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A NOTE ON THE OPTIMAL PORTFOLIO PROBLEM IN DISCRETE PROCESSES

Naoyuki Ishimura and Yuji Mita

We deal with the optimal portfolio problem in discrete-time setting. Employing the discrete Itô formula, which is developed by Fujita, we establish the discrete Hamilton–Jacobi–Bellman (d-HJB) equation for the value function. Simple examples of the d-HJB equation are also discussed.

Keywords: optimal portfolio problem, discrete Itô formula, discrete Hamilton–Jacobi–Bellman equation

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1. INTRODUCTION

In the study of the optimal behavior in economic problem under risk environment, it is important to investigate the optimal value function with respect to certain utility. When we formulate an optimization problem in terms of the stochastic control framework, the characterization of the optimality usually results in a form of the Hamilton–Jacobi–Bellman (HJB) equation for the corresponding value function. The HJB equation can reflect the optimal nature implied by the model; the solution tells us what is the optimal strategy.

In this note, we are concerned with the portfolio optimization problem under discrete-time circumstance. The market price process is modelled in discrete-time stochastic sequence. We derive the discrete Hamilton–Jacobi–Bellman (d-HJB) equation for the value function with respect to some utility functions. We also examine simple examples.

It is well known that the optimization problem under discrete-time stochastic processes has been already widely investigated and much progress has been made. The characterization of the optimality, which is essentially equivalent to the d-HJB equation of the current article, has been also well discussed. We refer for instance to Chapter III of an excellent book by Duffie [3].

The novelty of our research is to recast and reformulated the optimization problem in terms of the discrete Itô formula. The formula is recently discovered by Fujita and Kawanishi [5] and corresponds to the famous Itô formula in the continuous model. On the ground of this discrete Itô formula, we are able to build the theory of discrete-time portfolio optimization.

The paper is organized as follows. We recall the discrete Itô formula in Section 2. The d-HJB equation is then established in Section 3, which is followed in Section 4 by examples. Section 5 concludes the present article with discussions.

2. DISCRETE ITÔ FORMULA

We here recall the basic tool of our researches, namely, the discrete Itô formula. In the following presentation, we adopt the argument from recent work by Fujita and Kawanishi [5] (see also Fujita [4]).

Let $t = 0, 1, 2, \dots$ denote discrete time series and let $\{B_t\}_{t=0,1,2,\dots}$ with $B_0 = 0$ be the one-dimensional random walk [8]:

$$B_t = \sum_{n=1}^{t} Y_n, \tag{1}$$

where $\{Y_n\}_{n=1,2,\dots}$ are independent and identically distributed (i.i.d.) random variables. For simplicity, we assume

$$\mathrm{Prob}(Y_n = 1) = \mathrm{Prob}(Y_n = -1) = \frac{1}{2}, \qquad n = 1, 2, \dots, \tag{2}$$

throughout this paper. Precisely stated, we confine ourselves to treating the symmetric standard one-dimensional random walk. Generalizations are possible and will be revisited in Section 5.

The discrete Itô formula is then expressed as follows.

Theorem 1. (Fujita and Kawanishi [5])

(a) For any $f : \mathbb{Z} \to \mathbb{R}$, we have

$$f(B_{t+1}) - f(B_t) = \frac{f(B_t + 1) - f(B_t - 1)}{2} Y_{t+1} + \frac{f(B_t + 1) - 2f(B_t) + f(B_t - 1)}{2}.$$

(b) For any $f : \mathbb{Z} \times \mathbb{N} \to \mathbb{R}$, we have

$$\begin{aligned} f(B_{t+1}, t+1) - f(B_t, t) = {} & \frac{f(B_t + 1, t+1) - f(B_t - 1, t+1)}{2} Y_{t+1} \\ & + \frac{f(B_t + 1, t+1) - 2f(B_t, t+1) + f(B_t - 1, t+1)}{2} \\ & + f(B_t, t+1) - f(B_t, t). \end{aligned}$$

Now, the price process $\{X_t\}_{t=0,1,2\dots}$ with which we are concerned in this paper is assumed to be governed by the following discrete stochastic processes.

$$X_{t+1} - X_t = \mu(X_t, t) + \sigma(X_t, t)(B_{t+1} - B_t), \qquad t = 0, 1, 2, \dots, \tag{3}$$

where μ, σ are given continuous functions. For related model we refer to [7]. It can be seen that μ expresses the drift rate and σ means the volatility.

