KYBERNETIKA — VOLUME 45 (2009), NUMBER 5, PAGES 801 – 808

CONTROL A STATE–DEPENDENT DYNAMIC GRAPH TO A PRE–SPECIFIED STRUCTURE

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Recent years have witnessed an increasing interest in coordinated control of distributed dynamic systems. In order to steer a distributed dynamic system to a desired state, it often becomes necessary to have a prior control over the graph which represents the coupling among interacting agents. In this paper, a simple but compelling model of distributed dynamical systems operating over a dynamic graph is considered. The structure of the graph is assumed to be relied on the underling system's states. Then by following a proper protocol, the state-dependent dynamic graph is driven to a pre-specified structure. The main results are derived via Lasalle's Invariant Principle and numerical examples that find very good agreements with the analytical results are also included.

Keywords: state-dependent graph, Lasalle's Invariant Principle, dynamic system

AMS Subject Classification: 37F99

1. INTRODUCTION

In recent years, the coordination of distributed dynamic systems has attracted the attention of researchers from system theory [4, 5, 6, 8], biology [1, 3] and physics [9]. Particularly, it has witnessed an increasing interest in the interplay between information flows and system dynamics [8]. It is recognized that communication constraints may have a considerable impact on the performance of a distributed system.

In [9], Vicsek et al. proposed a simple but compelling model of n autonomous agents moving in a plane at the same speed but with different headings. They demonstrated that all agents could move in the same direction eventually. Then Jadbabaie, Lin, and Morse gave a theoretical explanation for this observed behavior and derived sufficient conditions under which the system can reach a consensus on headings [5]. This result was extended by Moreau who provided the necessary and sufficient condition for the convergence of individual agents' states to a common value [8]. The consensus problem over random networks was considered by Hatano and Mesbahi via notions from stochastic stability [4]. In addition, the relation between the rate of convergence and the algebraic connectivity of random graphs was established. In [6], Lu and Chen proposed a general method for synchronizing two chaotic systems based upon Kalman filtering and provided sufficient conditions for a driven system to track the states of a drive system asymptotically. Moreover, the consensus problems have received a lot of attention from biology researchers. Franks et al. demonstrated a speed versus accuracy trade-off in consensus decision making. They showed that house-hunting ant colonies choose a new nest more quickly in harsh conditions than in benign ones and are less discriminating [3]. A through review of the empirical and theoretical studies of consensus decision making can be found in [1]. For these systems, in order to steer the systems to a desired state or an objective, it often becomes necessary to have a prior control over the dynamic graph which represents the interaction among agents [7, 10].

The problem considered henceforth is obtained by tying the network structure to the dynamic states residing at nodes. Specifically, we study a scenario where the existence of an information channel between a pair of agents is determined by their states, i. e., if $||x_i - x_j|| \le r_i$, agent i can access the information of agent j. Here, x_i denotes the state of agent i and r_i represents the sensing radius of agent i. The symbol $||\cdot||$ is the Euclidean norm. The network structure has a blend of dynamic and combinatorial features and is called a state-dependent dynamic graph. Then the objective is to steer the dynamic graph to a pre-specified structure by designing a proper control law.

In fact, we are by no means the first one who have noticed the significance of controlling graphs. In [7], Meshbahi et al. considered graphs with incidence relations that are dictated by underlying dynamic states. Then the relation between the controllability of a distributed dynamic system and the corresponding graph was derived, pointing to a new research direction in system and control theory. Later, in [10], the problem of preserving the k-hop connectivity was considered. The main idea was to model connectivity as an invariance problem and transform it into a set of constraints on the control variables. Then the control law for a connectivity problem was obtained by minimizing a cost function.

The rest of this paper is organized as follows. In Section 2, we develop a general framework for our problem. The mathematical derivations of main results are presented in Section 3. Some numerical examples, including both one dimension and two dimensions, are presented in Section 4. Finally, Section 5 summarizes the main conclusions.

2. PRELIMINARIES AND PROBLEM SETUP

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of all natural numbers, all real numbers and all complex numbers respectively. We denote by I_n the $n \times n$ real identity matrix and the subscript variable n is omitted when no misunderstanding arises.

Graphs are a good choice to represent the relationships among group members (agents). Each agent is denoted by a node, the relation between two nodes is described by an arc[1]. It has been found that the dynamic behavior of a group of agents is closely related to the properties of the graph that represented the relations among agents [5, 8]. Next we will first survey some basic notions from graph theory.

