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ON A VARIANT OF THE LOCAL PROJECTION METHOD STABLE IN THE SUPG NORM

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We consider the local projection finite element method for the discretization of a scalar convection-diffusion equation with a divergence-free convection field. We introduce a new fluctuation operator which is defined using an orthogonal L^2 projection with respect to a weighted L^2 inner product. We prove that the bilinear form corresponding to the discrete problem satisfies an inf-sup condition with respect to the SUPG norm and derive an error estimate for the discrete solution.

Keywords: finite element method, convection-diffusion equation, stability, inf-sup condition, stabilization, SUPG method, local projection method, error estimates

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1. INTRODUCTION

In our recent work [7], we presented a novel analysis of local projection finite element methods applied to a scalar convection-diffusion-reaction equation

$$-\varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u + c\,u = f \quad \text{in } \Omega\,, \qquad\qquad u = u_b \quad \text{on } \partial\Omega\,. \tag{1}$$

We assumed that $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain with a polyhedral Lipschitz-continuous boundary $\partial\Omega$, ε is a positive constant, $\boldsymbol{b} \in W^{1,\infty}(\Omega)^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $u_b \in H^{1/2}(\partial\Omega)$. In addition, we used the frequently applied assumption that

$$c - \tfrac{1}{2}\operatorname{div}\boldsymbol{b} \geq \sigma_0 > 0 \quad \text{in } \Omega\,, \tag{2}$$

where σ_0 is a constant. Under these assumptions we proved that the bilinear form of a local projection stabilization satisfies an inf-sup condition in a norm which is stronger than the natural norm for which the bilinear form is coercive. Moreover, we proved that this stronger norm is equivalent to the norm of the streamline upwind / Petrov–Galerkin (SUPG) method if additional assumptions are satisfied. This important result implies that local projection methods are more stable than their coercivity suggests and that they often lead to the same error estimates as the SUPG method.

In the present paper, we extend the results of [7] to scalar convection-diffusion equations with a divergence-free convection field $\boldsymbol{b}$. Thus, we shall consider the following problem:

$$-\,\varepsilon\,\Delta u + \boldsymbol{b}\cdot\nabla u = f \quad \text{in } \Omega\,, \qquad\qquad u = u_b \quad \text{on } \partial\Omega\,. \tag{3}$$

We retain the above assumptions on the data, but instead of (2), we assume that

$$\operatorname{div}\boldsymbol{b} = 0 \quad \text{in } \Omega\,. \tag{4}$$

Then the problem (3) has still a unique solution in $H^1(\Omega)$ but the analysis of finite element discretizations of (3) is complicated by the fact that the natural norm corresponding to the respective bilinear form does not contain the $L^2(\Omega)$ norm. Let us stress that the case $c = 0$ is the most difficult one among the problems (1) with $c \geq 0$ if (4) holds.

The problem (3) is a basic model problem for many convection-diffusion phenomena arising in applications. It can also be viewed as a simplified model for a better understanding of numerical methods for the incompressible Navier–Stokes or Oseen equations.

We shall consider a new variant of the local projection method for which the fluctuation operator is defined using an orthogonal L^2 projection with respect to a weighted L^2 inner product. This allows us to prove that the underlying bilinear form satisfies an inf-sup condition with respect to the SUPG norm without any additional assumptions. We also present an error estimate which can be established in a simpler way than usual error estimates for local projection methods.

The plan of the paper is as follows. First, in the next section, we introduce the SUPG method and a general local projection discretization. Then, in Section 3, we define a special fluctuation operator and prove the stability of the local projection method with respect to the SUPG norm. Section 4 is devoted to the derivation of an error estimate and, in Section 5, we present numerical results. Finally, Section 6 contains our conclusions. Throughout the paper, we use standard notation for usual function spaces and norms, see, e. g., [5].

2. TWO STABILIZED DISCRETIZATIONS

Let $\mathscr{T}_h$ be a triangulation of Ω consisting of open shape-regular cells K possessing the usual compatibility properties. We set $h_K = \operatorname{diam}(K)$ for any $K \in \mathscr{T}_h$ and assume that $h_K \leq h$ for all $K \in \mathscr{T}_h$. Using the triangulation $\mathscr{T}_h$, we define a finite element space $W_h \subset H^1(\Omega)$, see, e. g., [5], and we set $V_h = W_h \cap H^1_0(\Omega)$. In addition, we introduce a function $\widetilde{u}_{bh} \in W_h$ such that its trace approximates the boundary condition u_b.

