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ON CHARACTERIZATION OF THE SOLUTION SET IN CASE OF GENERALIZED SEMIFLOW

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In the paper, a possible characterization of a chaotic behavior for the generalized semiflows in finite time is presented. As a main result, it is proven that under specific conditions there is at least one trajectory of generalized semiflow, which lies inside an arbitrary covering of the solution set. The trajectory mutually connects each subset of the covering. A connection with symbolic dynamical systems is mentioned and a possible numerical method of analysis of dynamical behavior is outlined.

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1. INTRODUCTION

When H. Poincare in 1881 – 1886 published his famous series of four memoirs related to topological analysis of non-linear dynamical systems and simultaneously a tract on celestial mechanics [27], a new fruitful era of research that addressed the complicated behavior of non-linear dynamical systems has begun. In that time, only three types of attractors were known, namely the fixed points, the periodic trajectories and the surfaces (in case of conservative systems).

In the beginning of the 1960's, American meteorologist E. N. Lorenz [17] has attracted much attention by his famous system of three nonlinear ODEs. Lorenz obtained this system by "truncating" the Navier–Stokes equations. Thought many scientists, especially experimentalists, knew this article, it is not too surprising that most mathematicians did not. Thus, when Ruelle and Takens proposed [28] specifically that turbulence was likely an instance of a "strange attractor". they did so without specific solutions of the Navier–Stokes equations, or truncated ones, in mind. In fact, it was the first rendezvous with serious and mathematically correct approach to the problems of the chaos when dealing with deterministic continuous/discrete dynamical systems. The natural question instantaneously rose up, namely whether that kind of behavior is exceptional in the nature or one has to await it rather often. In the case of Lorenz attractors, the affirmative answer to the later question has been done by R. F. Williams [33] who generalized previous result of J. Guckenheimer [13] and the main result is (roughly speaking): *There is uncountably many topologically mutually distinct Lorenz attractors.*

Since that time, retrospectively, one can observe two basic research areas related to chaotic systems : the first one is focused to the quest for possible new dynamical systems that produce a chaotic behavior while the second one concentrates on the problems of the numerical analysis of a particular and already known chaotic dynamical system [4, 5, 7, 29, 30].

Searching for new systems has been in principle bidirectional, too. The first direction has its roots, in fact, in physics. A lot of physical phenomena are described by a systems of nonlinear partial differential equations and to solve them, one has to reduce the infinite-dimensional description to a finite one. This is done usually by Galerkin method. As real physical systems are conservative, the first phase of their analysis is an attempt to prove the existence of an attractor and to access dimension of it at least theoretically and to access a neighborhood of that attracting set so that all trajectories that start in this neighborhood have as their target just the attractor. Equations of mathematical physics for which this program is carried out include the following: the reaction-diffusion equations from chemical dynamics and population growth, the incompressible Navier–Stokes equations, geophysical flows, the Kuramoto–Shivashinsky equation, the Cahn–Hilliard equation, the sine-Gordon equation, a nonlinear wave equations, the Ginzburg–Landau equation (a nonlinear Schrödinger equation) and a lot of other mutations of the just mentioned. Among of hundreds other sources, we can mention very nice compendium [32]. The second direction comes out of the original Lorenz system. The main task is to generalize the original structure of the right hand side to get a broad class of systems and trying to find out a new, non-homeomorphic, system while keeping a chaotic behavior. As a representatives of that direction, we can mention, e. g., [6, 8, 18]. A "proof" of chaotic behavior of some concrete system is in that cases usually provided by a numerical methods together with graphics. As a base point, the authors exploit the templates homotopical to the figures 8 or B and the main instrument of numerical computations are different Runge–Kutta difference formulas with automatic change of integration step, which is usually driven by an a posteriori error. There are hundreds of papers, which analyze different numerical methods of bifurcation analysis, computation of stable an unstable manifolds and we only mention a still growing book/lecture notes [12] and also, e. g., [31].

As a second way a perspective approach started in nineties of the last century. Dynamical systems may exhibit many beautiful and highly complicated behavior which are often difficult to capture analytically. With recent advances in computing power, numerical analysis is a useful approach, either as an initial investigation or to study systems for which direct analysis if difficult or even impossible. However, numerical computations require a number of properties which some of the systems do not initially posses. In the very least, they require finite dimensional systems on discretized, compact domains. In addition, sensitive dependence, one of the defining properties of chaotic systems, leads errors to blow up in time. This makes straight forward simulations of the system problematic if not misleading. Two key observations give hope that a reduced system, which captures the interesting dynamical properties exists. The first observation is that invariant sets often contain functions that are more regular than the typical functions of the natural phase space. This regularity, which has been shown to be a property of a large number of systems, allows for a restriction of the studied domain to a compact subset. The second observation is that dynamical systems are often low dimensional (e. g., fixed points, periodic orbits, homoclinic and heteroclinic orbits, horseshoes). An appropriate Galerkin projection of the full system onto a finite dimensional space should capture such low dimensional objects ([4, 5, 7, 29, 30]).

