in period t
a_t = consumption of capacitated resource in period t
R_t = amount of resource available in period t.

2. SIMPLIFYING ASSUMPTIONS

2.1. The capacity constraint

Without loss of generality, equation (2) can be substituted with $x_t \le \hat{R}_t$ where $\hat{R}_t = \frac{R_t}{a_t}$.

2.2. Given prices

If we assume that all prices $p_1, \ldots, p_T$ are given, let's say by $\hat{p}_1, \ldots, \hat{p}_T$, the objective (1) can be rewritten as:

$$\text{Max } Z = \sum_{t=1}^{T} (\alpha_t - \beta_t \cdot \hat{p}_t)\hat{p}_t - \sum_{t=1}^{T} [h_t I_t + c_t x_t] = C - \sum_{t=1}^{T} [h_t I_t + c_t x_t] \tag{7}$$

or

$$\text{Min } \hat{Z} = \sum_{t=1}^{T} [h_t I_t + c_t x_t]\,. \tag{8}$$

Additionally, defining;

$$\hat{D}_t = \alpha_t - \beta_t \cdot \hat{p}_t \tag{9}$$

problem (1) – (6) may be redefined as the following LP-problem:

$$\text{Min } \hat{Z} = \sum_{t=1}^{T} [h_t I_t + c_t x_t] \tag{10}$$

s.t.

$$x_t \le \hat{R}_t \quad \forall t \tag{11}$$
$$x_t + I_{t-1} - I_t = \hat{D}_t \quad \forall t \tag{12}$$
$$x_t \ge 0 \quad \forall t \tag{13}$$
$$I_t \ge 0, \quad \forall t. \tag{14}$$

2.3. Reasonable assumptions on c and h to support a fast problem-specific LP-algorithm

Logistics problems of this type ("Lot-sizing") will typically not have a very large time horizon. Consequently, making assumptions on stability of production and storage costs seems reasonable. We assume the following:

$$c_1 = c_2 = \ldots, c_T = c \text{ and } h_1 = h_2 = \ldots, h_T = h. \tag{15}$$

Given these assumptions, the objective (8) may be expressed:

$$\sum_{t=1}^{T} [h_t I_t + c_t x_t] = h \sum_{t=1}^{T} I_t + c \sum_{t=1}^{T} x_t. \tag{16}$$

Next, it is straightforward to realize by summing up the left and right side of equation (12) that;

$$\sum_{t=1}^{T} x_t = I_T - I_0{}^1 + \sum_{t=1}^{T} \hat{D}_t. \tag{17}$$

The right hand side of equation (17) is a constant, so is h and c. As a consequence, the objective may again be reformulated as:

$$\text{Min } \bar{Z} = \sum_{t=1}^{T} I_t \tag{18}$$

or verbally: minimizing total inventory. Now, relaxing the capacity constraints (11), the optimal solution to the remaining LP is straightforward:

$$x_t^* = \hat{D}_t \text{ and } I_t^* = 0, \forall t. \tag{19}$$

Taking the capacity constraints back into consideration it is (again) straightforward to realize that with the given objective, (minimal total storage) the optimal solution (to the constrained problem) is easily constructed as follows: Any capacity constraint violation can be "corrected" by producing necessary amounts in previous periods as close as possible to the period with capacity constraint violation as such a strategy will lead to total inventory minimization.

[1] I_0 is initial inventory and a ssumed a given constant optimization wise.

3. THE ALGORITHM

Now, a formalized algorithm for the optimal solution of the LP-problem (10) − (14) and added parametric constraints (15) can be formulated:

0. LET $x_t^* = \hat{D}_t, \forall t$
1. IF $x_t^* \leq \hat{R}_t, \forall t$ STOP (x_t^* is optimal)
2. IF next period is $T+1$ STOP
3. ELSE find next period, τ where $x_t^* > \hat{R}_t$ and produce a total of $x_t^* - \hat{R}_t$ in previous periods $\tau - 1, \tau - 2, \dots$ as close as possible to τ. (If impossible, problem is infeasible STOP)
4. SET $x_\tau^* = \hat{R}_\tau$ and update $x_{\tau-1}^*, x_{\tau-2}^*, \dots$ correspondingly
5. GOTO 2.