We shall assume that the space W_h has standard approximation properties, i. e., there exist $l \in \mathbb{N}$ and an interpolation operator

$$i_h \in \mathscr{L}(H^2(\Omega), W_h) \cap \mathscr{L}(H^2(\Omega) \cap H^1_0(\Omega), V_h)$$

satisfying

$$h_K^{-1}\,\|v-i_hv\|_{0,K}+|v-i_hv|_{1,K}+h_K\,|v-i_hv|_{2,K}\le C\,h_K^k\,|v|_{k+1,K}\qquad \forall\ v\in H^{k+1}(\Omega),\ K\in\mathscr{T}_h,\ k=1,\dots,l\,. \tag{5}$$

Furthermore, we set $\widetilde{u}_{bh}=i_h\widetilde{u}_b$ where $\widetilde{u}_b\in H^2(\Omega)$ is an extension of u_b. Thus, we have to assume that $u_b\in H^{3/2}(\partial\Omega)$. Let us emphasize that this assumption is made only for clarity of exposition and weaker assumptions on u_b would not cause any additional difficulties.

The simplest finite element discretization of (3) is the Galerkin discretization which is obtained by replacing the space $H_0^1(\Omega)$ in the weak formulation of (3) by its subspace V_h. This leads to the following discrete problem:

Find $u_h\in W_h$ such that $u_h-\widetilde{u}_{bh}\in V_h$ and

$$a^G(u_h,v_h)=(f,v_h)\qquad\forall\ v_h\in V_h\,,$$

where

$$a^G(u,v)=\varepsilon\,(\nabla u,\nabla v)+(\boldsymbol{b}\cdot\nabla u,v)$$

and $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$ or $L^2(\Omega)^d$. Integrating by parts, we obtain in view of (4)

$$a^G(v,v)=\varepsilon\,|v|_{1,\Omega}^2\qquad\forall\ v\in H_0^1(\Omega)\,, \tag{6}$$

which implies that the Galerkin discretization is uniquely solvable. If the solution of (3) satisfies $u\in H^{k+1}(\Omega)$ for some $k\in\{1,\dots,l\}$, then we have the error estimate (see, e. g., [5])

$$|u-u_h|_{1,\Omega}\le C\,h^k\left(1+\frac{h\,\|\boldsymbol{b}\|_{0,\infty,\Omega}}{\varepsilon}\right)|u|_{k+1,\Omega}\,,$$

where C is independent of h and the data of the problem. However, this estimate is useless if $\varepsilon\ll h\|\boldsymbol{b}\|_{0,\infty,\Omega}$, which we encounter in many applications. It is well known that, in this case, the Galerkin solution is usually globally polluted by spurious oscillations, cf., e. g., [9].

To enhance the stability and accuracy of the Galerkin discretization of (3) in the convection dominated regime, various stabilization strategies have been developed. One of the most popular approaches is the streamline upwind / Petrov–Galerkin (SUPG) method proposed by Brooks and Hughes [4]. The discrete problem reads:

Find $u_h\in W_h$ such that $u_h-\widetilde{u}_{bh}\in V_h$ and

$$a_h^{\rm SUPG}(u_h,v_h)=(f,v_h+\delta\,\boldsymbol{b}\cdot\nabla v_h)\qquad\forall\ v_h\in V_h\,,$$

where

$$a_h^{\rm SUPG}(u,v)=a^G(u,v)+\sum_{K\in\mathscr{T}_h}(-\varepsilon\,\Delta u+\boldsymbol{b}\cdot\nabla\,u,\delta\,\boldsymbol{b}\cdot\nabla v)_K$$

and $\delta\in L^\infty(\Omega)$ is a nonnegative stabilization parameter. As usual, $(\cdot,\cdot)_K$ denotes the inner product in $L^2(K)$ or $L^2(K)^d$. If

$$0\le\delta|_K\le\frac{h_K^2}{\varepsilon\,\mu^2}\qquad\forall\ K\in\mathscr{T}_h\,,$$

where μ is a constant from the inverse inequality

$$\|\Delta v_h\|_{0,K} \le \mu\, h_K^{-1}\, |v_h|_{1,K} \qquad \forall\; v_h \in V_h,\;\; K \in \mathscr{T}_h\,,$$

the bilinear form a_h^{SUPG} is coercive on V_h with respect to the norm

$$|||v|||_{\mathrm{SUPG}} = \left(\varepsilon\, |v|_{1,\Omega}^2 + \|\delta^{1/2}\, \boldsymbol{b}\cdot\nabla v\|_{0,\Omega}^2\right)^{1/2}. \tag{7}$$

More precisely, we have

$$a_h^{\mathrm{SUPG}}(v_h, v_h) \ge \frac{1}{2}\, |||v_h|||_{\mathrm{SUPG}}^2 \qquad \forall\; v_h \in V_h\,.$$

Thus, there exists a unique SUPG solution and, if $\delta > 0$, the SUPG method possesses a stronger stability in the streamline direction than the Galerkin discretization.

