KYBERNETIKA — VOLUME *46* (2010), NUMBER 3, PAGES 397–404

EIGENSPACE OF A CIRCULANT MAX–MIN MATRIX

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The eigenproblem of a circulant matrix in max-min algebra is investigated. Complete characterization of the eigenspace structure of a circulant matrix is given by describing all possible types of eigenvectors in detail.

Keywords: (max, min) algebra, circulant matrix, eigenvector

Classification: 08A72, 90B35, 90C47

1. INTRODUCTION

Eigenvectors of a max-min matrix characterize stable states of the corresponding discrete-events system. Investigation of the max-min eigenvectors of a given matrix is therefore of a great practical importance. The eigenproblem in max-min algebra has been studied by many authors. Interesting results were found in describing the structure of the eigenspace, and algorithms for computing the maximal eigenvector of a given matrix were suggested, see e.g. [1], [2], [3], [5], [7], [8], [9], [10]. The structure of the eigenspace as a union of intervals of increasing eigenvectors is described in [4].

By max-min algebra we understand a triple $(\mathcal{B}, \oplus, \otimes)$, where $\mathcal{B}$ is a linearly ordered set, and $\oplus = \max$, $\otimes = \min$ are binary operations on $\mathcal{B}$. The notation $\mathcal{B}(n, n)$ $(\mathcal{B}(n))$ denotes the set of all square matrices (all vectors) of given dimension n over $\mathcal{B}$. Operations $\oplus$, $\otimes$ are extended to matrices and vectors in a formal way.

The eigenproblem for a given matrix $A \in \mathcal{B}(n, n)$ in max-min algebra consists of finding a vector $x \in \mathcal{B}(n)$ (eigenvector) such that the equation $A \otimes x = x$ holds true. By the eigenspace of a given matrix we mean the set of all its eigenvectors.

In this paper the eigenspace structure for a special case of so-called circulant matrices is studied. Circulant matrices arise, for example, in applications involving the discrete Fourier transform and the study of cyclic codes for error correction, see [6]. The paper presents a detailed description of all possible types of eigenvectors of any given circulant matrix.

2. EIGENVECTORS OF CIRCULANT MATRICES

A square matrix is called circulant, if the input values in every row are the same as the values in the previous row, but they are cyclically shifted by one position to the right. Formally, matrix $A \in \mathcal{B}(n,n)$ is circulant if

$$a_{ij} = a_{i'j'}$$

whenever

$$i - i' \equiv j - j' \pmod{n} \ .$$

Hence, circulant matrix A is fully determined by its inputs $a_0, a_1, \dots, a_{n-1}$ in the first row. The input a_0 is the common value of all diagonal inputs, and similarly each a_i is the common value of all inputs on a line parallel to the matrix diagonal,

$$A(a_0, a_1, \dots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \dots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix} \ .$$

We shall use the notation $N = \{1, 2, \dots, n\}$ and $N_0 = \{0, 1, \dots, n-1\}$. Further we define, for a given circulant matrix $A = A(a_0, a_1, \dots, a_{n-1})$, a strictly decreasing sequence $M(A) = (m_1, m_2, \dots)$ of length $s(A)$ by recursion

$$m_r = \begin{cases} \max\{\, a_i;\ i \in N_0 \,\} & \text{for } r = 1 \\ \max\{\, a_i < m_{r-1};\ i \in N_0 \,\} & \text{for } r > 1 \end{cases}$$

Clearly, we have $m_1 > m_2 > \dots$ and the length $s(A)$ of the sequence $M(A)$ is the first s with the property $\{\, a_i;\ i \in N_0 \,\} = \{\, m_r;\ 1 \le r \le s \,\}$. For convenience, we shall use the notation $S(A) = \{1, 2, \dots, s(A)\}$. For any $r \in S(A)$ we denote by P_r the set of all positions of the value m_r in the first row of the matrix A, i.e.