The goal of this paper is a generalization of the results on chaotic behavior of single valued flows to the case of generalized semiflows (definition in Preliminaries). The main result states roughly that there is *at least one* trajectory of the generalized semiflow such that when one has arbitrary covering of the solution set with the surjectivity property (definition in Preliminaries) then the trajectory connects mutually all subsets of the covering in a finite time. As a result, the trajectory of the generalized dynamical semiflow can be described/coded through indexing set of the covering of the solution set and subsequently, the methods of symbolic dynamics can be used to analyze its dynamics. Moreover, the constructive proof of the main theorem leads to idea of numerical methods of the Conley index theory to reveal a topology of the solution set ([24, 25, 26]). So, *the novelty of our result can be seen as a generalization of the existing results for the case of multivalued, noncontinuous dynamical systems, moreover without any need to construct a specific selector in the multivalued case when analyzing solution sets including a possible chaotic behavior of them.*

The rest of the paper is organized as follows. Firstly, we give a short description of ideas that inspired us, then we present some definitions (section three) needed to formulate and to prove the main results in the fourth section.

2. INITIAL INSPIRATION

The main source of our inspiration is [9, 16], where concise description of a new approach on a sufficiently abstract level is presented and it seems to us to be worthwhile to remember it.

We think of $\Phi : X \to X$ as generating a dynamical system with $a' = \Phi(a)$. Let $\{\phi_k | k = 0, 1, 2, \ldots\}$ is complete orthogonal basis for X. Let

$$P_m : X \to X_m \triangleq span\,\{\phi_k | k = 0, 1, \ldots, m-1\}$$

be the orthogonal projection onto the first m modes. The standard Galerkin procedure suggests replacing the study of Φ by that of map $f^{(m)} : X_m \to X_m$ where $f^{(m)} \triangleq P_m \circ \Phi(\cdot, 0)$.

The problem is that if we study the dynamics using $f^{(m)}$ then we do not have any information concerning the errors introduced by the reduction to X_m and by the projection P_m. To get around this problem observe that we can write

$$\Phi(a) = \Phi(P_m a, 0) + (\Phi(a) - \Phi(P_m a, 0)).$$

In general we cannot hope to determine the right hand term exactly. However if we restrict our attention to a "small" set of a, then we may be able to obtain a useful bound on this term. With this in mind, let W be a compact subset of X_m and let V be a compact subset of $(I - P_m)X$. Then, $Z \triangleq W \times V$ is a compact subset of X.

Now assume that it can be shown that for all $a \in Z$

$$\|\Phi(a) - \Phi(P_m a, 0)\| < \epsilon.$$

Then, for all $a \in Z$, $\Phi(a)$ lies within an ϵ-ball of $\Phi(P_m a, 0)$. We want to recast these statements about bounds into the language of dynamical systems. Furthermore, we want this dynamics to be finite dimensional so that we can effectively analyze it. This leads us to consider multivalued or set valued maps $F : W \multimap \mathbb{R}^m$ with the property that for all $a \in Z$,

$$P_m \Phi(a) \in F(P_m a).$$

Perhaps it is worth noting at this point that if the images of F are "too large" then we will not be able to extract useful information from it. Thus, obtaining good bounds on $\|\Phi(a) - \Phi(P_m a, 0)\|$ is essential. At this point we have introduced two functions, the continuous map $f^{(m)} = P_m \circ \Phi(., 0) : W \to \mathbb{R}^m$, which we do not know explicitly, and a multivalued map $F : W \multimap \mathbb{R}^m$ which encloses $f^{(m)}$ in the sense that $f^{(m)}(P_m a) \in F(P_m a)$. It is the function F, which implicitly contains the error estimates that are to be analyzed. However, to directly manipulate an object with the computer it needs to have a combinatorial structure. To do that, W is decomposed into a cubical complex on which a combinatorial multivalued map $\mathcal{F}$ that takes grid elements to sets of grid elements is defined. As each grid element corresponds to a set in W, it is easy to pass from the combinatorial map $\mathcal{F}$ to the multivalued map F. The discussion up to now has described how one proceeds from infinite dimensional problem to a combinatorial object that can be analyzed using computer. The question that remains is how to use this combinatorial information to draw conclusions about the dynamics of Φ. The key tool is the Conley index theory, which is a topological generalization of Morse theory [22, 23]. In particular, it can be expressed in terms of homology, which is a combinatorial algebraic topological theory. Furthermore, the index can be used to prove the existence of specific dynamical structure such as fixed points, periodic orbits, heteroclinic orbits, and shift dynamics. The combinatorial map $\mathcal{F}$ is used to construct isolating neighborhoods and index pairs, and finally to compute the associated Conley index for the map $f^{(m)}$. The important theoretical considerations are that one can pass from $\mathcal{F}$ to a multivalued map F which is enclosure of $f^{(m)}$ and that the Conley index information is preserved through this transition. The final step is to then verify that the information given by the Conley index of $f^{(m)}$ on W may be lifted to the full map Φ on Z. We have moved the definition of the important concept of the Conley index to the Appendix.

