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A VALUE BASED ON MARGINAL CONTRIBUTIONS FOR MULTI–ALTERNATIVE GAMES WITH RESTRICTED COALITIONS

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This paper deals with cooperative games with n players and r alternatives which are called multi-alternative games. In the conventional multi-alternative games initiated by Bolger, each player can choose any alternative with equal possibilities. In actual social life, there exist situations in which players have some restrictions on their choice of alternatives. Considering such situations, we study restricted multi-alternative games. A value for a given game is proposed.

Keywords: game theory, cooperative game, multi-alternative game, restricted game, Banzhaf value

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1. INTRODUCTION

The cooperative game theory provides useful tools to analyze cost allocation, voting power, and so on. The problems to be analyzed by the cooperative game theory include n entities called players and are usually expressed by characteristic functions called games which map each subset of players to a real number. The solutions to the problems are given by value functions which assign a real number to each player. The real number called a value can show the cost borne by the player, power of influence, and so on depending on the problem setting. Several value functions have been proposed. As representative examples of value functions, the Shapley value [9] and the Banzhaf value [1, 6] are well-known. Each of them is uniquely specified by reasonable axiom systems.

In the conventional cooperative games, each player can take one from two options: cooperate and non-cooperate. However, in the real world problems, we may face a decision problem to choose one from several options. From this point of view, it is worthwhile to treat cooperative games in which each player has r options. Then multi-alternative games also called games with r alternatives have been proposed by Bolger [4]. A multi-alternative game is expressed by a generalized characteristic function which maps an arrangement showing all players' choices to an r-dimensional real vector. Bolger [4] proposed a generalized value function which maps a multi-alternative game to an n-dimensional real vector whose ith component shows the value of player i. This function is a generalization of the Shapley function. On the other hand, Ono [8] proposed a multi-alternative Banzhaf value (an MBZ value) as a generalization of the Banzhaf value.

The value functions/generalized value functions described above are considered under the assumption that all coalitions/arrangements are formed with equal possibilities. In the real world, there are many cases when this assumption does not hold. For example, when each player has its own ideology, he/she would be difficult to cooperate with players having totally different ideologies. Moreover, when a certain license is necessary to choose an option in a multi-alternative game, players without the licenses cannot choose it and then some arrangements cannot be realized.

Such asymmetries of players or restrictions on players' behaviors have been treated by introduction of the probability or restrictions on coalition/arrangement forming [2, 7]. For example, Myerson [7] introduced a restricted cooperation model derived from communication situations into the conventional cooperative games. In this model, only the coalitions which induce connected subgraphs are feasible. Other restricted cooperation models [2] were also proposed. However, no restricted cooperation model has been proposed in the framework of multi-alternative games, so far.

In this paper, we introduce a kind of restriction on arrangements into multi-alternative games. In our model, the choice of an alternative is restricted, while coalition forming is often restricted in the conventional restricted games. Such a restriction can be found in the real world. For example, when a license/skill is necessary for taking some alternatives, those alternatives cannot be chosen by unlicensed/unskillful players. Under the restrictions on choices, we propose a value based on marginal contributions for a given game. The value indicates an evaluation of an alternative by a player under the given game. It is shown that the value is proportional to the MBZ value when there is no restriction. Further, two axiom systems are given for the proposed value. One is composed of four axioms concerning null players, linearity, independence from unrelated players, and proportionality to welcome degree difference for voting games. The other is composed of four axioms concerning null players, linearity, proportionality to welcome degree difference for voting games and relation with arbitrariness.

In Section 2, we briefly introduce an extended multi-alternative games and related concepts given by Tsurumi et al. [10] and Bolger [4] and the Bolger value and the MBZ value are presented. In Section 3, we propose a restricted situation which is called *a restricted choice situation* and a value for multi-alternative games with the restricted situation. Further, related concepts and properties are presented. In Section 4, the proposed value is axiomatized. Two axiom systems are proposed. In Section 5, we give a numerical example which is called "Job Selection Game" to exemplify the usefulness of the restricted multi-alternative games and the proposed value. The concluding remarks are given in Section 6.

2. EXTENDED MULTI–ALTERNATIVE GAMES AND PREVIOUS VALUES

2.1. Extended multi-alternative games

In this section, we introduce the extended multi-alternative games proposed by Tsurumi et al. [10] which are extensions of multi-alternative games (games with r alternatives) by Bolger [4]. Extended multi-alternative games assume that each player chooses one from r ($r \geq 2$) alternatives or none of them while original multi-alternative games assume that each player always chooses one alternative. The extended multi-alternative games are mathematically characterized as follows:

Let $N = \{1, \dots, n\}$ be the set of players and $R = \{1, \dots, r\}$ the set of alternatives. Let Γ_j be the set of players who have chosen the alternative $j \in R$. A finite sequence of subsets of players, $\Gamma = (\Gamma_1, \dots, \Gamma_r)$, is called an arrangement. Each arrangement Γ satisfies $\Gamma_1 \cup \dots \cup \Gamma_r \subseteq N$ and $\Gamma_k \cap \Gamma_l = \emptyset$ ($\forall\, k \neq l$). Let Γ_0 be a subset of players who have chosen none of alternatives. Then we have $\Gamma_0 = N - \bigcup_{k \in R} \Gamma_k$. For the sake of convenience, we define $R_0 = \{0, 1, \dots, r\}$. We denote $\exists k \in R, S = \Gamma_k$ by $S \in \Gamma$. For any $S \in \Gamma$, we call (S, Γ) an embedded coalition (ECL). Let $E(N, R)$ be the set of ECLs and $A(N, R)$ the set of arrangements on N and R. Then a function $v : A(N, R) \to \mathbb{R}^r$ such that $v_k(\Gamma) = 0$ if $\Gamma_k = \emptyset$ is called an extended multi-alternative game on N with r alternatives, where $v(\Gamma) = (v_1(\Gamma), v_2(\Gamma), \dots, v_r(\Gamma))$ and $\mathbb{R}$ is the set of real numbers. Let $MG(N, R)$ be the set of extended multi-alternative games on N and R.