$$P_r = \{\, i \in N_0;\ a_i = m_r \,\}$$

and we define the greatest common divisors d_r, e_r as follows

$$d_r = \gcd(P_r \cup \{n\}) \ , \quad e_r = \gcd(d_1, d_2, \dots, d_r) = \gcd(e_{r-1}, d_r) \ .$$

Remark 2.1. The indices of matrix inputs a_i, as well as their positions, are numbers in $N_0 = \{0, 1, \dots, n-1\}$, while the row and columns of the matrix are indexed by numbers from 1 to n. Hence, for any $k \in N$, the kth row of A is of the form

$$A_k = (\dots, a_{k\,k}, a_{k\,k+1}, a_{k\,k+2}, \dots)$$

and for any position $p \in P_r$, we have $a_{k\,k+p} = m_r$ (as the matrix is circulant, the value of the column index $k + p$ is computed modulo n).

The following two lemmas will play key role in our investigations.

Lemma 2.2. Let circulant matrix $A = A(a_0, a_1, \dots, a_{n-1})$ be given, let x be eigenvector of A, let $k \in N$, $r \in S(A)$ and $p \in P_r(A)$. If $x_k < m_r$, then

$$x_k = x_{k+p} \ .$$

P r o o f. Let us assume first that $x_k < x_{k+p}$. Then we have, in view of Remark 2.1

$$x_k < m_r \otimes x_{k+p} = a_{k\,k+p} \otimes x_{k+p} \leq A_k \otimes x \ ,$$

which means that x cannot be eigenvector of A, a contradicton. We have proved $x_k \geq x_{k+p}$. By repeated use of this argument we get, in view of the cyclicity of A,

$$x_k \geq x_{k+p} \geq x_{k+2p} \geq \cdots \geq x_k$$

hence, the equality $x_k = x_{k+p}$ must hold true. □

Lemma 2.3. Let circulant matrix $A = A(a_0, a_1, \ldots, a_{n-1})$ be given, let x be eigenvector of A, let $k, l \in N$ and $r \in S(A)$. If $x_k < m_r$, then the following implications hold true

(i) if $k \equiv l \mod d_r$ then $x_k = x_l$,

(ii) if $k \equiv l \mod e_r$ then $x_k = x_l$.

P r o o f. (i) The value d_r is defined as the greatest common divisor of all positions in P_r and the dimension n. Hence, by the well-known theorem of the number theory, any sufficiently large integer multiple of d_r can be expressed as a linear combination of values in $P_r \cup \{n\}$ with non-negative coefficients. The assertion (i) is then obtained by repeated use of Lemma 2.2.

(ii) The assertion (ii) follows analogously from the definition of e_r and from the assertion (i). □

Theorem 2.4. Let circulant matrix $A = A(a_0, a_1, \ldots, a_{n-1})$ be given, let x be an eigenvector of A. Then $x_k \leq m_1$ holds true for every $k \in N$.

P r o o f. Let us assume, by contradiction, that $x_k > m_1$ for some $k \in N$. Then, by definition of m_1, the inequality $x_k > a_i$ holds for every $i \in N_0$, which gives $x_k > a_{kj}$ for every $j \in N$. Hence

$$x_k > \bigoplus_{j \in N} (a_{kj} \otimes x_j) = A_k \otimes x \ ,$$

i.e. $x_k \neq A_k \otimes x$ and, therefore, x is not eigenvector of A. □

Theorem 2.5. Let circulant matrix $A = A(a_0, a_1, \ldots, a_{n-1})$ be given, in which the diagonal input a_0 is greater than all the remaining inputs of the matrix. If a vector $x \in \mathcal{B}(n)$ has inputs fulfilling the inequalities $m_2 \leq x_k \leq m_1$ for every $k \in N$, then x is eigenvector of A.