[1]When the relation is bidirectional, it is described by an edge.

Then the definition of state-dependent graphs is given. Especially, we will focus on directed graphs.

A directed graph $\mathcal{G}$ consists of a non-empty finite set $\mathcal{V}$ of elements called nodes and a finite set $\mathcal{E}$ of ordered pairs of nodes called arcs. We call $\mathcal{V}$ the node set and $\mathcal{E}$ the arc set of $\mathcal{G}$. We write $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to indicate that $\mathcal{V}$ and $\mathcal{E}$ are the node set and the arc set of $\mathcal{G}$ respectively. For a node i in a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the set of its neighbors is defined as $\{j | j \in \mathcal{V}$ and $(j, i) \in \mathcal{E}\}$. Figure 1 shows an example of a directed graph.

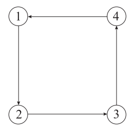


Fig. 1. An example of a directed graph. The node set is $\mathcal{V} = \{1, 2, 3, 4\}$ and the arc set is $\mathcal{E} = \{(1, 2), (2, 3), (3, 4), (4, 1)\}$.

A state-dependent graph is a mapping $\mathcal{G}_s$ from the distributed system state space X to the set of all labeled graphs of order N, $\mathcal{G}_N$. Here N is the number of agents in the system.

$$\mathcal{G}_s : X \to \mathcal{G}_N. \tag{1}$$

It is assumed that the node set $\mathcal{V}$ of these graphs is fixed; their edge set $\mathcal{E}(x(t))$ however is a function of the system state $x(t)$. Especially, in current paper, $\mathcal{E}(x(t))$ is specified as:

$$(j, i) \in \mathcal{E} \text{ iff } ||x_i(t) - x_j(t)|| \leq r_i, \tag{2}$$

where $x_i(t)$ is the state of agent i at time t and r_i denotes the sensing radius of agent i. Then the structure of a state-dependent graph is totally determined by the state of the distributed system. For further analysis, we define the state matrix $A = (a_{i,j})$ as follows: $a_{i,j} = x_i - x_j$ where $a_{i,j}$ denotes the (i, j)th element of matrix A and x_i is the state of agent i.

Suppose we want to steer the dynamic state-dependent graph to a pre-specified structure $\bar{\mathcal{G}} = \{\bar{\mathcal{V}}, \bar{\mathcal{E}}\}$ where $\bar{\mathcal{V}} = V$ and $\bar{\mathcal{E}}$ is pre-specified. For any such structure, we can construct a matrix $\bar{A} = (\bar{a}_{i,j})$, such that $\bar{a}_{i,j} = \bar{x}_i - \bar{x}_j$ and $(i, j) \in \bar{\mathcal{E}}$ iff $||\bar{a}_{i,j}|| \leq r_i$. Here $\bar{x}_i$ denotes the expected state of agent i. Then the state-dependent graph $\mathcal{G}_s(x(t))$ converges to the pre-specified topology $\bar{\mathcal{G}}$ iff A converges to the matrix $\bar{A}$. Here $\mathcal{G}_s$ is defined by (1).

To close this section, we formally define the notion of the convergence of dynamic state-dependent graphs.

Definition 1. Convergence of dynamic state-dependent graphs. Consider a distributed dynamic system which consists of n agents and let $\bar{\mathcal{G}}$ be the prescribed topology. We say the dynamic state-dependent graph, defined by (2), converges to $\bar{\mathcal{G}}$, iff $\lim_{t \to \infty} \mathcal{G}_s(x(t)) = \bar{\mathcal{G}}$.

3. MATHEMATICAL ANALYSIS

In this section, we aim at giving a protocol to ensure the convergence of a state-dependent graph to a pre-defined topology $\bar{\mathcal{G}}$.

The protocol is defined as follows:

$$\dot{x_i} = -\sum_{j\neq i}(x_i - x_j - \bar{a}_{i,j}). \tag{3}$$

For the sake of brevity, we define

$$r_{i,j} = x_i - x_j - \bar{a}_{i,j}, \tag{4}$$

then system (3) is rewritten as

$$\dot{x_i} = -\sum_{j\neq i} r_{i,j}. \tag{5}$$

For further analysis, we define a matrix $R = (r_{i,j})$, where $r_{i,j}$ is defined by Eq.(4). Here comes a lemma about $r_{i,j}$.