We note that Bolger [4] defined a multi-alternative game as a function which maps a pair of a set of players and an arrangement to a real number but we define an extended multi-alternative game by a payoff function which maps an arrangement to an r-dimensional vector following Tsurumi et al. [10]. Our definition as well as Tsurumi et al.'s makes clearer the alternative we evaluate.

In order to exemplify an extended multi-alternative game, we present the following example.

Example 1. [Job Selection Game] Three students A, B and C are considering to work part-time. There are two jobs 1 and 2 but students cannot take both. Then each student can take one job or nothing. They can take the same job. If only two students would take different jobs, the remaining student would not get any payoff but the students taking jobs would get some payoffs independently. The payoff does not depend on the job taken but on the student taking a job. The payoffs of students A, B and C would be 8, 6 and 4 units, respectively. If student A would work alone while students B and C would make the same choice, independent of the job taken by A, student A would get 5 units as a payoff. If students B and C would work together while student A would not work with them, independent of the job taken by them, students B and C would get 18 units as the total payoff. If student B would work alone while students A and C would make the same choice, independent of the job taken by B, student B would get 3 units as a payoff. If students A and C would work together while student B would not work with them, independent of the job taken by them, students A and C would get 25 units as the total payoff. If student C would work alone while students A and B would make the same choice, independent of the job taken by C, student C would get 1 unit as a payoff. If students A and B would work together while student C would not work with them, independent of the job taken by them, students A and B would get 30 units as the total payoff. If all students A, B and C would work together, independent of the job taken, they would get 50 units as the total payoff.

This game can be represented by the following extended multi-alternative game v for $k = 1, 2$ with $N = \{A, B, C\}$ and $R = \{1, 2\}$:

$$\begin{aligned}
&v_k(\Gamma) = 8, \text{ for } \Gamma_k = \{A\} \text{ and } |\Gamma_j| = 1, j = 1, 2,\\
&v_k(\Gamma) = 6, \text{ for } \Gamma_k = \{B\} \text{ and } |\Gamma_j| = 1, j = 1, 2,\\
&v_k(\Gamma) = 4, \text{ for } \Gamma_k = \{C\} \text{ and } |\Gamma_j| = 1, j = 1, 2,\\
&v_k(\Gamma) = 5, \text{ for } \Gamma_k = \{A\} \text{ and } (\{B, C\} \in \Gamma \text{ or } \emptyset \in \Gamma),\\
&v_k(\Gamma) = 3, \text{ for } \Gamma_k = \{B\} \text{ and } (\{A, C\} \in \Gamma \text{ or } \emptyset \in \Gamma),\\
&v_k(\Gamma) = 1, \text{ for } \Gamma_k = \{C\} \text{ and } (\{A, B\} \in \Gamma \text{ or } \emptyset \in \Gamma),\\
&v_k(\Gamma) = 30, \text{ for } \Gamma_k = \{A, B\},\\
&v_k(\Gamma) = 25, \text{ for } \Gamma_k = \{A, C\},\\
&v_k(\Gamma) = 18, \text{ for } \Gamma_k = \{B, C\},\\
&v_k(\Gamma) = 50, \text{ for } \Gamma_k = N,\\
&v_k(\Gamma) = 0, \text{ for other cases,}
\end{aligned}$$

where $|\Gamma_j|$ is the cardinality of Γ_j.

An important class of extended multi-alternative games is the set of voting games with r alternatives which are called extended multi-alternative voting games. Voting games with r alternatives are first considered by Bolger [3]. It is assumed that players can choose none of alternatives in the following description of multi-alternative voting games while the original multi-alternative voting games by Bolger do not assume.

There are n players and r alternatives. Let $N = \{1, \dots, n\}$ be the set of players and $R = \{1, \dots, r\}$ be the set of alternatives. Each player choose one of the r alternatives or none of them. We assume that only one alternative is elected. Let Γ_j be the set of players who choose alternative $j \in R$. The vector $\Gamma = (\Gamma_1, \dots, \Gamma_r)$ becomes an arrangement.

Let $\Gamma = (\Gamma_1, \dots, \Gamma_r)$ be an arbitrary arrangement. If alternative $j \in R$ is elected, we call (Γ_j, Γ) a pair of a winning coalition. If alternative j is not elected, we call (Γ_j, Γ) a pair of a losing coalition. Let WE be the set of pairs of winning coalitions. Let LE be the set of pairs of losing coalitions. Then the triple (N, R, WE) is called a voting game with r alternatives (or a multi-alternative voting game).

A multi-alternative voting game (N, R, WE) can be represented by a multi-alternative game v as follows:

$$v_k(\Gamma) = \begin{cases} 1 & \text{if } (\Gamma_k, \Gamma) \in WE, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $k \in R$.

2.2. Previous axioms and values

We describe axioms proposed previously and previous values for extended multi-alternative games. These axioms and values were proposed for multi-alternative games but we describe those with modification suitable for the extended multi-alternative games. In the modification, we regard an extended multi-alternative game as a multi-alternative game with $(r+1)$ alternatives $R_0 = \{0, 1, \ldots, r\}$ where the set of players with no choice takes zero value for any arrangement.

Let π^j , $j = 1, \ldots, r$, be a vector function which maps a multi-alternative game to an n-dimensional real vector whose i-th component shows the value of player i. The ith component of π^j is denoted by π_i^j.

Many axioms have been proposed as described in what follows.

Axiom 1. [j-efficiency] Value π^j satisfies

$$\sum_{i \in N} \pi_i^j(v) = v_j(\Gamma_{(N:j)}),$$

where $\Gamma_{(N:j)} = (\emptyset, \ldots, \emptyset, N, \emptyset, \ldots, \emptyset)$ (N is the $(j+1)$th component).

This axiom corresponds to the *efficiency* axiom in the conventional cooperative games.

The following three axioms correspond to axioms of *null player*, *linearity* and *symmetry* in the conventional cooperative games.

Axiom 2. [j-null player] Value π^j satisfies $\pi_i^j(v) = 0$ for any j-null player $i \in N$, where player i is a *j-null player* in v if and only if for all arrangements Γ satisfying $\Gamma_j \ni i$ and for all $k \neq j$

$$v_j(\Gamma) = v_j(\Gamma^{i \to k}),$$

where $\Gamma^{i \to k}$ is the arrangement obtained by changing player i's selection to the kth alternative in Γ ($i \notin \Gamma_k$).