4. RELAXING THE COST ASSUMPTIONS OF SUBSECTION 2.3

It may be interesting to judge the characteristics of our algorithmic framework if we allow a more general cost structure.

4.1. Constant production costs and time varying inventory costs

If we assume,

$$c_1 = c_2 = \dots, c_T = c \text{ and } h_1 \neq h_2 \neq \dots h_T \tag{20}$$

the basic arguments behind equations (16) through (18) holds. However, as the inventory costs now may vary over time, the objective of equation (18) must be changed to:

$$\text{Min } \hat{Z} = \sum_{t=1}^{T} h_t I_t. \tag{21}$$

Fortunately, the (new) objective of equation (21) does not imply changes in the algorithm of section 3. Obviously, the unconstrained "just in time solution" of equation (19) still holds with the objective (21) as minimal storage costs are obtained with no storage. Likewise, when adjusting production to fit capacity constrains, it is no point in spreading production over more periods than necessary. This will only lead to increased inventory as well as increased total inventory costs. Consequently, the objectives (21) and (18) with added constraints are both minimized through the algorithm of section 3.

4.2. Decreasing production costs and general storage costs

Based on the previous arguments, it is also straightforward to deduce that a relaxation of the constant production cost assumption to the following situation:

$$c_1 > c_2 > \cdots > c_T \text{ and } h_1 \neq h_2 \neq \dots h_T \tag{22}$$

will work just as well. Now, we open up for time-varying production costs, but with a decreasing pattern – i. e. $\frac{\mathrm{d}c(t)}{\mathrm{d}t} \leq 0$. Given this situation, the "just-in-time" unconstrained solution is still optimal and it is always cheapest to find the closest "move-point".

To sum up: Our proposed algorithm (of section 3) guarantees optimal solutions for all cases except (possibly) situations with increasing production costs. Additionally, such situations should be practically rare – in most situations one should be able to produce cheaper (not more expensive) over time. Still, if the need for increasing production costs are there, another assumption will prove handy.

4.3. A constant ratio between production and inventory costs

In Operations Management/Logistics it is fairly common to assume existence of an "inventory interest". Such an assumption means that the value of stored goods is the main component in computation of inventory costs. Following Nahmias [7] such an assumption implies[2]:

$$\frac{c_t}{h_t} = \text{ Constant } = c \Rightarrow c_t = c \cdot h_t. \tag{23}$$

Now, rewriting the inventory balance constrains (3) as:

$$x_t = \hat{D}_t + I_t - I_{t-1} \tag{24}$$

and substituting in for c_t from equation (23) into the objective (10), we get:

$$\hat{Z} = \sum_{t=1}^{T} \left[h_t I_t + c \cdot h_t (\hat{D}_t + I_t - I_{t-1}) \right]. \tag{25}$$

Now, assuming a given initial inventory I_0 and the elimination of a constant, the objective $\hat{Z}$ above may be replaced by the following:

$$\underline{Z} = \sum_{t=1}^{T} \hat{h}_t I_t \tag{26}$$

where

$$\hat{h}_t = (c+1)h_t + ch_{t+1} \text{ and } h_{T+1} = 0. \tag{27}$$

Finally, comparing the objectives $\underline{Z}$ of equation (26) and $\hat{Z}$ of equation (21), we observe structural equality and our algorithm would work also for the case with a constant ratio between production and inventory costs.

[2] In most reasonably competitive markets, the value of a product is proportional to the production costs. Of course, in a perfectly competitive market, price equals marginal costs and the assumption is "correct" if the main contribution to inventory costs is due to storage value.

5. MULTIPLE PRODUCTS

The single item LP of equations (10) – (14) is easily extendible to the multi item case:

$$\text{Min } \underline{\hat{Z}} = \sum_{i=1}^{I} \sum_{t=1}^{T} [h_{it} I_{it} + c_{it} x_{it}] \tag{28}$$

s.t.

$$\sum_{i=1}^{I} a_{it} x_{it} \leq R_t \quad \forall\, t \tag{29}$$

$$x_{it} + I_{i,t-1} - I_{it} = \hat{D}_{it} \quad \forall\, i, t \tag{30}$$

$$x_{it} \geq 0 \quad \forall\, t \tag{31}$$

$$I_{it} \geq 0, \quad \forall\, t. \tag{32}$$

In the above problem (equations (28) – (32)), a new subscript i is introduced, one for each product, with I as the total number of products. Additionally, we assume a common per period resource pool leading to the capacity constrains of equation (29). Finally, a_{it} represents the resource consumption by product i in time period t.

