QUASI–COPULAS WITH QUADRATIC SECTIONS IN ONE VARIABLE

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We introduce and characterize the class of multivariate quasi-copulas with quadratic sections in one variable. We also present and analyze examples to illustrate our results.

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1. INTRODUCTION

The concept of a *quasi-copula* was introduced in [1] (for the bivariate case) and [17] (for the multivariate case) in order to show that certain operations on univariate distributions are derivable from operations on random variables defined on the same probability space. Quasi-copulas have been used in probability theory (e. g. to express the pointwise best-possible bounds of nonempty sets of distribution functions: see [13, 15, 20]) and fuzzy set theory (e. g. in aggregation processes: see for instance [4, 7, 9, 11, 24]; in fact quasi-copulas constitute a special subclass of aggregation operators – see for instance [23]).

A stronger concept than that of a quasi-copula is the concept of a *copula*, which was introduced by A. Sklar in order to find the relationship between a distribution function and their margins (see [12, 25, 26]). A copula is the restriction to $[0,1]^n$ ($n \geq 2$) of a continuous multivariate distribution function whose univariate margins are uniform on $[0,1]$. Every copula is a quasi-copula, and there exist *proper* quasi-copulas, i. e., quasi-copulas which are not copulas.

In the literature we cannot find a great variety of classes and families of multivariate proper quasi-copulas (see [3, 5, 14, 21, 27]). Some types of quasi-copulas have been obtained from certain hypotheses about their sections of a special type (see [8, 10, 16, 19]). Our purpose is to follow this way in order to introduce and study a new class of multivariate quasi-copulas, namely the quasi-copulas with quadratic sections in one variable. In [22], we have studied the class of multivariate copulas with quadratic sections in one variable. In this paper we extend our study to proper quasi-copulas, providing examples of families which illustrate our results.

2. PRELIMINARIES

We first establish some notation. The interval $[0, 1]$ will be denoted by $\mathbb{I}$. Given two functions f_1 and f_2 with a common domain A, we write $f_1 \leq f_2$ if $f_1(x) \leq f_2(x)$ for all x in A. Let $n \geq 3$ be a natural number. If $\boldsymbol{u} = (u_1, \ldots, u_n)$ is in $\mathbb{I}^n$, $\boldsymbol{u}'$ will denote the point $(u_1, \ldots, u_{n-1})$ in $\mathbb{I}^{n-1}$. We may also write $\boldsymbol{u} = (\boldsymbol{u}', u_n)$. The point $(a, \ldots, a) \in \mathbb{I}^m$ will be denoted by $\boldsymbol{a}_m$. Finally, If $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{I}^n$ such that $\boldsymbol{u} \leq \boldsymbol{v}$ – i.e., $u_k \leq v_k$ holds for every $k = 1, \ldots, n$ –, then $[\boldsymbol{u}, \boldsymbol{v}]$ denotes the n-box $\times_{i=1}^n [u_i, v_i]$ in $\mathbb{I}^n$.

Let $n \geq 2$ be a natural number. An n-dimensional *copula* (briefly n-copula) is a function C from $\mathbb{I}^n$ to $\mathbb{I}$ satisfying the following conditions:

(C1) For every $\boldsymbol{u} \in \mathbb{I}^n$, $C(\boldsymbol{u}) = 0$ if at least one coordinate of $\boldsymbol{u}$ is equal to 0; and $C(\boldsymbol{u}) = u_k$ whenever all coordinates of $\boldsymbol{u}$ are equal to 1 except maybe u_k;

(C2) $V_C([\boldsymbol{a}, \boldsymbol{b}]) = \sum (-1)^{\kappa(\boldsymbol{c})} \cdot C(\boldsymbol{c}) \geq 0$ for every n-box $[\boldsymbol{a}, \boldsymbol{b}]$ in $\mathbb{I}^n$, where the sum is taken over all the *vertices* $\boldsymbol{c}$ of $[\boldsymbol{a}, \boldsymbol{b}]$ (i.e., each c_k is equal to either a_k or b_k) and $\kappa(\boldsymbol{c})$ is the number of indices k's such that $c_k = a_k$.