Axiom 3. [linearity] Value π^j satisfies $\pi^j(v+w) = \pi^j(v) + \pi^j(w)$ and $\pi^j(cv) = c \cdot \pi^j(v)$ for a *sum of extended multi-alternative games* $v+w$ and a *scalar multiplication of an extended multi-alternative game* cv, where, for extended multi-alternative games v and w, we define $v+w$ and cv by $(v+w)_j(\Gamma) = v_j(\Gamma) + w_j(\Gamma)$ and $(cv)_j = c \cdot v_j(\Gamma)$; $j = 1, \ldots, r$.

Axiom 4. [symmetry] Value π^j satisfies $\pi_i^j(v) = \pi_s^j(v)$ if players i and s are symmetric, where players $i \in N$ and $s \in N$ are said to be *symmetric* if and only if $v_j(\Gamma) = v_j(\Gamma')$ with arrangement Γ' obtained by interchange between players i and s in arrangement Γ.

Bolger [4] added another axiom called a *pivot move axiom.*

Axiom 5. [pivot move] Value π^j satisfies $\pi_i^j(v) = \pi_i^j(w)$ for extended multi-alternative games v and w such that

$$\sum_{k\neq j}\left(v_j(\Gamma) - v_j(\Gamma^{i\to k})\right) = \sum_{k\neq j}\left(w_j(\Gamma) - w_j(\Gamma^{i\to k})\right), \text{ for all } \Gamma \text{ such that } i \in \Gamma_j.$$

This claims that the values of player i to alternative j in games v and w should be same if sums of marginal contributions to jth alternative in those games are equal.

Moreover, the following axioms were considered.

Axiom 6. [mean of total contribution] Value π^j satisfies

$$\sum_{i\in N}\pi_i^j(v) = \frac{1}{(r+1)^{n-1}r}\bar{\eta}^j(v),$$

where $\bar{\eta}^j(v)$ is the sum of $\eta_i^j(v)$ which shows a contribution of player i to alternative j by his/her choice, i. e.,

$$\begin{aligned}\bar{\eta}^j(v) &= \sum_{i\in N}\eta_i^j(v),\\ \eta_i^j(v) &= \sum_{\Gamma\in A(N,R):i\in\Gamma_j}\sum_{k\neq j}(v_j(\Gamma) - v_j(\Gamma^{i\to k})).\end{aligned}$$

Note that $(r+1)^{n-1}r$ is the number of all arrangements such that $i \in \Gamma_j$.

Before describing Axiom 7, let us introduce an S-unanimity game v^S with $S \subseteq N$. An S-unanimity game v^S is defined by

$$v_j^S(\Gamma) = \begin{cases} 1 & \text{if } \Gamma_j \supseteq S \text{ and } j \neq 0,\\ 0 & \text{otherwise.}\end{cases}$$

This is an extended multi-alternative game in which the jth coalition Γ_j including S wins regardless of what arrangement occurs.

Now we are ready to introduce two kinds of unanimity axioms.

Axiom 7. Value π^j satisfies $\pi_i^j(v^S) = 1/|S|$ for all $i \in S(\subseteq N)$ and $j \neq 0$.

Axiom 8. Value π^j satisfies $\pi_i^j(v^S) = 1/(r+1)^{|S|-1}$ for all $i \in S(\subseteq N)$ and $j \neq 0$.

The derivations of Axioms 7 and 8 can be explained by two different probabilities to be a dictator for players in S, where a dictator for alternative $j \in R$ is a player $i \in N$ such that $v_j(\Gamma) = 1$ if and only if $i \in \Gamma_j$.

One out of $|S|$ players can be a dictator under the assumption that a coalition including S forms at the end. Then $1/|S|$ in Axiom 7 shows the probability to be a dictator. On the other hand, for a player in S to be a dictator without any assumption, he/she needs the agreements in choosing a common alternative of the other $(|S|-1)$ players in S. The probability of getting the agreements is $1/(r+1)^{|S|-1}$ which appears in Axiom 8. This is also a probability to be a dictator. The major difference between Axioms 7 and 8 is the following: $\pi_i^j(v^S)$ in Axiom 8 depends on the number of alternatives, r while $\pi_i^j(v^S)$ in Axiom 7 does not.

The Bolger value [4] and the MBZ value [8] have been proposed for multi-alternative games as extensions of the Shapley value and the Banzhaf value, respectively. These values are axiomatized by some of the axioms described above as shown below.

Theorem 1. [Bolger [4]] The value function $\theta^j(v)$, $j = 1, \dots, r$ defined as follows is the unique function satisfying Axioms 1 through Axiom 5:

$$\theta_i^j(v) = \sum_{\Gamma \in A(N,R):\Gamma_j \ni i} \sum_{k \neq j} \frac{(|\Gamma_j| - 1)!(n - |\Gamma_j|)!}{n! r^{n-|\Gamma_j|+1}} [v_j(\Gamma) - v_j(\Gamma^{i \to k})], \quad \forall i \in N, j \in R. \tag{2}$$

Theorem 2. [Ono [8]] The value function $\theta^j(v)$, $j = 1, \dots, r$ defined by (2) is the unique function satisfying Axioms $2, 3, 5$ and 7.

Theorem 3. [Ono [8]] The value function $\beta^j(v)$, $j = 1, \dots, r$ defined as follows is the unique function satisfying Axioms $2, 3, 4, 5$ and 6:

$$\beta_i^j(v) = \sum_{\Gamma \in A(N,R):\Gamma_j \ni i} \sum_{k \neq j} \frac{1}{(r+1)^{n-1} r} [v_j(\Gamma) - v_j(\Gamma^{i \to k})], \quad \forall i \in N, j \in R. \tag{3}$$

Theorem 4. [Ono [8]] The value function $\beta^j(v)$, $j = 1, \dots, r$ defined by (3) is the unique function satisfying Axioms $2, 3, 5$ and 8.

The difference between the Bolger value and the MBZ value can simply be said as follows: in the Bolger value, we consider permutations of players in coalitions while in the MBZ value, we consider only combinations of players in coalitions.