P r o o f. By definition of the sets P_r, the assumptions of the theorem give $P_1 = \{0\}$ and we have

$$A_k \otimes x = \bigoplus_{j \in N} (a_{kj} \otimes x_j) = (a_{kk} \otimes x_k) \oplus \bigoplus_{j \in N \setminus \{k\}} (a_{kj} \otimes x_j) \ .$$

Further we have

$$a_{kk} \otimes x_k = m_1 \otimes x_k = x_k \ ,$$

$$\bigoplus_{j \in N \setminus \{k\}} (a_{kj} \otimes x_j) \le \bigoplus_{j \in N \setminus \{k\}} (m_2 \otimes x_j) = m_2 \ ,$$

hence

$$x_k = a_{kk} \otimes x_k \le A_k \otimes x \le x_k \oplus m_2 = x_k \ .$$

for every $k \in N$, i.e. $A \otimes x = x$. □

Remark 2.6. In fact, Theorem 2.5 is a special case of the 'if' implication in Theorem 2.8. In Theorem 2.5 we have $P_1 = \{0\}$ and $d_1 = e_1 = n$, hence the assertions of Lemma 2.3 are fulfilled, in view of the fact that the equivalence relation modulo n is the identity relation on N_0.

Remark 2.7. On the other hand, if the maximal input of the circulant matrix is not unique, or if it is placed on other position than the diagonal one, then $0 < e_1 < n$ and the equivalence modulo e_1 differs from the identity relation on N_0. Hence, the inputs of any eigenvector cannot be arbitrary values in the interval $\langle m_2, m_1 \rangle$ but according to Lemma 2.3, some repetitions must occur, see Example 3.3.

Theorem 2.8. Let $A = A(a_0, a_1, \ldots, a_{n-1})$ be a circulant matrix. A vector $x \in \mathcal{B}(n)$ is eigenvector of A if and only if there is a partition $\mathcal{T}$, on N, such that for every class $T \in \mathcal{T}$ there exist $x(T) \in \mathcal{B}$ and $r(T) \in S(A)$, satisfying the following conditions

(i) $x_k = x(T) \le m_1$ for every $k \in T$,

(ii) $r(T) = \max\{ r \in S(A);\ x(T) < m_r \}$,

(iii) T is an equivalence class in N modulo $e_{r(T)}$.

Proof. (⇒) If x is eigenvector of A, then the conditions (i)–(iii) follow from Lemma 2.3 and Theorem 2.4.

(⇐) Let (i)–(iii) be satisfied. We remark that if $x(T) = m_1$, then, according to (ii), $r(T)$ is the maximum of the empty subset, which is the minimal element in $S(A)$, i.e. $r(T) = 1$ in this case.

Let $k \in N$ be arbitrary, but fixed. Then there is $T \in \mathcal{T}$ with $k \in T$. The position set P_1 is non-empty by definition, hence there is $p \in P_1$, and $a_p = m_1$. Therefore, $k \equiv k + p \mod e_{r(T)}$ and, as a consequence of conditions (i), (iii), we have

$$x_k = x_{k+p} = m_1 \otimes x_{k+p} = a_{k\,k+p} \otimes x_{k+p} \le \bigoplus_{j \in N} (a_{kj} \otimes x_j) = A_k \otimes x \ .$$

To prove the converse inequality, let us consider any index $j \in N$. If $j \in T$, then $x_j = x_k$, by (i). Thus,

$$\bigoplus_{j \in T} (a_{kj} \otimes x_j) = \bigoplus_{j \in T} (a_{kj} \otimes x_k) \le x_k \ .$$

On the other hand, if $j \notin T$, then j, k are not equivalent modulo $e_{r(T)}$. Therefore, the difference $p = j - k$ is not a multiple of the greatest common divisor $e_{r(T)}$, and by the well-known theorem of the number theory, the difference p cannot be expressed as a linear combination with integer coefficients, of the values in $P_1 \cup P_2 \cup \cdots \cup P_{r(T)} \cup \{n\}$, in view of the definition of $e_{r(T)}$. As a consequence we then have $a_p = m_q$ for some $q > r(T)$, which implies $m_q \le x(T)$, by assumption (ii). Therefore, $a_{kj} = a_{k\,k+p} = m_q \le x_k$. Thus, we have

$$\bigoplus_{j \in N \setminus T} (a_{kj} \otimes x_j) \le \bigoplus_{j \in N \setminus T} a_{kj} \le x_k \ .$$