The number $V_C([\boldsymbol{a}, \boldsymbol{b}])$ is known as the *C-volume* of $[\boldsymbol{a}, \boldsymbol{b}]$; in this paper we also consider the extension of this concept to other types of real-valued functions C on $\mathbb{I}^n$.

An n-dimensional *quasi-copula* (briefly n-quasi-copula) is a function Q from $\mathbb{I}^n$ to $\mathbb{I}$ satisfying condition (C1) for n-copulas and, in place of (C2), the weaker conditions:

(Q1) Q is non-decreasing in each variable;

(Q2) $|Q(\boldsymbol{u}) - Q(\boldsymbol{v})| \leq \sum_{i=1}^n |u_i - v_i|$ for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{I}^n$ (*1-Lipschitz condition*).

See [6] for the case $n = 2$, and [2] for $n > 2$; see also [12].

Let a be a real-valued function defined on $\mathbb{I}^n$. For every $k = 1, \ldots, n$ and $i_1, \ldots, i_k \in \mathbb{N}$ such that $1 \leq i_1 < \cdots < i_k \leq n$, $a_{i_1 \cdots i_k}$ denotes the function defined on $\mathbb{I}^k$ by $a_{i_1 \cdots i_k}(u_{i_1}, \ldots, u_{i_k}) = a(v_1, \ldots, v_n)$, where $v_j = u_{i_m}$ if $j = i_m$ for some $m = 1, \ldots, k$, and $v_j = 1$ otherwise. The functions $a_{i_1 \cdots i_k}$ are called as the k-dimensional *margins* of the function a. If $n \geq 3$ and $k = 2, \ldots, n$, then the k-dimensional margins of an n-quasi-copula (respectively n-copula) are k-quasi-copulas (respectively k-copulas).

Let C be a real-valued function defined on $\mathbb{I}^n$. For each $k = 1, \ldots, n$ and for every $\boldsymbol{v} = (u_1, \ldots, u_{k-1}, u_{k+1}, \ldots u_n) \in \mathbb{I}^{n-1}$, let $f_{k,\boldsymbol{v}}$ be the function defined on $\mathbb{I}$ by $f_{k,\boldsymbol{v}}(u_k) = C(\boldsymbol{u})$. The functions $f_{k,\boldsymbol{v}}$ are called *sections* of the function C *with respect to the variable* u_k.

In the sequel, we will only consider – without loss of generality – sections with respect to the last variable: all the results and examples in this paper about sections with respect to the last variable can be extended to sections with respect to any other variable.

3. QUASI–COPULAS WITH QUADRATIC SECTIONS

An n-quasi-copula C is said to have *quadratic sections* in the last variable if there exist three real-valued functions a, b and c defined on $\mathbb{I}^{n-1}$ so that

$$C(\boldsymbol{u}) = a(\boldsymbol{u}') \cdot u_n^2 + b(\boldsymbol{u}') \cdot u_n + c(\boldsymbol{u}') \quad \text{for all } \boldsymbol{u} \in \mathbb{I}^n. \tag{1}$$

In this section we characterize n-quasi-copulas with quadratic sections in the last variable. In [22], we have characterized n-copulas with quadratic sections in the last variable, for the case $n \geq 3$ (for $n = 2$, see [18]). For our purpose, it is useful to recall such characterization:

Theorem 1. A function C from $\mathbb{I}^n$ to $\mathbb{I}$ is an n-copula with quadratic sections in the last variable if, and only if, C is a function of the form

$$C(\boldsymbol{u}) = D(\boldsymbol{u}') \cdot u_n + a(\boldsymbol{u}') \cdot u_n(1 - u_n) \quad \text{for all } \boldsymbol{u} \in \mathbb{I}^n, \tag{2}$$

where D is an $(n\text{–}1)$-copula and a is a function satisfying the following two conditions:

$$a(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_{n-1}) = a(\boldsymbol{1}_{n-1}) = 0 \quad \text{for all } \boldsymbol{u}' \in \mathbb{I}^{n-1}, \ i = 1, \ldots, n-1; \tag{3}$$

and

$$|V_a(J')| \leq V_D(J') \quad \text{for all } (n-1)\text{-boxes } J' \subset \mathbb{I}^{n-1}.$$

Theorem 1 – and also the following results in this paper – are valid for the case $n = 2$ if we assume that the only 1-quasi-copula (which is also the only 1-copula) is the identity function Id $(\text{Id}(x) = x$ for every $x \in \mathbb{I})$, and the a-volume of some interval $[u, v]$ is $a(v) - a(u)$ for every real function a defined on $\mathbb{I}$.