Lemma 1. $\forall\, i, j, k \in \{1, 2, \cdots, n\}$, $r_{i,j} = r_{i,k} + r_{k,j}$, where n is the number of agents in the system.

Proof. The derivation of Lemma 1 is straightforward by taking the definitions of $r_{i,j}$ and $\bar{a}_{i,j}$ into the lemma. □

Next we will give a lemma which will be used in the stability analysis of system (3).

Lemma 2. Consider matrix R, $\forall\, i \in \{1, 2, \cdots, n\}$, $\sum_j r_{i,j} = 0$ if and only if $\forall\, i, j \in \{1, 2, \cdots, n\}$, $r_{i,j} = 0$.

Proof. Necessity (Proof by contradiction): Without loss of generality, assume that there is a pair (i, j) satisfying $r_{i,j} < 0$. Since $\sum_j r_{i,j} = 0$, the sum of the entries, except for $r_{i,j}$, in the ith row $\sum_{k\neq j} r_{i,k} > 0$. By Lemma 1, we have

$$(n-1)r_{i,j} = \sum_{k\neq j}(r_{i,k} + r_{k,j}) = \sum_{k\neq j} r_{i,k} + \sum_{k\neq j} r_{k,j}.$$

Then $r_{k,j} = -r_{j,k}$, $r_{j,j} = 0$, and $\sum_k r_{j,k} = 0$ yield

$$\sum_{k\neq j} r_{k,j} = 0.$$

Moreover, $\sum_{k\neq j} r_{i,k} > 0$ indicates that $r_{i,j} > 0$. Now a contradiction arises.

Sufficiency: The proof of sufficiency is rather straightforward, hence it is omitted here. □

Next, we will show that even by designing a semi-definite positive Lyapunov function, it is possible to use a blend of Lasalle's Invariant Principle and Lemma 2 to establish the stability results.

The main result of the current note is given below.

Theorem 1. For system (3), the dynamic state-dependent graph will converge to a pre-specified structure $\bar{\mathcal{G}}$.

P r o o f. Consider the following function

$$V \triangleq \sum_{i,j,i\neq j} V_{i,j},$$

with each $V_{i,j}$ defined by

$$V_{i,j} \triangleq \frac{1}{4}||x_i - x_j - \bar{a}_{i,j}||^2. \tag{6}$$

Thus

$$\frac{\partial V}{\partial x_i} = \sum_{j\neq i} \frac{1}{2}(x_i - x_j - \bar{a}_{i,j}) + \sum_{j\neq i} -\frac{1}{2}(x_j - x_i - \bar{a}_{j,i}). \tag{7}$$

Since $\bar{a}_{i,j} = -\bar{a}_{j,i}$,

$$-(x_j - x_i - \bar{a}_{j,i}) = x_i - x_j - \bar{a}_{i,j}. \tag{8}$$

Then we have

$$\frac{\partial V}{\partial x_i} = \sum_{j\neq i} (x_i - x_j - \bar{a}_{i,j}). \tag{9}$$

Since

$$\dot{x_i} = -\sum_{j\neq i} (x_i - x_j - \bar{a}_{i,j}),$$

we have

$$\dot{V} = \sum_i (\frac{\partial V}{\partial x_i})^T \dot{x}_i = -\sum_i \Big\| \sum_{j\neq i} r_{i,j} \Big\|^2 \leq 0. \tag{10}$$

According to Lasalle's Invariant Principle, system (3) will converge to the following invariant set

$$\{x| \quad \dot{V}(x) = 0\}, \tag{11}$$

which implies that $\forall i \sum_{j\neq i} r_{i,j} = 0$. Thus according to Lemma 2, $\forall i, j,\ r_{i,j} = 0$, which indicates that the pre-defined topology is reached asymptotically. □

Note that the statement $\dot{V} = 0$ is equal to $V = 0$. Hence, in the subsequent simulations, we will draw the curve of V instead of $\dot{V}$.

Corollary 1. For system (3), if initially the state-dependent graph achieves the desired topology $\bar{\mathcal{G}}$, in any subsequent time the desired topology is always kept.

P r o o f. If the initial state achieves the desired topology, we have $V = 0$. By (10), we have $\dot{V} = 0$, which implies that the desired topology is always kept. □

Remark 1. The significance of Corollary 1 is due to the fact that in many cases we need the state-dependent graphs to achieve some prescribed topologies, for instance connected, at all times other than asymptotically. In these cases, Corollary 1 will meet our requirements.