Summarizing we get

$$x_k \le A_k \otimes x = \bigoplus_{j \in T} (a_{kj} \otimes x_j) \oplus \bigoplus_{j \in N \setminus T} (a_{kj} \otimes x_j) \le x_k \ .$$

The fixed index $k \in N$ is arbitrary, hence we have proved $A \otimes x = x$. □

3. EXAMPLES OF EIGENVECTORS

This section contains several examples of eigenvectors of a circulant matrix. The examples illustrate Theorem 2.5, Theorem 2.8 and Remark 2.7.

Example 3.1. Let $n = 12$ and let $A = A(15, 1, 3, 4, 3, 0, 7, 1, 1, 4, 2, 2)$ be a circulant matrix generated by inputs on positions $(0, 1, 2, \ldots, 10, 11)$ in the first row. Then the strictly decreasing sequence of inputs has the form $M(A) = (m_1, m_2, \ldots, m_7) = (15, 7, 4, 3, 2, 1, 0)$. The maximal input $m_1 = 15$ is on the diagonal, i.e. on position 0 and nowhere else, the second largest input has the value $m_2 = 7$. Hence, in view of Theorem 2.5, any vector with arbitrary inputs from interval $\langle 7, 15 \rangle$, e.g. $x = (11, 9, 8, 14, 11, 12, 15, 7, 8, 8, 10, 7)^T$, is an eigenvector of A.

$$\begin{pmatrix} 15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 \\ 2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 \\ 2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 \\ 4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 \\ 1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 \\ 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 \\ 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 \\ 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 \\ 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 \\ 4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 \\ 3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 \\ 1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 \end{pmatrix} \otimes \begin{pmatrix} 11 \\ 9 \\ 8 \\ 14 \\ 11 \\ 12 \\ 15 \\ 7 \\ 8 \\ 8 \\ 10 \\ 7 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \\ 8 \\ 14 \\ 11 \\ 12 \\ 15 \\ 7 \\ 8 \\ 8 \\ 10 \\ 7 \end{pmatrix}$$

Example 3.2. In this example we show further eigenvectors of the matrix $A = A(15, 1, 3, 4, 3, 0, 7, 1, 1, 4, 2, 2)$ from the previous example. If an eigenvector should contain inputs not belonging to the interval $\langle m_2, m_1 \rangle = \langle 7, 15 \rangle$, then in view of

Theorem 2.2, such inputs cannot be larger than $m_1 = 15$. Hence such inputs must be less than the value $m_2 = 7$, and some repetitions must occur, by Lemma 2.3.

The position sets for particular inputs are $P_1 = \{0\}$ for $m_1 = 15$, $P_2 = \{6\}$ for $m_2 = 7$, $P_3 = \{3, 9\}$ for $m_3 = 4$, $P_4 = \{2, 4\}$ for $m_4 = 3$, $P_5 = \{10, 11\}$ for $m_5 = 2$, $P_6 = \{1, 7, 8\}$ for $m_6 = 1$ and $P_7 = \{5\}$ for $m_7 = 0$. By definition of the greatest common divisors d_r, e_r we get

$$
\begin{aligned}
d_1 &= \gcd(P_1 \cup \{n\}) = \gcd(0, 12) = 12 & e_1 &= 12 \\
d_2 &= \gcd(P_2 \cup \{n\}) = \gcd(6, 12) = 6 & e_2 &= \gcd(d_1, d_2) = \gcd(12, 6) = 6 \\
d_3 &= \gcd(P_3 \cup \{n\}) = \gcd(3, 9, 12) = 3 & e_3 &= \gcd(e_2, d_3) = \gcd(6, 3) = 3 \\
d_4 &= \gcd(P_4 \cup \{n\}) = \gcd(2, 4, 12) = 2 & e_4 &= \gcd(e_3, d_4) = \gcd(3, 2) = 1
\end{aligned}
$$