Let the solution of (3) satisfy $u \in H^{k+1}(\Omega)$ for some $k \in \{1, \dots, l\}$ and let the stabilization parameter δ be constant on each element of $\mathscr{T}_h$. Then, applying the techniques presented in [9], we derive that the SUPG solution satisfies the error estimate

$$|||u - u_h|||_{\mathrm{SUPG}} \le C\, h^k \left(\sum_{K\in\mathscr{T}_h} \gamma_K\, |u|_{k+1,K}^2\right)^{1/2},$$

where

$$\gamma_K = \varepsilon + \|\delta^{1/2}\, \boldsymbol{b}\|_{0,\infty,K}^2 + \frac{h_K^2\, \|\boldsymbol{b}\|_{0,\infty,K}^2}{\max\{\varepsilon, \|\delta^{1/2}\, \boldsymbol{b}\|_{0,\infty,K}^2\}}\,.$$

Balancing the two terms in γ_K containing δ, we deduce that

$$\delta|_K \approx \frac{h_K^2}{\max\{\varepsilon,\, h_K \|\boldsymbol{b}\|_{0,\infty,K}\}}\,.$$

Then $\gamma_K \lesssim \varepsilon + 2\, h_K \|\boldsymbol{b}\|_{0,\infty,K}$ and hence

$$|||u - u_h|||_{\mathrm{SUPG}} \le C\, h^k\, (\varepsilon + h\, \|\boldsymbol{b}\|_{0,\infty,\Omega})^{1/2}\, |u|_{k+1,\Omega}\,,$$

where C is independent of h and the data of the problem. This is a significant improvement in comparison to the Galerkin method since the constant in the estimate for the streamline derivative of the error now does not deteriorate for decreasing ε. Moreover, spurious oscillations are suppressed and they are localized only along sharp layers.

During the last decade, stabilization techniques based on local projections (LP) have become very popular, see, e. g., [2], [3], and [8]. To formulate a LP method, we introduce a discontinuous finite element space $D_h \subset L^2(\Omega)$ and denote by π_h a projection operator which maps the space $L^2(\Omega)$ onto D_h. Furthermore, we define the so-called fluctuation operator $\kappa_h = id - \pi_h$, where id is the identity operator on $L^2(\Omega)$. Then the local projection discretization of (3) considered in this paper reads:

Find $u_h \in W_h$ such that $u_h - \widetilde{u}_{bh} \in V_h$ and

$$a_h^{LP}(u_h, v_h) = (f, v_h) \qquad \forall\, v_h \in V_h\,, \tag{8}$$

where

$$a_h^{LP}(u, v) = a^G(u, v) + (\kappa_h(\boldsymbol{b} \cdot \nabla u), \tau\, \kappa_h(\boldsymbol{b} \cdot \nabla v))$$

and $\tau \in L^\infty(\Omega)$ is a nonnegative stabilization parameter. In view of (6) it is obvious that the bilinear form a_h^{LP} is coercive on V_h with respect to the local projection norm

$$|||v|||_{LP} = \left(\varepsilon\, |v|_{1,\Omega}^2 + \|\tau^{1/2}\, \kappa_h(\boldsymbol{b} \cdot \nabla v)\|_{0,\Omega}^2\right)^{1/2}\,.$$

This assures the existence of a unique solution of the local projection discretization.

It was demonstrated in [6] that the stabilization parameter τ should be chosen analogously as for the SUPG method. Therefore, we shall assume that there exists a constant $C_1 \geq 1$ such that

$$\frac{1}{C_1}\, \frac{h_K^2}{\max\{\varepsilon, h_K\, \|\boldsymbol{b}\|_{0,\infty,K}\}} \leq \tau|_K \leq C_1\, \frac{h_K^2}{\max\{\varepsilon, h_K\, \|\boldsymbol{b}\|_{0,\infty,K}\}} \qquad \forall\, K \in \mathscr{T}_h\,. \tag{9}$$

Then, under suitable assumptions on the spaces W_h and D_h, the fluctuation operator κ_h and the data of the problem (3) (see the following sections), it was proved in [6] that the solution of the local projection discretization (8) satisfies the error estimate

$$|||u - u_h|||_{LP} \leq C\, h^k\, (\varepsilon + h)^{1/2}\, \|u\|_{k+1,\Omega}\,.$$

As we see, the convergence rate is the same as for the SUPG method. However, the norm in which the error is measured seems to be weaker than the SUPG norm since the local projection term in $|||\cdot|||_{LP}$ measures only the fluctuations. We shall show in the following sections that a suitable definition of the fluctuation operator enables us to prove the convergence also with respect to the SUPG norm.