3. THE PROPOSED VALUE FOR RESTRICTED MULTI–ALTERNATIVE GAMES

In the conventional extended multi-alternative games, each player can choose any alternative from a given set of alternatives. However, in the real world, there exists a situation where some alternatives cannot be chosen by all players. For example, in Job Selection Game described in the previous section, some students cannot take some jobs due to their inabilities or conflicts with regular lessons. In order to treat such situations, we formulate restricted games with r alternatives (restricted multi-alternative games).

In this paper, we consider the restriction on the selection of alternatives for each player. Let R_i be the set of alternatives which player $i \in N$ can choose. Obviously, we have $R_i \subseteq R_0$ and $0 \in R_i$, $\forall\, i \in N$. Especially, $R_i = R_0$ holds if player i can choose any alternatives and $R_i = \{0\}$ holds if player i can choose none of alternatives. Then the set of feasible arrangements, W, is defined by

$$W = \{\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_r) \mid \forall\, j \in R\, \forall\, i \in \Gamma_j;\ j \in R_i\}. \tag{4}$$

We call the set of feasible arrangement W *a restricted choice situation*. Let $AR(N, R)$ be the set of restricted choice situations. We characterize a multi-alternative game with a restricted choice situation as a pair (v, W) where $v \in MG(N, R)$ and $W \in AR(N, R)$.

In restricted games such as those derived from *communication situations* by Myerson [7], restricted situations are focused on the relations among players. However, in restricted games derived from *restricted choice situations*, restricted situations are focused on the ability of each player.

Now, we propose a value for multi-alternative games with restricted choice situations. For convenience, we define a subset $W_{i,j}$ of a restricted choice situation $W \in AR(N, R)$ where a player $i \in N$ chooses the jth alternative by

$$W_{i,j} = \{\Gamma \in W \mid i \in \Gamma_j\}.$$

We define a function $f^j : MG(N, R) \to (\mathbb{R}^n)^{AR(N,R)}$ $(j = 1, \ldots, r)$ by its ith component,

$$f_i^j(v)(W) = \begin{cases} \displaystyle\sum_{\substack{\Gamma \in W \\ \Gamma_j \ni i}} \sum_{\substack{k \in R_0 - \{j\} \\ \Gamma^{i \to k} \in W}} \frac{1}{|W|}[v_j(\Gamma) - v_j(\Gamma^{i \to k})], & \text{if } W_{i,j} \neq \emptyset, \\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

Let us interpret the function defined by (5). The term $v_j(\Gamma) - v_j(\Gamma^{i \to k})$ can be interpreted as the marginal contribution of player i to Γ_j. $|W|$ shows the number of feasible arrangements. Therefore, the weight $\frac{1}{|W|}$ means that each feasible arrangement is formed with equal probability. Then $f_i^j(v)(W)$ is the expected value of the marginal contributions of player i to alternative j in restricted game (v, W).

Theorem 5. When $W = A(N, R)$, the proposed value $f_i^j(v)(W)$ is proportional to the MBZ value $\beta_i^j(v)$ in (3). More specifically, $f_i^j(v)(A(N, R)) = \frac{r}{r+1}\beta_i^j(v)$. Namely, the normalized $f_i^j(v)(W)$ equals to the normalized MBZ value of player i to alternative j.

P r o o f. it is clear from the definitions of those values. □

Proposition 1. Let $v^1, v^2 \in MG(N,R)$ and $W \in AR(N,R)$. Then the following holds.

$$\sum_{\substack{k\in R_0-\{j\}\\ \Gamma^{i\to k}\in W}} v_j^1(\Gamma) - v_j^1(\Gamma^{i\to k}) = \sum_{\substack{k\in R_0-\{j\}\\ \Gamma^{i\to k}\in W}} v_j^2(\Gamma) - v_j^2(\Gamma^{i\to k})$$

for any $\Gamma \in W$ satisfying $i \in \Gamma_j$,

$$\Rightarrow f_i^j(v^1)(W) = f_i^j(v^2)(W) \quad \forall i \in N, j \in R.$$

Proof. it is clear from the definition of f^j. □

This proposition is a generalization of Axiom 5 to multi-alternative games with restricted choice situations. When there are no restrictions of alternatives, this property is one of axioms of the Bolger value and an MBZ value.

In the rest of this section, we give some concepts associated to the axiom system of value f^j.

First, a concept related to the basis of extended multi-alternative games is provided as a direct extension of Bolger's concept. To describe this, we introduce a game $v^{T,\Gamma}$ defined by

$$v_j^{T,\Gamma}(\Gamma^*) = \begin{cases} 1 & \text{if } \Gamma^* = \Gamma \text{ and } \Gamma_j = T, \\ 0 & \text{otherwise,} \end{cases}$$

where v is an extended multi-alternative game, Γ is an arrangement and T is a coalition such that $T \in \Gamma$ ($T \neq \emptyset$).

Lemma 1. [Bolger [4]] The collection $\{v^{T,\Gamma}\}$ of all such games serves as a basis for the vector space of all extended multi-alternative games. Namely, if v is an extended multi-alternative game, we may write

$$v_j = \sum_{T\in\Gamma, \Gamma\in A(N,R)} v_j(\Gamma) v_j^{T,\Gamma}.$$

Now, we provide some modified concepts for extended multi-alternative games with restricted choice situations.

Definition 1. [*j-null player* for restricted multi-alternative games]
Let $v \in MG(N,R)$, $W \in AR(N,R)$, $i \in N$ and $j \in R$. Player i is called *a j-null player* on (v, W) if and only if the following holds:

$$\text{if } W_{i,j} \neq \emptyset \text{ then } v_j(\Gamma) - v_j(\Gamma^{i\to k}) = 0, \quad \forall \Gamma \in W_{i,j}, k \in R_0 - \{j\}, \Gamma^{i\to k} \in W.$$

Note that player i is a j-null player if $W_{i,j} = \emptyset$.

This concept is a generalization of *j-null players* of Bolger's multi-alternative games.

Definition 2. Let $v \in MG(N,R)$, $W \in AR(N,R)$ and $i \in N$. Then player i is called an *unrelated player* if $R_i = \{0\}$.