The just described abstract schema was successfully applied to some equations like Cahn–Hilliard equation modeling the process of phase separation of a binary alloy at a fixed temperature or Swift–Hohenberg equation describing the onset off Rayleigh–Bénard heat convection, [10, 11].

As far as is to the author known, all examples that have been published in the literature till now are single valued equations. We would like to generalize the above described theory to the case of multivalued ones represented by generalized semiflows. The generalized semiflows cover such objects as, e. g. differential inclusions, e. g., see [1]. We establish some existence result concerning the trajectories of the generalized semiflows and we will discuss some structural properties of them. The results are described in the fourth section but first of all, we will define some objects and their properties in the third section.

3. PRELIMINARIES

Let us define the basic objects of our investigations, e. g., see [2].

Definition 1. A *generalized semiflow* $\mathcal{G}$ on X is a family of maps $\varphi : [0, \infty) \to X$ (called *solutions*) satisfying the hypotheses:

(H1) (*Existence*) For each $z \in X$ there exists at least one $\varphi \in \mathcal{G}$ with $\varphi(0) = z$.

(H2) (*Translates of solutions are solutions*) If $\varphi \in \mathcal{G}$ and $\tau \geq 0$, then $\varphi^\tau \in \mathcal{G}$, where $\varphi^\tau(t) \triangleq \varphi(t + \tau), t \in [0, \infty)$.

(H3) (*Concatenation*) If $\varphi, \psi \in \mathcal{G}, t \geq 0$, with $\psi(0) = \varphi(t)$ then $\theta \in \mathcal{G}$, where

$$\theta(\tau) \triangleq \begin{cases} \varphi(\tau) & \text{for} \quad 0 \leq \tau \leq t, \\ \psi(\tau - t) & \text{for} \quad t < \tau \end{cases}$$

(H4) (*Upper-semicontinuity with respect to initial data*) If $\varphi_j \in \mathcal{G}$ with $\varphi_j(0) \to z$ then there exists a subsequence φ_μ of φ_j and $\varphi \in \mathcal{G}$ with $\varphi(0) = z$ such that $\varphi_\mu(t) \to \varphi(t)$ for each $t \geq 0$.

Remark. Let $\mathcal{G}$ be a generalized semiflow and let $E \subset X$. Define for $t \geq 0$

$$T(t)\, E \triangleq \{\varphi(t)\, | \varphi \in \mathcal{G} \text{ with } \varphi(0) \in E\},$$

so that $T(t) : 2^X \to 2^X$, where 2^X is the space of all subsets of X. It follows from (H2), (H3) that $\{T(t)\}_{t \geq 0}$ defines a semigroup on 2^X. Note that (H4) implies that $T(t)\{z\}$ is compact for each $z \in X, t \geq 0$.

Notation. The expression $\varphi(\cdot) \in \mathcal{G}(x)$ means the solution $\varphi(\cdot)$ that starts at $x \in X$.

If for each $z \in X$ there is exactly one $\varphi \in \mathcal{G}$ with $\varphi(0) = z$ then $\mathcal{G}$ is called a *semiflow*.

Definition 2. The generalized semiflow $\mathcal{G}$ is said to be *upper-semicompact* from X to $\mathcal{C}([0, \infty); X)$ ($\mathcal{C}$ means a space of continuous mappings from $[0, \infty)$ into X) if for any solution $x_n \in X$ converging to $x \in X$ and for any generalized semiflow $\varphi_n(\cdot) \in \mathcal{G}$ starting at x_n, there exists a subsequence of $\varphi_n(\cdot)$ converging to a generalized semiflow $\varphi(\cdot) \in \mathcal{G}$ uniformly on compact intervals.