3. STABILITY WITH RESPECT TO THE SUPG NORM

The local projection method was introduced by Becker and Braack [1] as a two-level approach since the space W_h was constructed on a mesh obtained by refining the triangulation $\mathscr{T}_h$. The space W_h then consists of continuous functions which are polynomials of degree l on each element of the finer triangulation whereas the space D_h consists of discontinuous functions which are polynomials of degree $l-1$ on each element of the triangulation $\mathscr{T}_h$ (in case of quadrilateral or hexahedral elements the polynomial degree is considered for each variable and possibly on the reference element). The refinement of $\mathscr{T}_h$ is constructed in such a way that the interior of each element $K \in \mathscr{T}_h$ contains one new vertex created by this refinement. Therefore, for any $K \in \mathscr{T}_h$, we can define a bubble function b_K with $\operatorname{supp} b_K \subset \overline{K}$ which is piecewise (multi)linear with respect to the finer triangulation and hence its product with any function from D_h belongs to W_h. Thus, we have the property

$$\forall\, K \in \mathscr{T}_h \quad \exists\, b_K \in H_0^1(K) \cap C(\overline{K}): \qquad 0 < b_K \leq 1 \ \text{ in } \ K\,, \quad b_K \cdot D_h \subset W_h\,, \tag{10}$$

where $b_K \cdot D_h$ is regarded as a space of functions vanishing outside K. Recently, a one-level approach was introduced in [8] where both D_h and W_h are constructed on $\mathscr{T}_h$ but, using the same space D_h as before, the space W_h is defined by enriching elementwise the polynomials of degree l by $b_K \cdot (D_h|_K)$ with a polynomial bubble function b_K on K of the lowest possible degree. Thus, the condition (10) is satisfied also in this case. We refer to [8] for a detailed description of various pairs of finite element spaces W_h, D_h applicable to the local projection method.

Now we assume that we are given finite element spaces W_h and D_h satisfying (10) and we fix one function b_K from (10) for each $K \in \mathscr{T}_h$. We assume that there exists a constant $C_2 \geq 1$ independent of h such that

$$\|q\|_{0,K} \leq C_2 \, \|b_K^{1/2} \, q\|_{0,K} \qquad \forall \ q \in D_h, \ K \in \mathscr{T}_h \, . \tag{11}$$

If all functions b_K are generated by one function defined on the reference element, the inequality (11) can be proved by transforming the norms in (11) on the reference element and applying equivalence of norms on finite-dimensional spaces. We shall also need the inverse inequality

$$|v_h|_{1,K} \leq C_3 \, h_K^{-1} \, \|v_h\|_{0,K} \qquad \forall \ K \in \mathscr{T}_h, \, v_h \in W_h \, , \tag{12}$$

which can be proved analogously as (11). Again, the constant C_3 is assumed to be independent of h.

Let us consider any $K \in \mathscr{T}_h$ and define the bilinear form $(\cdot, \cdot)_{b,K}$ by

$$(u, v)_{b,K} = (b_K \, u, v)_K \, .$$

Then it is easy to see that $(\cdot, \cdot)_{b,K}$ is an inner product on $L^2(K)$. We define the operator $\pi_K : L^2(K) \to D(K) \equiv D_h|_K$ as the projection onto the finite-dimensional space $D(K)$ which is orthogonal with respect to $(\cdot, \cdot)_{b,K}$. Thus, π_K satisfies

$$(\pi_K \, v, q)_{b,K} = (v, q)_{b,K} \qquad \forall \ v \in L^2(K), \ q \in D(K) \, . \tag{13}$$

Clearly,

$$\|b_K^{1/2} \, \pi_K \, v\|_{0,K} \leq \|b_K^{1/2} \, v\|_{0,K} \qquad \forall \ v \in L^2(K) \, .$$

This inequality together with (10) and (11) implies that, for any $v \in L^2(K)$,

$$\|b_K \, \pi_K \, v\|_{0,K} \leq \|b_K^{1/2} \, \pi_K \, v\|_{0,K} \leq \|v\|_{0,K} \, , \tag{14}$$

$$\|\pi_K \, v\|_{0,K} \leq C_2 \, \|v\|_{0,K} \, . \tag{15}$$

Note also that

$$(v, b_K \, \pi_K \, v)_K = (v, \pi_K \, v)_{b,K} = (\pi_K \, v, \pi_K \, v)_{b,K} = \|b_K^{1/2} \, \pi_K \, v\|_{0,K}^2$$

and hence, in view of (11),

$$(v, b_K \, \pi_K \, v)_K \geq C_2^{-2} \, \|\pi_K \, v\|_{0,K}^2 \, . \tag{16}$$

Using the operators π_K, we define the operator π_h introduced in the previous section by $(\pi_h \, v)|_K = \pi_K \, (v|_K)$ for any $v \in L^2(\Omega)$ and $K \in \mathscr{T}_h$. The fluctuation operator used in the local projection discretization (8) is defined by means of this operator π_h.