The main results in this section can find applications, for instance, in the rendezvous problem. Consider a group of unmanned vehicles which aims at reaching a rendezvous point. The information flows among vehicles can be described by a state-dependent graph, called communication graph [2]. To get to the rendezvous point, a sufficient condition is to guarantee connectivity maintenance. According to Corollary 1, if the communication graph is initially connected, it remains connected throughout the system evolution under the proposed protocol, which indicates that rendezvous is reached.

4. NUMERICAL SIMULATIONS

In this section, we verify the stability results obtained in Section 3 by numerical examples. In our simulations, both one dimension and two dimensions cases are considered.

There are three nodes, each with one dimension, in the first example. The state equations of the nodes are described by:

$$\dot{x_i} = -\sum_{j \neq i} (x_i - x_j - \bar{a}_{i,j}) \tag{12}$$

for $i = 1, 2, 3$, where $\bar{a}_{i,j}$ is the (i, j)th entry of a pre-defined matrix $\bar{A}$ which is defined as follows:

$$\bar{A} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \end{pmatrix}.$$

In this case, function V defined in the proof of Theorem 1 is

$$V = \sum_{i \in \{1,2,3\}, j \in \{1,2,3\}, \quad i \neq j} V_{i,j}, \tag{13}$$

where $V_{i,j}$ is defined by (6). Figure 2 illustrates the curve of the function V. It is clear that as time goes by, $V(t)$ asymptotically reduces to 0, which indicates that the pre-defined topology is reached asymptotically.

In the second example, the pre-defined topology is defined to be the name of our university which consists of two characters in Chinese. In this example, there are 51 nodes, 30 nodes for the first character and 21 nodes for the second character. For each node, its state is represented by a two-dimensional vector, depicting the x coordinate and y coordinate of the node moving in the plane. The step-size (a parameter specified in Runge Kutta method in approximating solutions to differential equations) in the example is 0.0005 seconds. Figure 3 shows consecutive snapshots of the proximity structure for 51 agents in free-space using Protocol (3).

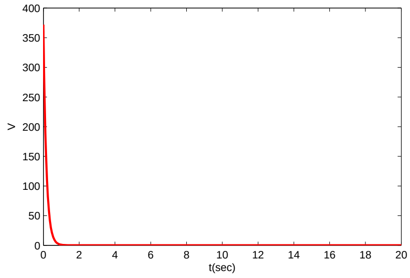
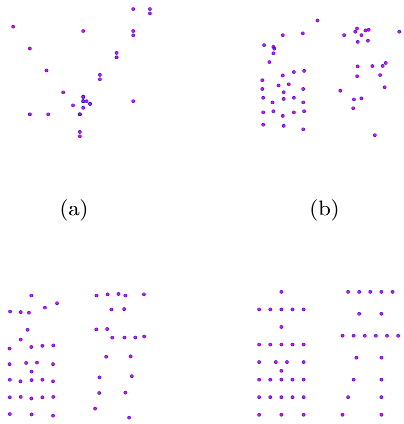


Fig. 2. The plot of function V vs time t.



(a) (b)

(c) (d)

Fig. 3. The example of forming the Chinese name of our university.

The initial positions, which is highly skew from the desired topology and illustrated by Figure 3 (a), are chosen randomly. And we note that the pre-specified topology is reached in Figure 3 (d) and maintained thereafter.

5. CONCLUSION

Motivated by a class of problems associated with control of distributed dynamic systems, we considered graphs with incidence relations that are dictated by the underlying dynamic states, state-dependent graphs. In particular, we considered the problem of controlling the structure of dynamic graphs so that the resulting motion always make the structure of dynamic state-dependents graphs converge to a pre-defined topology. The potential application of the main results includes: flocks of mobile agents equipped with sensing and communication devices, rendezvous in space, distributed sensor fusion in sensor networks, synchronization of coupled oscillators and etc.

ACKNOWLEDGMENT

The authors would like to thank the anonymous referees for their comments and suggestions.

This work was partly supported by the National Natural Science Foundation of P. R. China (No. 60774088), the Program for New Century Excellent Talents of High Education of China (No. NCET-2005-229), the Application Base and Frontier Technology Research Project of Tianjin (08JCZDJC21900), and the Science & Technology Research Key Project of Education Ministry of China (No. 107024).

(Received June 19, 2008.)

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