Concerning the process (3), we obtain the next proposition, which will be useful in the sequel.

Proposition 1. (a) For any $f : \mathbb{Z} \to \mathbb{R}$, we have

$$\begin{aligned} f(X_{t+1}) - f(X_t) = {} & \frac{f(X_t + \mu_t + \sigma_t) - f(X_t + \mu_t - \sigma_t)}{2} Y_{t+1} \\ & + f(X_t + \mu_t) - f(X_t) \\ & + \frac{f(X_t + \mu_t + \sigma_t) - 2f(X_t + \mu_t) + f(X_t + \mu_t - \sigma_t)}{2}, \end{aligned} \tag{4}$$

where the use of abbreviations $\mu_t := \mu(X_t, t)$, $\sigma_t := \sigma(X_t, t)$ are made.

(b) For any $f : \mathbb{R} \times \mathbb{N} \to \mathbb{R}$, we have

$$\begin{aligned} f(X_{t+1}, t+1) - f(X_t, t) = {} & \frac{f(X_t+\mu_t+\sigma_t, t+1) - f(X_t+\mu_t-\sigma_t, t+1)}{2} Y_{t+1} \\ & + (f(X_t + \mu_t, t+1) - f(X_t, t+1)) \\ & + \frac{f(X_t+\mu_t+\sigma_t, t+1) - 2f(X_t+\mu_t, t+1) + f(X_t+\mu_t-\sigma_t, t+1)}{2} \\ & + (f(X_t, t+1) - f(X_t, t)). \end{aligned} \tag{5}$$

For convenience of notation we write the latter part in the right hand side of (5) as follows:

$$\begin{aligned} \mathcal{L}_X f(X_t, t) := {} & f(X_t + \mu_t, t+1) - f(X_t, t+1) \\ & + \frac{f(X_t + \mu_t + \sigma_t, t+1) - 2f(X_t + \mu_t, t+1) + f(X_t + \mu_t - \sigma_t, t+1)}{2} \\ & + f(X_t, t+1) - f(X_t, t). \end{aligned}$$

Remark. If the time step is taken to be $\delta \mathrm{d}t$ so that (3) turns into

$$X_{t+\delta \mathrm{d}t} - X_t = \mu(X_t, t)\delta \mathrm{d}t + \sigma(X_t, t)\sqrt{\delta \mathrm{d}t}(B_{t+\delta \mathrm{d}t} - B_t),$$

we then find, parallel to (4),

$$\begin{aligned} & f(X_{t+\delta \mathrm{d}t}) - f(X_t) \\ & = \frac{f(X_t + \mu_t \delta \mathrm{d}t + \sigma_t \sqrt{\delta \mathrm{d}t}) - f(X_t + \mu_t \delta \mathrm{d}t - \sigma_t \sqrt{\delta \mathrm{d}t})}{2} Y_{t+\delta \mathrm{d}t} \\ & \quad + f(X_t + \mu_t \delta \mathrm{d}t) - f(X_t) \\ & \quad + \frac{f(X_t + \mu_t \delta \mathrm{d}t + \sigma_t \sqrt{\delta \mathrm{d}t}) - 2f(X_t + \mu_t \delta \mathrm{d}t) + f(X_t + \mu_t \delta \mathrm{d}t - \sigma_t \sqrt{\delta \mathrm{d}t})}{2}. \end{aligned}$$

Consequently, as $\delta \to 0$, we recover the well known Itô formula

$$\mathrm{d}f(X_t) = f'(X_t)\sigma_t \mathrm{d}W_t + f'(X_t)\mu_t \mathrm{d}t + \frac{1}{2} f''(X_t)\sigma_t^2 \mathrm{d}t,$$

where $\{W_t\}$ denotes a standard Wiener process. This is just a heuristic argument, which, however, justifies our discretization scheme at the same time.