Unrelated players are the players who cannot choose any alternatives and j-null players for all $j \in R$ because $W_{i,j} = \emptyset$ for all $j \in R$.

4. AXIOMATIC APPROACHES

In this section, we give axioms which are reasonable for a value function to restricted multi-alternative games. First, we consider four axioms concerning null players, linearity, the independence from unrelated players, and the proportionality to total deducted welcome difference in voting games. The first two axioms are generalizations of those of the Bolger value and the MBZ value. That is, Axiom 9 and 10 are generalizations of Axiom 2 and 3 for multi-alternative games with restricted choice situations.

Let π^j be a vector function from $MG(N,R)$ into $(\mathbb{R}^n)^{AR(N,R)}$. The ith component of π^j is denoted by π_i^j. Note that for any $v, w \in MG(N,R)$, $v + w \in MG(N,R)$ holds.

Axiom 9. [j-null player] Given $v \in MG(N,R)$, $i \in N$, $j \in R$ and $W \in AR(N,R)$, the following holds:

$$\pi_i^j(v)(W) = 0 \Leftrightarrow i \text{ is a } j\text{-}\textit{null player} \text{ on } W$$

Axiom 10. [Linearity] Given v^1, $v^2 \in MG(N,R)$ c_1, $c_2 \in \mathbb{R}$ and $W \in AR(N,R)$, the following holds:

$$\pi^j(c_1 v^1 + c_2 v^2)(W) = c_1 \pi^j(v^1)(W) + c_2 \pi^j(v^2)(W), \ j = 1, \ldots, r$$

Axiom 11. [Independence from unrelated players] Let $v \in MG(N,R)$ and $W \in AR(N,R)$. Let us add an unrelated player $n+1$ to the set of players N, and we denote v' the $(n+1)$-person game. Then, the following holds:

$$\pi_i^j(v', W) = \pi_i^j(v, W), \quad \forall\, i \in N, \ \forall\, j \in R.$$

Axiom 11 means that the value is not changed by the addition of unrelated players to a game. Note that because $R_{n+1} = \{0\}$, the set of feasible arrangements W does not change between the $(n+1)$-person game v' and the original n-person game v in Axiom 11.

Axiom 12 described in what follows is a property with respect to multi-alternative voting games. If Γ_j changed from a winning coalition to a losing coalition by player i's moving from Γ_j to Γ_k ($k \in R_0 - \{j\}$), the movement is called a negatively influential movement under arrangement Γ. On the contrary, if Γ_j changed from a losing coalition to a winning coalition by player i's moving from Γ_j to Γ_k ($k \in R_0 - \{j\}$), the movement is called a positively influential movement under arrangement

Γ. Under an arrangement Γ, let $M^-_{i,j}(\Gamma|v)$ be the number of negatively influential movements of player i from Γ_j and $M^+_{i,j}(\Gamma|v)$ the number of positively influential movements of player i from Γ_j. Using $M^-_{i,j}(\Gamma|v)$ and $M^+_{i,j}(\Gamma|v)$, we define

$$M_{i,j}(v) = \begin{cases} \sum\limits_{\substack{\Gamma\in W \\ i\in\Gamma_j}} (M^-_{i,j}(\Gamma|v) - M^+_{i,j}(\Gamma|v)) & \text{if there exists } \Gamma \text{ such that } \Gamma_j \ni i, \\ 0 & \text{otherwise.} \end{cases}$$

Then Axiom 12 is given as follows.

Axiom 12. [Proportionality to welcome degree difference] Let $W \in AR(N,R)$, $\Gamma \in W$, $i,s \in N$ and $j \in R$, and let v be a voting game with r alternatives. If $M_{i,j}(v) - M_{s,j}(v) = l$, we have

$$\pi^j_i(v)(W) = \pi^j_s(v)(W) + \frac{l}{|W|}.$$

Let us interpret Axiom 12. First, $M^-_{i,j}(\Gamma|v)$ and $M^+_{i,j}(\Gamma|v)$ can be interpreted as welcome and unwelcome degrees of player i to alternative j under arrangement Γ, respectively. Then the difference $M^-_{i,j}(\Gamma|v) - M^+_{i,j}(\Gamma|v)$ can show the deducted welcome degree of player i to alternative j under arrangement Γ. Accordingly, $M_{i,j}(v)$ stands for a total deducted welcome degree of i to j, which may indicate the power to victory of player i by choosing alternative j. Axiom 12 shows that the difference of values between players i and s should be proportional to the difference between their total deducted welcome degrees, more specifically, it should be the ratio of the difference between their total deducted welcome degrees to the number of feasible arrangements.

Theorem 6. Function f^j, $j = 1,\ldots,r$ defined by (5) is the unique function which satisfies Axioms 9 through 12.

P r o o f. By definition of f^j, it is easy to show that the function f^j satisfies Axioms 9 through 12. Hence, we prove that $\pi^j : MG(N,R) \to (\mathbb{R}^n)^{AR(N,R)}$, $j = 1,\ldots,r$ which satisfies Axioms 9 through 12 is nothing but f^j, $j = 1,\ldots,r$.

Let $v \in MG(N,R)$, $W \in AR(N,R)$, $\Gamma \in W$ and $T \in \Gamma$. From Axiom 11, we can assume that at least one unrelated player is included in the set of players without loss of generality. Note that the value of an unrelated player is zero, i. e., $\pi^j_i(v)(W) = 0$ from Axiom 9 because an unrelated player is a j-null player for all $j \in R$.

First, we consider $\pi^j_i(v^{T,\Gamma})(W)$ for all $i \in N$ when $T = \Gamma_j$.

For player $i \in N$, there are four possibilities: (i) i is an unrelated player, (ii) $i \in T$, (iii) $i \in \Gamma'_j - T$ for some $\Gamma' \in W$ and (iv) $j \notin R_i$, where Γ'_j is the jth component of Γ'.

In case (i): because player i is an unrelated player, from definition, $M_{i,j}(v^{T,\Gamma}) = 0$ ($j \neq 0$) and, from Axiom 9, $\pi^j_i(v^{T,\Gamma})(W) = 0$.