Now we are in the position to prove the main result of this paper.

Theorem 1. Let the finite element spaces W_h and D_h satisfy (10), (11) and (12). Let the fluctuation operator κ_h be defined using the operators π_K from (13). Let the stabilization parameter τ be constant on each element of the triangulation $\mathscr{T}_h$ and satisfy (9). Then the bilinear form a_h^{LP} satisfies

$$\sup_{v_h\in V_h} \frac{a_h^{LP}(u_h,v_h)}{|||v_h|||_{\mathrm{SUPG}}} \geq \beta\,|||u_h|||_{\mathrm{SUPG}} \qquad \forall\ u_h \in V_h\,, \tag{17}$$

where $\beta = 1/(\sqrt{2}+\sqrt{10}\,C_1\,C_2^3\,C_3)^2$ and the norm $|||\cdot|||_{\mathrm{SUPG}}$ is defined by (7) with $\delta=\tau$.

P r o o f. Consider any $u_h \in V_h$. We shall construct a function $v_h \in V_h$ such that

$$a_h^{LP}(u_h,v_h) \geq |||u_h|||_{\mathrm{SUPG}}^2 \qquad \text{and} \qquad |||u_h|||_{\mathrm{SUPG}} \geq \beta\,|||v_h|||_{\mathrm{SUPG}}\,. \tag{18}$$

The inequalities (18) immediately imply the inf-sup condition (17).

First, we define functions $w_h \in D_h$ and $z_h \in V_h$ by

$$w_h|_K = \tau_K^{1/2}\,\pi_K(\boldsymbol{b}\cdot\nabla u_h)\,, \qquad z_h|_K = C_2^2\,\tau_K^{1/2}\,b_K\,w_h|_K \qquad \forall\ K\in\mathscr{T}_h\,,$$

where $\tau_K = \tau|_K$. Then, according to (16),

$$(\boldsymbol{b}\cdot\nabla u_h, z_h)_K \geq \|w_h\|_{0,K}^2 \qquad \forall\ K\in\mathscr{T}_h\,.$$

Consequently,

$$\begin{aligned} a_h^{LP}(u_h,z_h) &\geq \|w_h\|_{0,\Omega}^2 + \varepsilon\,(\nabla u_h,\nabla z_h) + (\kappa_h(\boldsymbol{b}\cdot\nabla u_h),\tau\,\kappa_h(\boldsymbol{b}\cdot\nabla z_h)) \\ &\geq \|w_h\|_{0,\Omega}^2 - |||u_h|||_{LP}\,|||z_h|||_{LP}\,. \end{aligned} \tag{19}$$

Employing the inverse inequality (12), we obtain for any $K\in\mathscr{T}_h$

$$|z_h|_{1,K} \leq C_3\,h_K^{-1}\,\|z_h\|_{0,K} = C_2^2\,C_3\,h_K^{-1}\,\tau_K^{1/2}\,\|b_K\,w_h\|_{0,K}$$

and hence, in view of (10) and (14),

$$|z_h|_{1,K} \leq C_2^2\,C_3\,h_K^{-1}\,\tau_K^{1/2}\,\|w_h\|_{0,K} \tag{20}$$

and

$$|z_h|_{1,K} \leq C_2^2\,C_3\,h_K^{-1}\,\tau_K\,\|\boldsymbol{b}\cdot\nabla u_h\|_{0,K}\,. \tag{21}$$

Since $\|\kappa_h(\boldsymbol{b}\cdot\nabla z_h)\|_{0,K} \leq 2\,C_2\,\|\mathbf{b}\|_{0,\infty,K}\,|z_h|_{1,K}$ due to (15), it follows from (20) and (9) that

$$|||z_h|||_{LP}^2 \leq \zeta\,\|w_h\|_{0,\Omega}^2 \qquad \text{with} \qquad \zeta = 5\,C_1^2\,C_2^6\,C_3^2\,.$$

Therefore,

$$|||u_h|||_{LP}\,|||z_h|||_{LP} \leq \tfrac{1}{2}\,\zeta\,|||u_h|||_{LP}^2 + \tfrac{1}{2}\,\|w_h\|_{0,\Omega}^2$$

and (19) implies that

$$a_h^{LP}(u_h,z_h) \geq \tfrac{1}{2}\,\|w_h\|_{0,\Omega}^2 - \tfrac{1}{2}\,\zeta\,|||u_h|||_{LP}^2\,.$$