3. DISCRETE HAMILTON–JACOBI–BELLMAN EQUATION

In this section, we wish to analyze the optimal portfolio problem in discrete-time setting. We begin with introducing a controlled price process $\{X_t\}_{t=0,1,2,\dots}$ of the form

$$X_{t+1} - X_t = \mu(X_t, t, u_t) + \sigma(X_t, t, u_t)(B_{t+1} - B_t), \qquad t = 0, 1, 2, \dots, \tag{6}$$

where $\{u_t\}_{t=0,1,2,\dots}$ stand for the adapted control; namely, u_t is measurable with respect to $\sigma(B_k \,|\, k = 1, 2, \dots, t)$ $(t = 1, 2, \dots)$. The involvement of the control variable u_t in (3) is for the sake of generality. Its interpretation should be considered for each specific model. We will give examples in the next section.

Our aim is then to determine the adapted control $\{u_t\}_{t=0,1,2,\dots}$ which maximizes certain functional; that is, we want to solve the problem:

$$V(x, t) := \sup_{\{u_s\}_{s=t}^{T-1}} J\Big(x, t, \{u_s\}_{s=t}^{T-1}\Big), \tag{7}$$

where we have put

$$J(x, t, \{u_s\}_{s=t}^{T-1}) := \mathrm{E}^{x,t}\Big[\sum_{k=t}^{T-1} U_1(X_k, k, u_k) + U_2(X_T, T) \,\Big|\, X_t = x\Big],$$

with positive $T \in \mathbb{N}$. Here U_1, U_2 are utility functions, which are customarily assumed to be an increasing and strictly concave function in X_k. We remark that in this formulation the introduction of the utility U_1 is just due to the fact that it is a standard control framework; its interpretation is limited in the context of portfolio optimization.

Now we derive the discrete Hamilton–Jacobi–Bellman (d-HJB) equation, which features a property of the value function $V(x, t)$ and hence gives a solution to the stochastic control problem (7). It should be noted that the assumption of symmetry in (1), (2) is not essential.

Theorem 2. (d-HJB equation) We have for $t = 0, 1, \dots, T-1$,

$$\begin{aligned} &\sup_{\{u_s\}_{s=t}^{T-1}} \{\mathcal{L}_X^u V(x,t) + U_1(x,t,u_t)\} := \\ &\sup_{\{u_s\}_{s=t}^{T-1}} \Big\{ V(x+\mu_t, t+1) - V(x, t+1) \\ &\qquad + \frac{V(x+\mu_t+\sigma_t, t+1) - 2V(x+\mu_t, t+1) + V(x+\mu_t-\sigma_t, t+1)}{2} \\ &\qquad + V(x, t+1) - V(x,t) + U_1(x,t,u_t)\Big\} = 0, \\ &V(x,T) = U_2(x,T), \end{aligned} \tag{8}$$

where we have put

$$\mu_t := \mu(x, t, u_t), \qquad \sigma_t := \sigma(x, t, u_t).$$

P r o o f. We just give a sketch of proof.

Since $\{X_t\}$ is Markovian, the so-called Bellman principle [2],[6] is in order; we infer that

$$V(x,t) = \sup_{\{u_s\}_{s=t}^{T-1}} \mathrm{E}^{x,t}\Big[U_1(x,t,u_t) + V(X_{t+1},t+1)\Big],$$
$$V(x,T) = U_2(x,T).$$

Application of Proposition 1 to $V(X_{t+1},t+1)$ then completes the demonstration. □

The so-called verification theorem is also possible, which is read as follows.