In case (ii): because $i \in T = \Gamma_j$, we have $M^-_{i,j}(\Gamma|v^{T,\Gamma}) = |R_i| - 1 = |\{\Gamma^{i\to k} \in W, k \in R_0 - \{j\}\}|$. Moreover, for any $\Gamma' \in W$ such that $i \in \Gamma'_j \cap T$ and $\Gamma' \neq \Gamma$, $M^-_{i,j}(\Gamma'|v^{T,\Gamma}) = 0$ and, for any $\Gamma' \in W$ such that $i \in \Gamma'_j \cap T$ $M^+_{i,j}(\Gamma'|v^{T,\Gamma}) = 0$. Then we have $M_{i,j}(v^{T,\Gamma}) = M^-_{i,j}(\Gamma|v^{T,\Gamma}) = |R_i| - 1 = |\{\Gamma^{i\to k} \in W, k \in R_0 - \{j\}\}|$.

In case (iii), i. e., $i \in \Gamma'_j - T$ for some $\Gamma' \in W$: we have two possibilities for such Γ', $\Gamma' = \Gamma^{i\to j}$ and $\Gamma' \neq \Gamma^{i\to j}$. When $\Gamma' = \Gamma^{i\to j}$, we have $M^-_{i,j}(\Gamma'|v^{T,\Gamma}) = 0$ and $M^+_{i,j}(\Gamma'|v^{T,\Gamma}) = 1$. When $\Gamma' \neq \Gamma^{i\to j}$, we have $M^-_{i,j}(\Gamma'|v^{T,\Gamma}) = 0$ and $M^+_{i,j}(\Gamma'|v^{T,\Gamma}) = 0$ because $i \in \Gamma'_j - T$. From those results, we have $M_{i,j}(v^{T,\Gamma}) = -M^+_{i,j}(\Gamma^{i\to j}|v^{T,\Gamma}) = -1$.

In case (iv): because $j \notin R_i$, by the definition, $M_{i,j}(v^{T,\Gamma}) = 0$.

Let $s \in N$ be an unrelated player, we have

$$M_{i,j}(v^{T,\Gamma}) - M_{s,j}(v^{T,\Gamma}) = \begin{cases} |\{\Gamma^{i\to k} \in W, k \in R_0 - \{j\}\}| & \text{if } i \in T, \\ -1 & \text{if } i \notin T,\ \Gamma^{i\to j} \in W, \\ 0 & \text{if } i \notin T,\ \Gamma^{i\to j} \notin W. \end{cases}$$

Therefore, from Axiom 12, we obtain

$$\pi^j_i(v^{T,\Gamma})(W) = \begin{cases} \displaystyle\sum_{\Gamma^{i\to k} \in W, k \in R_0 - \{j\}} \frac{1}{|W|} & \text{if } i \in T, \\ -\frac{1}{|W|} & \text{if } i \notin T,\ \Gamma^{i\to j} \in W, \\ 0 & \text{if } i \notin T,\ \Gamma^{i\to j} \notin W. \end{cases} \tag{6}$$

Next, we consider $\pi^j_i(v^{T,\Gamma})(W)$ for all $i \in N$ when $T \neq \Gamma_j$. In this case, for any $i \in N$ such that i is not an unrelated player, we have $v^{T,\Gamma}_j(\Gamma) = v^{T,\Gamma}_j(\Gamma^{i\to k}) = 0$ for any k such that $i \notin \Gamma_k$ and $\Gamma^{i\to k} \in W$. Because an unrelated player is a j-null player, all players are j-null players. Then, from Axiom 9, we have

$$\pi^j_i(v^{T,\Gamma})(W) = 0 \quad \forall i \in N. \tag{7}$$

Unifying equation (6) and (7),

$$\pi^j_i(v^{T,\Gamma})(W) = \begin{cases} \displaystyle\sum_{\Gamma^{i\to k} \in W, k \in R_0 - \{j\}} \frac{1}{|W|} & \text{if } T = \Gamma_j,\ i \in T \\ -\frac{1}{|W|} & \text{if } T = \Gamma_j,\ i \notin T, \Gamma^{i\to j} \in W \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Using the values $\pi^j_i(v^{T,\Gamma})(W)$, we finally obtain $\pi^j_i(v)(W)$ for arbitrary $v \in MG(N,R)$ and $W \in AR(N,R)$. From Axiom 10 and Lemma 1, for any $v \in$

$MG(N,R)$ and $W \in AR(N,R)$ such that $W_{i,j} \neq \emptyset$,

$$\begin{aligned}
&\pi_i^j(v)(W) \\
&= \sum_{\Gamma \in W, T \in \Gamma} v(T,\Gamma)\pi_i^j(v^{T,\Gamma})(W) \\
&= \sum_{\Gamma \in W, \Gamma_j \ni i} \sum_{k \in R_0 - \{j\}, \Gamma^{i \to k} \in W} v_j(\Gamma) \cdot \frac{1}{|W|} - \sum_{\Gamma \in W, \Gamma_j \not\ni i, \Gamma^{i \to j} \in W} v_j(\Gamma) \cdot \frac{1}{|W|} \\
&= \frac{1}{|W|} \sum_{\Gamma \in W, \Gamma_j \ni i} \Big(\sum_{k \in R_0 - \{j\}, \Gamma^{i \to k} \in W} v_j(\Gamma) - \sum_{k \in R_0 - \{j\}, \Gamma^{i \to k} \in W} v_j(\Gamma^{i \to k}) \Big) \\
&= f_i^j(v)(W).
\end{aligned}$$

Moreover, for any $v \in MG(N,R)$ and $W \in AR(N,R)$ such that $W_{i,j} = \emptyset$, $\pi_i^j(v)(W) = f_i^j(v)(W) = 0$ from Axiom 9.

This completes the proof. □

In the rest of this section, we show that the proposed value can be axiomatized by another axiom system which contains a new axiom. The new axiom is related to the arbitrariness in selection of an alternative. The arbitrariness is defined in the following definition.