In view of (6), we have $a_h^{LP}(u_h, u_h) = |||u_h|||_{LP}^2$ and hence, defining $v_h \in V_h$ by

$$v_h = 4\, z_h + 2\,(1+\zeta)\, u_h\,,$$

we obtain

$$\begin{aligned}\tfrac{1}{2}\, a_h^{LP}(u_h, v_h) &\geq \|w_h\|_{0,\Omega}^2 + |||u_h|||_{LP}^2 \\ &= \varepsilon\, |u_h|_{1,\Omega}^2 + \|\tau^{1/2}\, \pi_h(\boldsymbol{b}\cdot\nabla u_h)\|_{0,\Omega}^2 + \|\tau^{1/2}\, \kappa_h(\boldsymbol{b}\cdot\nabla u_h)\|_{0,\Omega}^2\,,\end{aligned}$$

which gives the first inequality in (18) by the triangular inequality. Finally, let us prove the second inequality in (18). Using (9), we obtain for any $K \in \mathscr{T}_h$

$$\tau_K \|\boldsymbol{b}\cdot\nabla z_h\|_{0,K}^2 \leq \tau_K\, \|\boldsymbol{b}\|_{0,\infty,K}^2\, |z_h|_{1,K}^2 \leq C_1\, h_K\, \|\boldsymbol{b}\|_{0,\infty,K}\, |z_h|_{1,K}^2$$

and hence, in view of (21) and again (9), we derive

$$\varepsilon\, |z_h|_{1,K}^2 + \tau_K \|\boldsymbol{b}\cdot\nabla z_h\|_{0,K}^2 \leq 2\, C_1^2\, C_2^4\, C_3^2\, \tau_K\, \|\boldsymbol{b}\cdot\nabla u_h\|_{0,K}^2\,.$$

Thus

$$|||z_h|||_{\mathrm{SUPG}} \leq \sqrt{2}\, C_1\, C_2^2\, C_3\, |||u_h|||_{\mathrm{SUPG}}\,,$$

which shows that, for any $\beta \leq 1/(2 + 2\,\zeta + 4\,\sqrt{2}\, C_1\, C_2^2\, C_3)$, the second inequality in (18) holds. □

4. ERROR ANALYSIS

In this section, we shall prove an error estimate for the solution of the local projection discretization (8). Thanks to the inf-sup condition established in Theorem 1, the proof can be carried out easier than for general local projection methods, cf., e. g., [6], [8]. Similarly as for the space W_h, we shall assume that there exists an interpolation operator $j_h \in \mathscr{L}(L^2(\Omega), D_h)$ such that

$$\|q - j_h q\|_{0,K} \leq C\, h_K^k\, |q|_{k,K} \qquad \forall\; q \in H^k(\Omega),\;\; K \in \mathscr{T}_h,\;\; k = 1, \dots, l\,, \tag{22}$$

where the integer l is the same as for W_h.

It is well known that local projection stabilizations lead to nonconsistent discretizations. Indeed, the weak solution u of the problem (3) satisfies

$$a_h^{LP}(u, v_h) = (f, v_h) + s_h(u, v_h) \qquad \forall\; v_h \in V_h\,,$$

where

$$s_h(u, v) = (\kappa_h(\boldsymbol{b}\cdot\nabla u), \tau\, \kappa_h(\boldsymbol{b}\cdot\nabla v))\,.$$

Consequently, the solution u_h of the local projection discretization (8) obeys the relation

$$a_h^{LP}(u - u_h, v_h) = s_h(u, v_h) \qquad \forall\; v_h \in V_h\,.$$

From this we deduce using Theorem 1 and the triangular inequality that

$$\begin{aligned}\beta\, |||u - u_h|||_{\mathrm{SUPG}} &\leq \beta\, |||u - i_h u|||_{\mathrm{SUPG}} \\ &\quad + \sup_{v_h \in V_h} \frac{a_h^{LP}(u - i_h u, v_h)}{|||v_h|||_{\mathrm{SUPG}}} + \sup_{v_h \in V_h} \frac{s_h(u, v_h)}{|||v_h|||_{\mathrm{SUPG}}}\,,\end{aligned} \tag{23}$$

where we now assume that $u \in H^2(\Omega)$. The estimate of the second term on the right-hand side of (23) will be based on the following lemma.

Lemma 1. Under the assumptions of Theorem 1, we have for any $w \in H^1(\Omega)$ and $v \in H_0^1(\Omega)$

$$a_h^{LP}(w,v) \le C \left(\sum_{K\in\mathscr{T}_h} \lambda_K \, (h_K^{-2}\, \|w\|_{0,K}^2 + |w|_{1,K}^2) \right)^{1/2} |||v|||_{\mathrm{SUPG}}\,, \tag{24}$$

where $\lambda_K = \varepsilon + h_K\, \|\boldsymbol{b}\|_{0,\infty,K}$ and $C = 4\sqrt{2\,C_1}\,C_2^2$.