Theorem 3. Let $W(x,t)$ solves the discrete Hamilton–Jacobi–Bellman equation (8):

$$\sup_{\{u_s\}_{s=t}^{T-1}} \{\mathcal{L}_X^u W(x,t) + U_1(x,t,u_t)\} = 0,$$
$$W(x,T) = U_2(x,T).$$

Then we have

$$W(x,t) \geq J(x,t,\{u_s\}_{s=t}^{T-1}), \tag{9}$$

for every $x \in \mathbb{R}$, $t = 0,1,2,\ldots,T-1$ and adapted $\{u_t\}$. Furthermore, if for every $x \in \mathbb{R}$, $t = 0,1,2,\ldots,T-1$ there exists an adapted $\{u_t^*\}$ with

$$u_k^* \in \arg \sup_{\{u_l\}_{l=k}^{T-1}} (\mathcal{L}_X^u W(k,X_k^*) + U_1(x,X_k^*,u_k)),$$

for every $t \leq k \leq T$, where X_k^* is the controlled process corresponding to u_k^* through (6), then we obtain

$$W(x,t) = V(x,t) = J(x,t,\{u_s\}_{s=t}^{T-1}).$$

P r o o f. It suffices to verify (9); that is

$$W(x,t) \geq \mathrm{E}^{x,t}\Big[\sum_{k=t}^{T-1} U_1(X_k,k,u_k) + U_2(X_T,T)\,\Big|\, X_t = x\Big], \tag{10}$$

for every adapted $\{u_t\}$. Since W is a solution of the d-HJB equation, we see that

$$\mathcal{L}_X^u W(x,t) + U_1(x,t,u_t) \leq 0. \tag{11}$$

Application of the discrete Itô formula (4) yields

$$W(X_T,T) - W(X_t,t)$$
$$= \sum_{k=t}^{T-1} \mathcal{L}_X^u W(X_k,k) + \sum_{k=t}^{T-1} \frac{W(X_k+\mu_k+\sigma_k,t+1) - W(X_k+\mu_k-\sigma_k,t+1)}{2} Y_{k+1}. \tag{12}$$

Taking expectations in (12) and taking into account of the inequality (11), we infer that the desired inequality (10) holds true. This finishes the proof. □

4. EXAMPLES

We present two examples of the d-HJB equation to illustrate our theory.

Example 1. We assume in (6) that $\mu \equiv 0$ and $\sigma(X,t,u) = \sigma u X$ with positive constant σ. The underlying model thus becomes

$$X_{t+1} - X_t = \sigma u_t X_t (B_{t+1} - B_t).$$

The control u_t states the fraction of money invested in the stocks. It is to be noted that these dynamics correspond to a stock evolving like $S_{t+1} = S_t + \sigma S_t(B_{t+1} - B_t)$ and a risk-free asset with zero interest rate. As to utility functions, we take $U_1 \equiv 0$ and $U_2 = \sqrt{x}$. Therefore, the d-HJB equation (8) becomes

$$\begin{aligned} &\sup_{\{u_s\}_{s=t}^{T-1}} \Bigg\{ \frac{V((1+\sigma u_t)x, t+1) - 2V(x,t+1) + V((1-\sigma u_t)x, t+1)}{2} \\ &\qquad\qquad + V(x,t+1) - V(x,t) \Bigg\} = 0, \\ &V(x,T) = \sqrt{x}. \end{aligned} \tag{13}$$

We will seek a solution of the form

$$V(x,t) = g(t)\sqrt{x}, \tag{14}$$

where $g(T) = 1$. Inserting (14) into (13) we deduce that

$$\sup_{u_t} \left\{ g(t+1) \frac{\sqrt{(1+\sigma u_t)x} + \sqrt{(1-\sigma u_t)x}}{2} - g(t)\sqrt{x} \right\} = 0.$$

The maximization is attained by the optimal constant strategy $u_t \equiv 0$ and hence we obtain $g(t) \equiv 1$ as well as $V(x,t) \equiv \sqrt{x}$.

Example 2. We assume in (6) that $\mu(X,t,u) = u$ and $\sigma(X,t,u) = \sigma u$ with $\sigma > 1$. The underlying model thus becomes

$$X_{t+1} - X_t = u_t + \sigma u_t (B_{t+1} - B_t).$$

The control u_t means the amount of money in the stock at time t, if we interpret the dynamics of the stock price as $S_{t+1} = S_t + S_t 1 + \sigma S_t(B_{t+1} - B_t)$ and consider a risk-free asset with null interest rate. As to utility functions, we take $U_1 \equiv 0$ and $U_2 = \sqrt{x}$ as before. Therefore, the d-HJB equation (8) becomes

$$\begin{aligned} &\sup_{\{u_s\}_{s=t}^{T-1}} \Bigg\{ V(x+u_t, t+1) - V(x,t+1) + \\ &\qquad \frac{V(x+u_t+\sigma u_t, t+1) - 2V(x+u_t, t+1) + V(x+u_t-\sigma u_t, t+1)}{2} \\ &\qquad + V(x,t+1) - V(x,t) \Bigg\} = 0, \\ &V(x,T) = \sqrt{x}. \end{aligned} \tag{15}$$