Now, moving back to the original assumptions in subsection 2.3, it is obvious that relaxation of the capacity constraint (29) leaves us with I decoupled LPs, structurally identical to the singe item case. Hence, given the original assumption of time constant inventory and production costs, the algorithmic structure of section 3 may be applied with minor modifications. These modifications involve the following argument: Look at a time period (say τ) where the capacity constraint is violated. Then we must move to a time period t ($< \tau$) (with spare production capacity) as close to τ as possible to keep inventory at a minimum. Now, we must determine which product we should start producing, and it is obviously cost effective to choose the product (with unsatisfied demand in τ) with the smallest ratio $\frac{c_{it}}{a_{it}}$. This secures cost minimization. Such a procedure must (obviously) be repeated until all infeasibilities are removed. The remaining solution is then optimal.

Surely, the above procedure is too simple for the more general cases of section 4. However, as discussed previously, logistic time horizons are short, so cost stationarity (constant costs) may be a reasonable practical assumption. Refer also to section 7 for further discussion on this matter.

6. SOME SIMPLE NUMERICAL EXPERIMENTS

Tables 1 and 2[3] show the results of some simple[4] numerical experiments. Our algorithm was implemented in Fortran 95 and speed[5] was compared to state of the

[3] k and m mean 10^3 and 10^6 respectively.

[4] The data used was based on simple randomization, and the main variable dimension was the length of the time horizon. As such, the reported performance comparisons, should not be taken as any kind of proof. However, the simplicity of our algorithm indicates by itself, that it should compare favourably with any Simplex-based algorithm. Our cases are of course available for inspection for any interested readers.

[5] Surely, our initial assumption of a relatively short time horizon still holds, so these examples should not be taken as any encouragement to apply such long time horizons in practical applications.

Table 1. A single item example – speed in CPU-secs.

	$T = 10k$	$T = 100k$	$T = 1m$
CPLEX	0.219	1.766	31.156
Algorithm	0.031	0.093	0.672
Change (%)	700 %	1893 %	4637 %

Table 2. A multi item ($I = 10$) example – speed in CPU-secs.

	$T = 1k$	$T = 10k$	$T = 100k$	$T = 1m$
CPLEX	0.188	2.281	41.953	N/A
Algorithm	0.078	0.141	0.953	40.045
Change (%)	240 %	1623 %	4401%	N/A %

art optimization software – CPLEX. As the tables indicate, our algorithm performs very favourably against CPLEX.

7. CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

As demonstrated above, the proposed algorithm is able to solve very large single (and multi) item lot-size problems with negligible set-up costs under certain cost function assumptions. Furthermore, such problems may arise as sub-problems in certain dynamic pricing problems. Additionally, a closer inspection of the LP in equations (28) – (32) also reveals a fixed set-up (or negligible set-up cost) CLSP problem. The CLSP or Capacitated Lot-Size Problem problem is a MILP which has been intensively studied in OR/Mathematical Programming research literature. An excellent survey may be found in [3]. Complexity issues are discussed by Florian et. al. in [4] while some noteworthy solution attempts may be examined in [2, 11] and [9]. Different sets of test cases may be found in [1]. Many of the available test cases apply cost stationarity, making our multi-item cost function assumptions relevant.

One interesting (and promising) candidate for further research is to try to use our fixed set-up specialized LP algorithm as a sub-problem solver in a Lagrangian relaxation procedure relatively similar to the approach in [5]. By substituting the "Thomas Algorithm" [10] with the likewise efficient Wagner/Whitin algorithm [12], a similar procedure as in [5] may be tested.

Obviously, it should be possible also to relax the strict cost function assumptions in our multi item version – perhaps with some relatively simple algorithmic modifications. Still, this approach is mainly based on specialized cost structures, so it seems unlikely that a generalized procedure based on these principles are readily available.

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