P r o o f. Consider any $w \in H^1(\Omega)$ and $v \in H_0^1(\Omega)$. In view of (4), we obtain

$$a_h^{LP}(w,v) = \varepsilon\,(\nabla w, \nabla v) - (w, \boldsymbol{b}\cdot\nabla v) + s_h(w,v)\,.$$

Since τ is piecewise constant, we derive using (15)

$$a_h^{LP}(w,v) \le 4\,C_2^2\, |||w|||_{\mathrm{SUPG}}\, |||v|||_{\mathrm{SUPG}} - (w, \boldsymbol{b}\cdot\nabla v)\,.$$

The assumption (9) implies that, for any $K \in \mathscr{T}_h$,

$$\|\tau^{1/2}\, \boldsymbol{b}\cdot\nabla w\|_{0,K}^2 \le \tau_K\, \|\boldsymbol{b}\|_{0,\infty,K}^2\, |w|_{1,K}^2 \le C_1\, h_K\, \|\boldsymbol{b}\|_{0,\infty,K}\, |w|_{1,K}^2\,,$$

where again $\tau_K = \tau|_K$. Therefore,

$$|||w|||_{\mathrm{SUPG}} \le \left(\sum_{K\in\mathscr{T}_h} (\varepsilon + C_1\, h_K\, \|\boldsymbol{b}\|_{0,\infty,K})\, |w|_{1,K}^2 \right)^{1/2} . \tag{25}$$

Choosing any $K \in \mathscr{T}_h$ and using the estimates

$$(w, \boldsymbol{b}\cdot\nabla v)_K \le \varepsilon^{-1/2}\, \|\boldsymbol{b}\|_{0,\infty,K}\, \|w\|_{0,K}\, \varepsilon^{1/2}\, |v|_{1,K}\,,$$
$$(w, \boldsymbol{b}\cdot\nabla v)_K \le \tau_K^{-1/2}\, \|w\|_{0,K}\, \|\tau^{1/2}\, \boldsymbol{b}\cdot\nabla v\|_{0,K}\,,$$

we get

$$(w, \boldsymbol{b}\cdot\nabla v)_K \le \|\boldsymbol{b}\|_{0,\infty,K}\, \varrho_K^{-1/2}\, \|w\|_{0,K}\, (\varepsilon\, |v|_{1,K}^2 + \|\tau^{1/2}\, \boldsymbol{b}\cdot\nabla v\|_{0,K}^2)^{1/2}$$

with $\varrho_K = \max\{\varepsilon, \tau_K \|\boldsymbol{b}\|_{0,\infty,K}^2\}$. Since τ_K satisfies (9), we deduce that $\varrho_K \ge h_K \|\boldsymbol{b}\|_{0,\infty,K}/C_1$ and, consequently,

$$(w, \boldsymbol{b}\cdot\nabla v) \le \left(\sum_{K\in\mathscr{T}_h} C_1\, \|\boldsymbol{b}\|_{0,\infty,K}\, h_K^{-1}\, \|w\|_{0,K}^2 \right)^{1/2} |||v|||_{\mathrm{SUPG}}\,.$$

This proves the lemma. □

Theorem 2. Let the assumptions of Theorem 1 be satisfied and let (5) and (22) hold. Let the solution of (3) satisfy $u \in H^{k+1}(\Omega)$ for some $k \in \{1,\dots,l\}$ and let $\boldsymbol{b}\cdot\nabla u \in H^k(\Omega)$. Then the solution of the local projection discretization (8) satisfies

$$|||u-u_h|||_{\mathrm{SUPG}} \le C\,h^k\,(\varepsilon+h\,\|\boldsymbol{b}\|_{0,\infty,\Omega})^{1/2}\,|u|_{k+1,\Omega} + C\,h^{k+1/2}\left(\sum_{K\in\mathscr{T}_h}\frac{|\boldsymbol{b}\cdot\nabla u|^2_{k,K}}{\|\boldsymbol{b}\|_{0,\infty,K}}\right)^{1/2},$$

where C is independent of h and the data of the problem (3).

Proof. Since $\kappa_h q_h = 0$ for any $q_h \in D_h$, we obtain

$$s_h(u,v_h) \le 4\,C_2^2\,\|\tau^{1/2}\,(\boldsymbol{b}\cdot\nabla u - j_h(\boldsymbol{b}\cdot\nabla u))\|_{0,\Omega}\,|||v_h|||_{\mathrm{SUPG}} \qquad \forall\, v_h \in V_h\,.$$

Thus, the theorem follows easily from (23), (25), (24), (5), (22), and (9). □

We see that if the term
$$\sum_{K\in\mathscr{T}_h}\frac{|\boldsymbol{b}\cdot\nabla u|^2_{k,K}}{\|\boldsymbol{b}\|_{0,\infty,K}}$$
can be bounded independently of h (e. g., if $\boldsymbol{b}\neq\boldsymbol{0}$ in $\overline{\Omega}$), we have an analogous error estimate as for the SUPG method.