We will seek a solution of the form

$$V(x,t) = g(t)\sqrt{x}, \tag{16}$$

where $g(T) = 1$. Inserting (16) into (15) we infer that

$$\sup_{\{u_s\}_{s=t}^{T-1}} \Big\{ g(t+1)\frac{\sqrt{x+(1+\sigma)u_t} + \sqrt{x-(\sigma-1)u_t}}{2} - g(t)\sqrt{x} \Big\} = 0.$$

The maximization is attained by the optimal strategy

$$u_t = \frac{2}{\sigma^2-1}X_t. \tag{17}$$

Placing (17) back into (15) we obtain

$$g(t) = \Big(\frac{1}{2}\Big)^{T-t}\Big(\sqrt{\frac{\sigma+1}{\sigma-1}} + \sqrt{\frac{\sigma-1}{\sigma+1}}\Big)^{T-t},$$

and the corresponding $V(x,t)$.

5. DISCUSSIONS

We have developed the theory of discrete-time portfolio optimization. On the basis of the discrete Itô formula, we deduce the discrete Hamilton–Jacobi–Bellman (d-HJB) equation for the corresponding value function under the stochastic control framework, and establish the relevant verification theorem. Simple examples are also exhibited.

Generalizations may be performed toward several directions. As an example, we first point out that the basic random walk we consider in (1)(2) can be extended to, for instance

$$\mathrm{Prob}(Y_n = a) = p, \qquad \mathrm{Prob}(Y_n = -b) = 1-p,$$

with a,$b > 0$ and $0 < p < 1$. The discrete Itô formula then turns out to be

$$\begin{aligned} &f(B_{t+1}) - f(B_t) \\ &\quad = \frac{f(B_t+a) - f(B_t-b)}{a+b}Y_{t+1} + \frac{bf(B_t+a) - (a+b)f(B_t) + af(B_t-b)}{a+b}. \end{aligned}$$

The Doob–Meyer decomposition is also possible in this case, which is

$$\begin{aligned} f(B_t) = &\sum_{n=0}^{t-1} \frac{f(B_n+a) - f(B_n-b)}{a+b}(Y_{n+1} - (a+b)p + b) \\ &+ \sum_{n=0}^{t-1} \Big\{ \frac{bf(B_t+a) - (a+b)f(B_t) + af(B_t-b)}{a+b} \\ &\qquad + \frac{f(B_n+a) - f(B_n-b)}{a+b}((a+b)p - b) \Big\} + f(0). \end{aligned}$$

Further extensions, including the formulation of d-HJB as in Theorem 3, are naturally figured out; however, the corresponding optimality formulas become a little involved. We just remark that the validity of Theorem 3 does not depend on our particular assumption of the symmetry of basic random walk.

There remains several issues to be pursued forward. Firstly, we should investigate whether there exists an existence theory for the d-HJB equation or not. We remark that although various methods have been introduced so far and substantial progress has been achieved as to the HJB equation, we may understand that the analysis of the HJB equation has stayed as main difficulties of this subject. We want to know the discrete version, hopefully with a convergence result, really makes the situation much easier to handle. Secondly, the risk measuring quantity such as the one in [1] may be possible or not. Lastly, we should involve more examples, especially those of practical importance. These points will be our next themes for future research.

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Naoyuki Ishimura, Department of Mathematics, Graduate School of Economics, Hitotsubashi University, Kunitachi, Tokyo 186-8601. Japan.
ishimura@econ.hit-u.ac.jp

Yuji Mita, Graduate School of Economics, Hitotsubashi University, Kunitachim Tokyo 186-8601. Japan.
e-mail: em071027@g.hit-u.ac.jp