Definition 3. [Total Marginal Weakness and Arbitrariness] Let $W \in AR(N,R)$ and $\Gamma \in W$. Then we define the total marginal weakness $MW_j(\Gamma,W)$ and the arbitrariness $AB_j(v,W)$ by

$$MW_j(\Gamma,W) = \sum_{k \in R_0 - \{j\}} (m^{j \to k}(\Gamma,W) - m^{k \to j}(\Gamma,W)), \tag{9}$$

$$AB_j(v,W) = \frac{1}{|W|} \sum_{\Gamma \in W} MW_j(\Gamma,W) v_j(\Gamma), \tag{10}$$

where $m^{j \to k}(\Gamma,W) = |\{i \in \Gamma_j | \Gamma^{i \to k} \in W\}| \quad \forall k \in R_0 - \{j\}$.

$m^{j \to k}(\Gamma,W)$ shows the number of players who can change his/her choice from alternative j to alternative k under arrangement $\Gamma \in W$. The difference $(m^{j \to k}(\Gamma,W) - m^{k \to j}(\Gamma,W))$ can be interpreted as the marginal weakness in keeping the number of supporters on alternative j. Then $MW_j(\Gamma,W)$ can be interpreted as the *total marginal weakness* of alternative j. On the other hand, if a strong alternative j provides a large payoff $v_j(\Gamma)$ players tend to select alternative j, while if a weak alternative k provides a small payoff $v_k(\Gamma)$ players tend to avoid alternative k. However, if a weak alternative j provides a large payoff $v_j(\Gamma)$, the selection of alternative j is more arbitrary. From this point of view, $AB_j(v,W)$ can be understood as *arbitrariness*.

With respect to the concept of the total marginal weakness, we obtain the following proposition.

Proposition 2. Let $W \in AR(N, R)$ and $\Gamma^1, \Gamma^2 \in W$. If $\Gamma^2_j \subseteq \Gamma^1_j$ for some $j \in R$, the following holds:

$$MW_j(\Gamma^2, W) \leq MW_j(\Gamma^1, W).$$

Proof. Let $W \in AR(N, R)$ and consider two arrangements Γ^1, Γ^2 satisfying $\Gamma^2_j \subseteq \Gamma^1_j$ for some $j \in R$. Using the property of W defined by (4), from $\Gamma^2_j \subseteq \Gamma^1_j$, we have

$$m^{j\to k}(\Gamma^2, W) \leq m^{j\to k}(\Gamma^1, W), \quad \forall\, k \in R_0 - \{j\}.$$

Thus, we have

$$\sum_{k\in R_0-\{j\}} m^{j\to k}(\Gamma^2, W) \leq \sum_{k\in R_0-\{j\}} m^{j\to k}(\Gamma^1, W). \tag{11}$$

On the other hand, because $i \in \Gamma^1_j - \Gamma^2_j$ can move to the jth coalition in Γ^2 while he/she cannot in Γ^1, we have

$$\sum_{k\in R_0-\{j\}} m^{k\to j}(\Gamma^1, W) \leq \sum_{k\in R_0-\{j\}} m^{k\to j}(\Gamma^2, W). \tag{12}$$

Combining (11) and (12), we obtain the proposition. □

Proposition 2 shows that an alternative becomes weaker if the coalition choosing the alternative larger.

Now, we describe a new axiom related to the arbitrariness.

Axiom 13. Let $v \in MG(N, R)$ and $W \in AR(N, R)$. Then the following holds:

$$\sum_{i\in N} \pi^j_i(v)(W) = AB_j(v, W).$$

This axiom requires that the sum of values to alternative j of all players equals to the arbitrariness in selection of alternative j. In this sense, we may regard the value $\pi^j_i(v)(W)$ as the degree of discretion.

In the Bolger value, the sum of the values of all players equals to the payoff of the grand coalition as is given in Axiom 1. In the MBZ value, the sum of the values of all players equals to the mean of total contributions of all players as is given in Axiom 6. Axiom 13 corresponds to those axioms in the Bolger and MBZ values.

We obtain the following lemma.

Lemma 2. Function f^j, $j = 1, \dots, r$ defined by (5) satisfies Axiom 13.

Proof. Given $v \in MG(N,R)$ and $W \in AR(N,R)$, we obtain

$$\begin{aligned}
&\sum_{i \in N} f_i^j(v)(W) \\
&= \frac{1}{|W|} \sum_{i \in N} \sum_{\Gamma \in W, \Gamma_j \ni i} \sum_{k \in R_0 - \{j\}, \Gamma^{i \to k} \in W} (v_j(\Gamma) - v_j(\Gamma^{i \to k})) \\
&= \frac{1}{|W|} \left(\sum_{\Gamma \in W} \sum_{i \in \Gamma_j} \sum_{k \in R_0 - \{j\}, \Gamma^{i \to k} \in W} v_j(\Gamma) - \sum_{\Gamma \in W} \sum_{i \in \Gamma_j} \sum_{k \in R_0 - \{j\}, \Gamma^{i \to k} \in W} v_j(\Gamma^{i \to k}) \right) \\
&= \frac{1}{|W|} \left(\sum_{\Gamma \in W} \sum_{i \in \Gamma_j} \sum_{k \in R_0 - \{j\}, \Gamma^{i \to k} \in W} v_j(\Gamma) - \sum_{\Gamma \in W} \sum_{k \in R_0 - \{j\}} \sum_{i \in \Gamma_k, \Gamma^{i \to j} \in W} v_j(\Gamma) \right) \\
&= \frac{1}{|W|} \left(\sum_{\Gamma \in W} \sum_{k \in R_0 - \{j\}} m^{j \to k}(\Gamma, W) v_j(\Gamma) - \sum_{\Gamma \in W} \sum_{k \in R_0 - \{j\}} m^{k \to j}(\Gamma, W) v_j(\Gamma) \right) \\
&= AB_j(v, W)
\end{aligned}$$

□

Now we have the following uniqueness theorem which shows another axiom system of the proposed value.

Theorem 7. Function f^j, $j = 1, \dots, r$ defined by (5) is the unique function satisfying Axioms 9, 10, 12 and 13.

Proof. From Lemma 2, it suffices to prove the uniqueness of f^j, $j = 1, \dots, r$.

To this end, we only prove $\pi^j(v^{T,\Gamma})(W)$ defined by (6) is the unique function satisfying Axioms 12 and 13 for $v \in MG(N,R)$, $W \in AR(N,R)$, $\Gamma \in W$ and $T \in \Gamma$ when $T = \Gamma_j$. This is because the other part of the proof can be performed in the same way as the proof of Theorem 6.