Definition 3. Let D be a closed set and let us consider a sequence of nonempty closed subsets $S_n \subset D$, $n \in \mathbb{N} \cup 0$, $\mathcal{S} = \{S_n\}$, such that $S_n \cap S_{n+1} \neq \emptyset$. Let $\varphi(\cdot) \in \mathcal{G}(x)$ be a solution. We say that $\mathcal{S}$ *forms a* $\varphi(\cdot)$*-chain* when there exists a nondecreasing sequence of times $0 \leq t_0 \leq t_1 \leq \ldots \leq t_n \leq \ldots$ such that for all $n \geq 0$, for any $t \in [t_n, t_{n+1}]\,, \varphi(t) \in S_n$ and $\varphi(t_{n+1}) \in S_{n+1}$.

Definition 4. Let D be a closed set and let us consider a sequence of nonempty closed subsets $S_n \subset D$ and we assume that there exists $T < +\infty$ such that for each nonnegative n and for each $z \in S_{n+1}$ there exists $x \in S_n$ with solution $\varphi_n(\cdot) \in S_n$ and exists $\tau \in [0, T)$ with $\varphi(\tau) = z$, then the system $\mathcal{S} = \{S_n\}$ is called to be $T-$surjective under $\mathcal{G}$. When $T \to +\infty$, then the system $\mathcal{S} = \{S_n\}$ is called to be *surjective under* $\mathcal{G}$.

Definition 5. Let $D \subset X$ be a constrained set. A solution $\varphi(\cdot)$ is *locally positively D-invariant* when there exists $T > 0$ such that for each $t \in [0, T]$ we have $\varphi(t) \in D$. When $T = +\infty$ we call $\varphi(\cdot)$ *positively D-invariant*, compare, e. g., [3, 14, 15]. When all $\varphi(\cdot) \in \mathcal{G}$ are (locally) positively D-invariant, we say that generalized semiflow $\mathcal{G}$ is (locally) positively D-invariant.

Below, we give some examples to illustrate the concept of the *generalized semiflow*.

Example 1. The *generalized* semiflow is a far-going generalization of the usual conception of the (semi)dynamical systems. We have avoided the smoothness even the continuity of the defining tangent vector manifold. The dynamics of the process can be discontinuous even multivalued set-mapping. From this point of view, the traditional dynamical systems analyzed, e. g., in [4, 5, 6, 19, 20] are special cases of the general concept – the continuous and/or smooth dynamical system are embedded into the set of the generalized semiflows. The objects of our investigation can be represented, e. g., by a three-dimensional inclusion

$$\begin{aligned} \dot{x}(t) &= -y(t) - u(t) \\ \dot{y}(t) &= x(t) + ay(t) \\ \dot{u}(t) &\in [-c, c] \end{aligned}$$

or, generally, when one analyzes a control system with multivalued feedback like

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ u(t) &\in U(x(t))\,, \end{aligned}$$

where $x(\cdot) \in X$ and $u(\cdot) \in \mathcal{U}$ and X and $\mathcal{U}$ are finite-dimensional vector spaces with, generally, different dimensions. One can encounter such a type of generalized semiflows, e. g., in economy.

Example 2. As a second example ([1, 21]), we have in mind the evolution inclusions of the type

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} \in A(y(t)) + F(y(t)), \ y(0) = y_0 \in X,$$

where X is a Banach space, $A : D(A) \subset X \to 2^X$ is an dissipative operator and $F : X \to 2^X$ is a multivalued Lipschitz map with nonempty, bounded, convex, closed values. An important particular case is $A = \partial\varphi$, where $\partial\varphi$ is the subdifferential of a proper, convex, lower semicontinuos function φ. As a representative of that abstract definition, one can consider boundary value problem

$$\frac{\partial y}{\partial t} \in \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial v}{\partial x_j}\right) + a_0(x)y + f(y) + h, \text{ on } \Omega \times (0,T)$$

$y = 0$ on $\partial\Omega \times (0,T)$, $y(x,0) = u_0(x)$ on Ω.

That type of equations belongs to the reaction-diffusion type of equations of mathematical physics. Equations describe such phenomena like fluid flow, chemical reactions, generally energy transport within a stochastic environment.An example arrives directly from the physics, namely from the theory of incompressible Navier–Stokes equations ([2]). Let $f \in L^2(\Omega)^3$ and consider the incompressible Navier–Stokes equations $u_t + (u \cdot \nabla)u = \nu\Delta u - \nabla p + f$, $\operatorname{div} u = 0$ with boundary condition $u|_{\partial\Omega} = 0$, where $\nu > 0$ is a constant (kinematic viscosity), u means the flow velocity, p means the pressure and f are external forces. As usual, we switch to the weak formulation of the problem. We use the function spaces $\mathcal{V} = \{u \in C_0^\infty(\Omega)^3 \,|\, \operatorname{div} u = 0\}$, $H =$ closure of $\mathcal{V}$ in $L^2(\Omega)^3$, $V = \left\{u \in H_0^1(\Omega)^3 | \operatorname{div} u = 0\right\}$. The weak formulation of the Navier–Stokes equations on the space of, just defined, solenoidal functions can be found anywhere, e. g., [2, 32]. It can be proved that the set of weak solutions, G_{NS}, is a generalized semiflow on H, e. g., [2].