Clearly, the further computation gives $e_5 = e_6 = e_7 = 1$. By Lemma 2.3, any input $x_k < m_r$ must be repeated in x after e_r positions. In particular, inputs less than value $m_2 = 7$ must be repeated after 6 positions, inputs less than $m_3 = 4$ must be repeated on every third position. However, inputs which are not less than $m_2 = 7$ can be arbitrary. The above conditions are satisfied e.g. by vector $x = (3, 6, 5, 3, 11, 11, 3, 6, 5, 3, 10, 7)^T$, which is therefore an eigenvector of A, in view of Theorem 2.8.

$$
\begin{pmatrix}
15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 \\
2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 \\
2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 \\
4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 \\
1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 & 1 \\
1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 & 7 \\
7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 & 0 \\
0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 & 3 \\
3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 4 \\
4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 \\
3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 \\
1 & 3 & 4 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15
\end{pmatrix}
\otimes
\begin{pmatrix} 3 \\ 6 \\ 5 \\ 3 \\ 11 \\ 11 \\ 3 \\ 6 \\ 5 \\ 3 \\ 10 \\ 7 \end{pmatrix}
=
\begin{pmatrix} 3 \\ 6 \\ 5 \\ 3 \\ 11 \\ 11 \\ 3 \\ 6 \\ 5 \\ 3 \\ 10 \\ 7 \end{pmatrix}
$$

We may note that if an eigenvector x of A should contain an input $x_k < m_4 = 3$, then such an input would be repeated after every $e_4 = 1$ position, in other words, the eigenvector would have only that single input, i.e. it would be a constant vector.

Example 3.3. Last example illustrates Remark 2.7 by analyzing eigenvectors of the matrix $B = B(15, 1, 3, 15, 3, 0, 7, 1, 1, 4, 2, 2)$, which differs from matrix A in a single input, namely $b_3 = 15$. Thus, the maximal input of the matrix B is placed on the diagonal position 0 and also on a non-diagonal position 3. We have $P_1 = \{0, 3\}$ for $m_1 = 15$ and $e_1 = d_1 = \gcd(0, 3, 12) = 3$. Therefore, Theorem 2.5 cannot be applied, and the input values belonging to the interval $\langle m_2, m_1\rangle = \langle 7, 15\rangle$ must be repeated after $e_1 = 3$ positions. In fact, the same is true for all input values in the interval $\langle m_4, m_1\rangle = \langle 3, 15\rangle$, because it can be easily computed that $e_2 = e_3 = 3$.

$$\begin{pmatrix} 15 & 1 & 3 & 15 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 \\ 2 & 15 & 1 & 3 & 15 & 3 & 0 & 7 & 1 & 1 & 4 & 2 \\ 2 & 2 & 15 & 1 & 3 & 15 & 3 & 0 & 7 & 1 & 1 & 4 \\ 4 & 2 & 2 & 15 & 1 & 3 & 15 & 3 & 0 & 7 & 1 & 1 \\ 1 & 4 & 2 & 2 & 15 & 1 & 3 & 15 & 3 & 0 & 7 & 1 \\ 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 15 & 3 & 0 & 7 \\ 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 15 & 3 & 0 \\ 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 15 & 3 \\ 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 & 15 \\ 15 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 & 3 \\ 3 & 15 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 & 1 \\ 1 & 3 & 15 & 3 & 0 & 7 & 1 & 1 & 4 & 2 & 2 & 15 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 11 \\ 5 \\ 3 \\ 11 \\ 5 \\ 3 \\ 11 \\ 5 \\ 3 \\ 11 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ 5 \\ 3 \\ 11 \\ 5 \\ 3 \\ 11 \\ 5 \\ 3 \\ 11 \\ 5 \end{pmatrix}$$

ACKNOWLEDGEMENT

The support of the Czech Science Foundation under Grant 402/09/0405 is kindly acknowledged.

(Received April 14, 2010)

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