To satisfy Axiom 13, we should have

$$\sum_{i \in N} \pi_i^j(v^{T,\Gamma})(W) = \sum_{k \in R_0 - \{j\}} \frac{m^{j \to k}(T, W) - m^{k \to j}(T, W)}{|W|}. \tag{13}$$

We can verify that (6) satisfies Axiom 12 because (6) shows $\pi_i^j(v^{T,\Gamma})(W) = M_{i,j}(v^{T,\Gamma})/|W|$. Moreover, because we have

$$\sum_{i \in T} \pi_i^j(v^{T,\Gamma})(W) = \sum_{i \in \Gamma_j} \sum_{\Gamma^{i \to k} \in W, k \in R_0 - \{j\}} \frac{1}{|W|} = \sum_{k \in R_0 - \{j\}} \frac{m^{j \to k}(\Gamma, W)}{|W|}, \tag{14}$$

$$\sum_{i \notin T} \pi_i^j(v^{T,\Gamma})(W) = \sum_{i \notin \Gamma_j} \sum_{\Gamma^{i \to j} \in W} -\frac{1}{|W|} = \sum_{k \in R_0 - \{j\}} -\frac{m^{k \to j}(\Gamma, W)}{|W|}, \tag{15}$$

(6) satisfies Axiom 13.

Suppose that there exists another function $\pi_i'^j$ satisfying Axioms 12 and 13. Then $\pi_i'^j$ should be expressed by $\pi_i'^j(v^{T,\Gamma})(W) = M_{i,j}(v^{T,\Gamma})/|W| + \delta$ for some $\delta \in \mathbb{R} - \{0\}$ from Axiom 12. We obtain

$$\sum_{i \in N} \pi_i^j(v^{T,\Gamma})(W) \tag{16}$$

$$= \sum_{k \in R_0 - \{j\}} \frac{m^{j \to k}(T, W) - m^{k \to j}(T, W)}{|W|} + n\delta. \tag{17}$$

Because $\delta \neq 0$, (13) cannot be satisfied. This contradicts the fact that $\pi_i'^j$ satisfies Axiom 13. □

5. NUMERICAL EXAMPLE

We calculate the proposed value f_i^j in Job Selection Game described in Example 1 and demonstrate the effect by a restriction. We compare the values in two different situations: a situation where all students can choose all jobs and a situation when student A cannot choose job 2.

Table 1. The proposed values
when all students can choose all jobs.

	Job 1		Job 2	
Player	Value	Normalized	Value	Normalized
A	11.407	0.413	11.407	0.413
B	8.963	0.327	8.963	0.327
C	6.963	0.254	6.963	0.254
Total	27.333	1	27.333	1

Table 2. The proposed values
when student A cannot choose job 2.

	Job 1		Job 2	
Player	Value	Normalized	Value	Normalized
A	8.555	0.309	0	0
B	10.722	0.387	5.444	0.569
C	8.388	0.303	4.111	0.430
Total	27.666	1	9.555	1

The proposed values are shown in Tables 1 and 2. Table 1 shows the proposed values when all students can choose all jobs while Table 2 shows the proposed values when student A cannot choose job 2. Because v is symmetric with respect to jobs, the proposed values are same independent of the jobs students choose when all students can choose all jobs. As shown in Table 2, the value of student A in Job 2 is zero, this would be natural from the restriction that A cannot take Job 2. The value of student A decreases not only in Job 2 but also in Job 1 by the restriction. This would be a reflection of Axiom 12 by the decrement of possible movements of student A. By the comparison between Tables 1 and 2, we can observe that the restriction can strongly change the strength (value) of players (students).

6. CONCLUDING REMARKS

We have investigated extended multi-alternative games with *restricted choice situations*. We have proposed a value based on marginal contributions for restricted multi-alternative games. Two systems of axioms have been given to characterize the value uniquely. In numerical example, we observe that the restriction can strongly change the strength of players.

Table 3. The comparison of axioms satisfied with the values for multi-alternative games.

Axiom	Bolger		MBZ		f^j	
j-null player	◯	◯	◯	◯	◯	◯
Linearity	◯	◯	◯	◯	◯	◯
Symmetry	◯	○	◯	○	○	○
j-efficiency	◯	○	×	×	×	×
Mean of total contributions	×	×	◯	○	×	×
Arbitrariness of the selection	×	×	×	×	○	◯
Pivot move	◯	◯	◯	◯	○	○
Proportionality to welcome degree difference	×	×	×	×	◯	◯
Independence from unrelated players	–	–	–	–	◯	○
$\pi_i^j(v^S) = 1/\lvert S\rvert$	○	◯	×	×	×	×
$\pi_i^j(v^S) = 1/(r+1)^{\lvert S\rvert-1}$	×	×	○	◯	×	×

In order to compare values proposed for extended multi-alternative games with previously proposed Bolger and MBZ values, the satisfied axioms are shown in Table 3. In Table 3, "◯" means that the axiom is one of the axiom systems of the value, "○" means that the value satisfies the axiom, "×" means that the value does not satisfies the axiom and "–" means that the axiom cannot be applicable in the setting of games on which values are considered.

The axioms of j-efficiency, mean of total contribution and arbitrariness of the selection correspond with one another because they show the properties of the sum of values. The last two axioms correspond with each other. Moreover, as we show in Theorem 5, the proposed value is proportional to the MBZ value. Therefore, the proposed value satisfies $\pi_i^j(v^S) = r/(r+1)^{|S|}$ for all $i \in S$.

Though the proposed value is proportional to the MBZ value, the systems of axioms do not totally correspond with each other. This implies that we obtain new systems of axioms for the MBZ value and those for the proposed values. The former can be obtained rather easily by multiplying $r/(r+1)$ to constants in axioms of the arbitrariness of the selection and the proportionality to total welcome difference.

However, the latter needs more investigation because the proposed value is considered under a restricted situation. This would be one of the future topics about the proposed value.

Moreover, we introduced a special restriction expressed by (4). The investigations on value functions under more generalized restrictions would be also future topics.

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