4. MAIN RESULTS

The main result is formulated as follows:

Theorem. Let D be a compact subset. We assume

1. generalized semiflow $\mathcal{G}$ is positively D-invariant and upper semicompact,
2. there is a index set $\mathcal{I}$ such that D is covered by a family of closed subsets $\mathcal{S} = \{S_\alpha\}_{\alpha\in\mathcal{I}}$, $D \subset \bigcup_{\alpha\in\mathcal{I}} S_\alpha$,
3. let $\mathcal{S}$ be T-surjective under $\mathcal{G}$, i. e. $T < +\infty$.

Then for any sequence $\{\alpha_0, \alpha_1, \dots, \alpha_n, \dots\} \subset \mathcal{I}$, there exists *at least one* solution $\varphi(\cdot) \in \mathcal{G}$ and a nondecreasing sequence $0 \le t_0 \le t_1 \le \dots \le t_n \le \dots$ such that system $\mathcal{S}$ is surjective under $\varphi(\cdot) \in \mathcal{G}$.

To prove the Theorem, we firstly formulate a technical Lemma.

Lemma. Let D be a compact subset. We assume

1. generalized semiflow $\mathcal{G}$ is positively D-invariant and upper semicompact,
2. we consider a sequence of compact subsets $\mathcal{S} = \{S_n\}$ such that $S_n \subset D$,
3. let $\mathcal{S}$ be T-surjective under $\mathcal{G}$, i. e. $T < +\infty$.

Then there exists at least one solution $\varphi(\cdot) \in \mathcal{G}$ such that $\mathcal{S}$ forms a $\varphi(\cdot)$-chain.

Proof. The main idea of the proof is to glue together local solutions between neighboring subsets S_n. We will construct an inverse iterated process gluing together subsequently $S_n, S_{n-1}, \ldots, S_0$ to find out whether there is a nonempty subset of S_0, which can act as a set of initial point from which an arbitrary target point in S_n can be reached. To accomplish this part of the proof, we recurrently define sets

$$\mathcal{U}_{n,n-1} \triangleq \{x \in S_{n-1} |\ \exists\ \tau \in [0,T]\ \exists \varphi(\cdot) \in \mathcal{G} \Rightarrow \varphi(t) \in S_{n-1}\ t \in [0,T]\ \varphi(\tau) \in S_n\}$$
$$\mathcal{U}_{n,j} \triangleq \{x \in S_j |\ \exists\ \tau_j \in [0,T]\ \exists \varphi(\cdot) \in \mathcal{G}\ \varphi(\cdot) \in S_j(x) \Rightarrow \varphi(\tau_j) \in \mathcal{U}_{n,j+1}\}$$

where $j = n-2, n-1, \ldots, 0$.

Thus the set $\mathcal{U}_{n,0}$ is the set of initial states $x_0 \in S_0$ from which at least one solution φ goes successively through all $S_j,\ j = 1, \ldots, n$. Assumption 3 of Lemma implies that $\mathcal{U}_{n,j} \neq \emptyset$ and closed due to assumption of upper semicompactnes. We observe that filtration $\mathcal{U}_{n,0}, n \in \mathbb{N} \cup 0$ forms a non increasing family

$$\ldots\ \mathcal{U}_{n,0} \supset \mathcal{U}_{n-1,0} \supset \ldots \supset \mathcal{U}_{0,0}$$

and since D is compact, the intersection $\mathcal{U}_\infty \triangleq \bigcap_{n=0}^{\infty} \mathcal{U}_{n,0} \neq \emptyset$,

Now, to prove the existence and surjectivity of the $\varphi(\cdot) \in \mathcal{G}$, we will, by backward's way, construct a solution that starts at $x \in \mathcal{U}_\infty$ when n is kept fixed. The proof is based on gluing technique and classical mathematical induction with respect to the local time in order to get the time-global solution over the whole $\varphi(\cdot)$-chain formed by $\mathcal{S}$.