5. NUMERICAL RESULTS

Our numerical tests show that the local projection operator considered in this paper leads to similar results as the common local orthogonal L^2 projection operator. We shall demonstrate it for the following setting of the problem (3).

Example 1. We consider the problem (3) in $\Omega = (0,1)^2$ with
$$\varepsilon = 10^{-8}\,, \qquad \boldsymbol{b} = (1,0)\,, \qquad f = 1\,, \qquad u_b = 0\,.$$

The solution of Example 1 possesses an exponential boundary layer at $x = 1$ and parabolic boundary layers at $y = 0$ and $y = 1$. Outside the layers, the solution is very close to the function $u_0(x,y) = x$.

We shall present numerical results for both the one-level approach and the two-level approach of the local projection method (see Section 3). The one-level method is defined using a triangulation $\mathcal{T}_h$ consisting of 20×20 equal squares. The space W_h is constructed using the Q_2 element enriched by three bubble functions on each element K of $\mathcal{T}_h$. Choosing functions $b_K \in Q_2(K)$ satisfying (10), these three bubble functions are $b_K\,x$, $b_K\,y$ and $b_K\,x\,y$. The two-level method uses a triangulation $\mathcal{T}_h$ consisting of 10×10 equal squares for constructing the space D_h. The space W_h is constructed on the same triangulation as in the one-level case using the Q_2 element. For any $K \in \mathcal{T}_h$, the function b_K satisfying (10) is piecewise bilinear with respect to a decomposition of K into four equal squares. For both methods, the projection space D_h is constructed using the Q_1 element and the stabilization parameter is defined by

$$\tau|_K = \frac{1}{15}\,\min\left\{\frac{h_K}{\|\boldsymbol{b}\|_{0,\infty,K}},\frac{h_K^2}{6\,\varepsilon}\right\} \qquad \forall\, K\in\mathcal{T}_h\,.$$

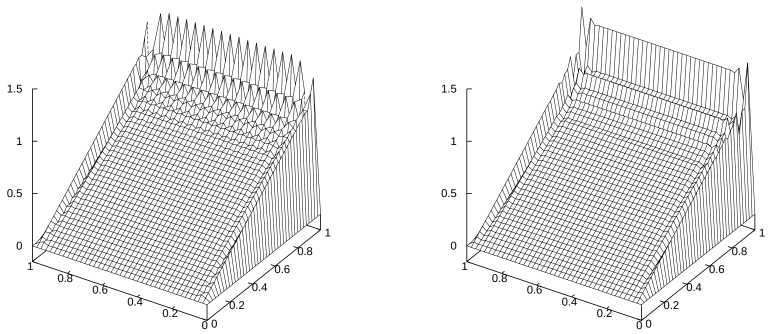

Fig. 1. LP solutions of Example 1 defined using local orthogonal L^2 projections: the one-level approach (left) and the two-level approach (right).

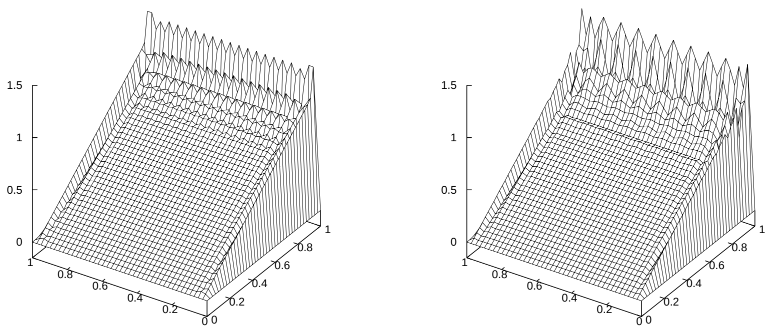

Fig. 2. LP solutions of Example 1 defined using the local projection operator from Section 3: the one-level approach (left) and the two-level approach (right).

The discrete solutions obtained are depicted in Figure 1 and Figure 2. The one-level solution is visualized without the additional bubbles so that the corresponding function belongs to the space W_h used for computing the two-level solution. The lines in the figures connect the values of the solutions at vertices, midpoints of edges and centres of elements of the 20×20 mesh. We observe that the solutions are not significantly influenced by the choice of the local projection operator. It is important that, like for residual-based stabilizations, spurious oscillations are localized along boundary layer regions.

6. CONCLUSIONS

In this paper, we proposed a new fluctuation operator in the local projection finite element method for the numerical solution of scalar convection-diffusion equations. This operator enabled us to prove stability and error estimates with respect to the SUPG norm for general divergence-free convection fields. Numerical results show that the local projection method with the new fluctuation operator still leads to numerical solutions with oscillations localized to layer regions.

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