So, let as have $x \in \mathcal{U}_\infty$ when n is kept fixed. There exist $\psi_1(\cdot) \in \mathcal{G}$ and $\delta_1^n \in [0,T]$ such that $\psi_1(\delta_1^n) \in \mathcal{U}_{n,1}$. We set $t_1^n \triangleq \delta_1^n$, $\varphi_1^n = \psi_1(t_1^n)$ and $\varphi_n(t) \triangleq \psi_1(t)$ on $[0, t_1^n]$. Now we suppose that we have already constructed $\varphi_n(\cdot)$ on the interval $\left[0, t_j^n\right]$ such that $\varphi_n\left(t_j^n\right) \in \mathcal{U}_{n,j} \subset S_j$ for $j = 1, \ldots, k$. As $\varphi_n(t_k^n) \in \mathcal{U}_{n,k}$, there exist $\psi_{k+1}(\cdot) \in \mathcal{G}(t_k^n)$ and $\delta_{k+1}^n \in [0,T]$ such that $\psi_{k+1}\left(\delta_{k+1}^n\right) \in \mathcal{U}_{n,k+1}$. We set $t_{k+1}^n \triangleq t_k^n + \delta_{k+1}^n$ and $\varphi_n(t + t_k^n) \triangleq \psi_{k+1}(t)$ on $\left[t_k^n, t_{k+1}^n\right]$. When $k = n$, we can prolongate $\varphi_n(\cdot)$ to $[t_n^n, +\infty[$ by any solution that starts at $\varphi_n(t_n^n)$ at time t_n^n.

We have proved that there exists $\varphi_n(\cdot) \in \mathcal{G}(x)$ and a sequence of $t_n^j \in [0, jT]$ such that

$$\forall\, j = 1, \ldots, n,\ \varphi_n\left(t_n^j\right) \in \mathcal{U}_{n,j} \subset S_j \text{ and } \forall\, t \in \left[t_n^{j-1}, t_n^j\right],\ \varphi_n(t) \in S_j.$$

We suppose that generalized semiflow is upper semicompact. That assumption implies that sequence $\varphi_n(\cdot) \in \mathcal{G}$ is compact in the space $\mathcal{C}([0,\infty); X)$. Then a subsequence $\varphi_m(\cdot)$ converges to some solution $\varphi(\cdot) \in \mathcal{G}(x)$ starting at x uniformly on compact intervals. Thus, we can extract successive converging subsequences of

$\delta_j^n \in [0,T]$ converging to δ_j when $n \geq j \to +\infty$ and setting $t_{j+1} \triangleq t_j + \delta_j$, we can conclude that $\varphi(t_j) \in S_j$. □

Now, one can see that Theorem is an easy consequence of the Lemma.

Proof. When we associate the sequence $\{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\}$ with the sequence of subsets $S_n \triangleq S_{\alpha_n}$, we observe that assumptions of the Lemma are satisfied. □

We give an very informal illustrative example to show what the theorem tells to us ([34]). The example comes to us from ancient times, namely the iterative Newton's method for determining the zeros of a polynomial. Everybody very well knows that the initial point of the iterative process should be sufficiently close to a zero of the polynomial we try to find. Then the iterates will converge to the zero set of that polynomial. But the term "sufficiently close" makes very often a lot of troubles as a very strong "separation of zeros" theorems have to be used prior to start the iteration process. So, the eligible question arise: what happens if the initial point is not "sufficiently close" to the zero set of the polynomial. We a little bit formalize the just mentioned scenario (see the following Figure 1).

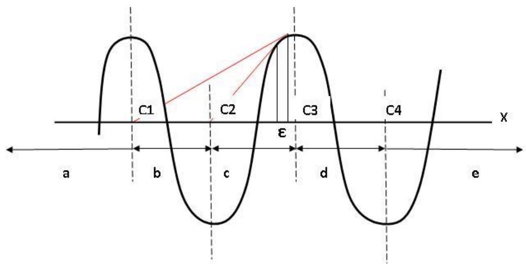

Fig. 1.

Let us have a polynomial f of the fifth degree with five real separated zeros. The zeros are separated into 5 intervals defined by critical points of the polynomial $c_i, i = 1,2,3,4$ marked by $\{a,b,c,d,e\}$. The main iteration process is the very well known mapping $T : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ are real numbers, specifically $Tx = x - \frac{f(x)}{f'(x)}$, where $f^{'}$ is the derivative of f and T is sometimes called the Newton transform of f. One can see that to catch a zero in the interval b starting the iteration process from the interval c, there is a very tiny interval $\epsilon \in c$, where the the iteration step finishes inside the interval b (the red lines). Now, each iteration $\{x_1, x_2, \ldots, x_n, \ldots\}$ forming a discrete trajectory of a discrete dynamical process, can be coded otherwise, namely by an interval in which the iteration lies. That means, in our case we have three symbols – the alphabet $\{b,c,d\}$ and, for example, if $x_1 \in c, x_2 \in b, x_3 \in d, x_4 \in b, \ldots$ then we get a symbolic expression of the trajectory as $\{c,b,d,b,...\}$. The transition from the iterative description to the symbolic one represents set-valued mapping – the generalized semiflow. Let $\mathcal{S} = \{b,c,d\}^N$ means a set of all possible sequences, each entry of them is one of the three symbols. Now, we are ready to show the meaning of the just proved theorem. It means that when we choose *arbitrary* symbolic sequence then it has to be in the image of the Newton transform. To show that, we use the "iterated inverse image" approach, [34], in fact, we mimic the proof of the theorem. The beginning of that process is depicted in Figure 1. To get the symbol b, one can start inside the interval/symbol c, more precisely inside $\epsilon \in c$. As a second step, we need to reach the interval $\epsilon \in c$ from subsequent symbol of the arbitrary symbolic sequence, we have chosen at the beginning. But this can be accomplished in the similar way as it was previously due to smoothness of the Newton transform. In this way, we can continue as long as the arbitrary chosen symbol sequence is not exhaust. The process is, in some way, similar to the Cantor set construction. Much more deeply is the Newton method, from the symbolic dynamics point of view, analyzed in, already mentioned, [34].

5. OBSERVATIONS AND OUTLOOKS

In this section we give some conclusions and future outlooks of the just proved theorem. Thought the proof of the theorem is rather easy and geometrically clear, we can do some far-going conclusions.

As a first observation, the Theorem's proof gives some guidelines how to get a solution of the generalized semiflow systems numerically. Selecting an enough broad bounded and closed region of the phase space, we can cover it by a cubical complex as it was roughly mentioned in the Initial inspirations. It can be accomplished due to assumptions of the Theorem. As a next step, we can make use of the, in the Initial inspirations mentioned, results and numerical algorithms of the computational homology and of the Conley index theory, which should be generalized for the case of generalized semiflows. As a result, the topological structure of the solutions of the generalized semiflows should be obtained.

As a second observation, the indexing set $\mathcal{I} = \{\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots\}$, which describes the covering of the phase space, forms an alphabet, which codes the trajectories that start in $x \in \mathcal{U}_\infty$ in order to get a code word. The trajectories's coding provides a straight way to utilize known methods of symbolic dynamics. By analysis of the code word there are known topological methods, which can reveal whether there are fixed points, periodic trajectories even if the trajectories embody a chaotic behavior.

Both observations are just now under further investigation and the results are scheduled for publishing in the near future.

APPENDIX

In the Appendix, we give a definition of the concept of *Conley index* and some of its basic properties (see [23]). The index is a generalization of the very well known concept of the Morse index of a hyperbolic equlibrium point of a vector field, defined as the dimension of its unstable manifold at that point. Conley index is the homotopy class of the space obtained from an isolated invariant set by collapsing the exit set to a point. The fundamental idea of the Conley index is to characterize an isolated invariant set by the behavior of the flow on the boundary of a neighborhood. More precisely:

Isolating neighborhood. A compact set $N \subset X$, where X denotes a locally compact metric space, is an *isolating neighborhood*, if $\mathrm{Inv}(N, \varphi) := \{x \in N | \varphi(\mathbb{R}, x) \subset N\} \subset \mathrm{int}\, N$, where $\varphi(\mathbb{R} \times X) \to X$ is a flow satisfying $\varphi(0, x) = x$ and $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$, and $\mathrm{int}\, N$ denotes the interior of N.

Isolated invariant set (i.i.s.). S is an *isolated invariant set* if $S = \mathrm{Inv}(N)$ for some isolating neighborhood N.

Exit set. $N_e \subset N$ is defined as follows $N_e = \{x \in N | \varphi([0, t], x) \bigcap N \nsubseteq N, \forall\, t \geq 0\}$.

Isolating block. An *isolating block* for the i.i.s. S is an isolating neighborhood N such that the flow has no "internal tangencies" with the boundary of N − there is no trajectory segment touching the boundary of N from the inside. More precisely, let $x \in \partial N$, where ∂N means the boundary of N, then there exists some $\epsilon \geq 0$ such that $\varphi((-\epsilon, 0), x) \subset N$ and $\varphi((0, \epsilon), x) \subset N$.

Pointed space. A *pointed space* (Y, y_0) is a topological space Y with a distinguished point $y_0 \in Y$.

Given a pair (N, L) of spaces with $L \subset N$, $N/L := (N \setminus L) \bigcup [L]$, where $[L]$ denotes the equivalence class of points in L in the equivalence relation: $x \sim y$ iff $x = y$ or $x, y \in L$.

Let (X, x_0) and (Y, y_0) be pointed topological spaces and let $f, g : (X, x_0) \to (Y, y_0)$ be a continuous functions. Implicitly, the assumption that $f(x_0) = g(x_0) = y_0$ is supposed.

Homotopy of functions. We say that f is *homotopic* to g, denoted by $f \sim y$, if there exists a continuous function $F : X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, $F(x_0, s) = y_0$, $0 \leq s \leq 1$.

Obviously, $\sim$ is an equivalence relation. The equivalence class of f in this relation is called the *homotopy class* of f and denoted $[f]$.

Homotopy of spaces. Two pointed topological spaces (X, x_0) and (Y, y_0) are *homotopic* , $(X, x_0) \sim (Y, y_0)$, if there exists $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (X, x_0)$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$.

Homotopy defines an equivalence class on the set of topological spaces, e. g., $\mathbb{R}^2 \setminus \{0\} \sim S^1$.

Index pair. Let S be an i.i.s. A pair of compact sets (N, L) where $L \subset N$ is called an *index pair* for S if

1. $S = \mathrm{Inv}\,(cl\,(N \setminus L))$ and $N \setminus L$ is a neighborhood of S (cl means a closure),

2. L is *positively invariant* in N; that is given $x \in L$ and $\varphi([0,t],x) \subset N$, then $\varphi([0,t],x) \subset L$.

3. L is an exit set for N.

Conley index. Let S be an i.i.s. (see previous definition). The *(homotopy) Conley index* of S is $h(S) = h(S,\varphi) \sim (N/L,[L])$ – the Conley index is a homotopy type of the pointed space N/L. One can prove that

1. Given an isolated invariant set S, there exists an index pair.

2. The Conley index is well defined: let (N,L) and $(N^{'},L^{'})$ be index pairs for an i.i.s. S. Then $(N/L,[L]) \sim (N^{'}/L^{'},[L^{'}])$.

The following Theorem states the simplest example of how the Conley index is used to make assertions about the dynamics of φ:

Theorem – Ważewski property. If the Conley index of N is not trivial, then $\mathrm{Inv}(N,\varphi) \neq \oslash$.

The second important Theorem leads to the conclusion that the Conley index of the dynamical system generated by a sufficiently good numerical approximation will be the same as that of the original system:

Theorem – Continuation property. If N is an isolating neighborhood for a continuous family of maps φ_λ, e. g. $\mathrm{Inv}(N,\varphi_\lambda) \subset \mathrm{int}\, N$ for $\lambda \in [0,1]$, then the Conley index of N under φ_{λ_0}, is the same as the Conley index of N under φ_{λ_1}, where $\lambda_0, \lambda_1 \in [0,1]$.

Example. We demonstrate the Conley homotopy index concept on the well known 2-dimensional hyperbolic system (see Figure 1), which can be seen as a topologically equivalent flow in the neighborhood of the origin when the fixed point of a original system is hyperbolic fixed point due to Hartman–Grobman theorem (see, e. g. [15]). The origin represents the hyperbolic fixed point of the dynamical system. As one can see, every box round the origin that has the origin as its interior point, constitute an isolating block N. The upper and the lower sides of the box (the bold dash lines) compose the exit sets N_e. Here, the exit set is composition of two disjoint sets. Now, accordingly to the above mentioned definitions, we collapse (homotopy equivalence) each of the exit sets to a point (see Figure 3) – one gets the first picture. The tilde between pictures means homotopy. Subsequently, the two point that form northern and southern poles can be by another homotopy equivalence identified so that, as a result, we get a 1-sphere with a point – pointed space $(S_1,*)$. The star $*$ represents the northern pole in the last picture of the Figure 3. So the Conley index has a homotopy type of the circle with a distinguished point. Due to isomorphism of the 1-sphere with $\mathbb{Z}^1$ (1-dimensional integers), we conclude an agreement with the result of the Morse theory, which states 1-dimensional unstable manifold in that case.

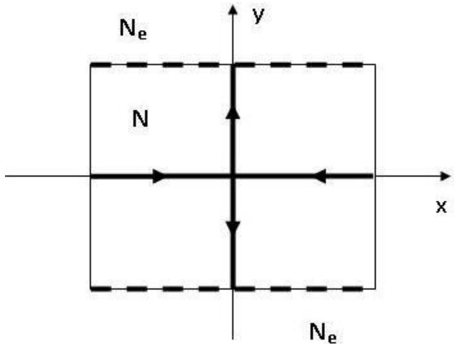

Fig. 2.

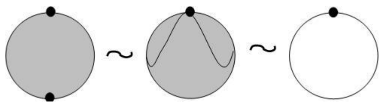

Fig. 3.

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