The following lemma and its corollary show that n-quasi-copulas with quadratic sections in the last variable have a similar form to n-copulas with those sections.

Lemma 2. Let a, b and c be three real-valued functions defined on $\mathbb{I}^{n-1}$, and C the function defined by (1). Then, the following statements are equivalent:

(i) C satisfies condition (C1) and its margin $C_{1\cdots n-1}$ is an $(n-1)$-quasi-copula.

(ii) $a(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_{n-1}) = a(\boldsymbol{1}_{n-1}) = 0$ for all $\boldsymbol{u}' \in \mathbb{I}^{n-1}$ and each $i = 1, \ldots, n-1$, $c(\boldsymbol{u}') = 0$ for all $\boldsymbol{u}' \in \mathbb{I}^{n-1}$, and there exists an $(n-1)$-quasi-copula D such that $b = D - a$.

P r o o f. If the function C defined by (1) satisfies condition (i), then the following equalities hold for every $\boldsymbol{u} \in \mathbb{I}^n$: (a) $C(1, \ldots, 1, u_n) = a(\boldsymbol{1}_{n-1}) \cdot u_n^2 + b(\boldsymbol{1}_{n-1}) \cdot u_n + c(\boldsymbol{1}_{n-1}) = u_n$; (b) $C(\boldsymbol{u}', 0) = c(\boldsymbol{u}') = 0$; and (c) $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = a(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_{n-1}) \cdot u_n^2 + b(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_{n-1}) \cdot u_n = 0$ for all $i = 1, \ldots, n-1$. Hence we have that $a(\boldsymbol{1}_{n-1}) = a(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_{n-1}) = 0$

for all $\boldsymbol{u}' \in \mathbb{I}^{n-1}$ and each $i = 1, \ldots, n-1$. On the other hand, if we take $D = C_{1\ldots n-1}$, then $D(\boldsymbol{u}') = C(\boldsymbol{u}', 1) = a(\boldsymbol{u}') + b(\boldsymbol{u}')$ for all $\boldsymbol{u}' \in \mathbb{I}^{n-1}$, which completes the proof of the necessary condition. Similarly, the sufficient condition can be easily proved. $\qquad\square$

In the following corollary we replace a by $-a$, for convenience.

Corollary 3. If C is an n-quasi-copula with quadratic sections in the last variable, then C is a function of the form (2), where D is an $(n-1)$-quasi-copula and a is a function satisfying condition (3).

Observe that if a is the null function in (2) we obtain n-quasi-copulas with linear sections in the last variable, a trivial case that might be excluded of our study: for every $(n-1)$-quasi-copula (respectively copula) D it is immediate that the function C given by $C(\boldsymbol{u}) = D(\boldsymbol{u}') \cdot u_n$ ($\boldsymbol{u} \in \mathbb{I}^n$) is an n-quasi-copula (respectively copula).

In order to shorten the statements and the proofs of the following results, we introduce some notation: If g is a function defined on $\mathbb{I}^m$, $\boldsymbol{u} \in \mathbb{I}^m$ and $u_i < v_i \leq 1$ for one index $i = 1, 2, \ldots, m$, then the subtraction $g(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots u_m) - g(\boldsymbol{u})$ will be denoted by $\Delta g_i(\boldsymbol{u})$.

The following theorem characterizes quasi-copulas with quadratic sections in the last variable.

Theorem 4. A function C from $\mathbb{I}^n$ to $\mathbb{I}$ is an n-quasi-copula with quadratic sections in the last variable if, and only if, C is a function of the form (2), where D is an $(n-1)$-quasi-copula, a is a real-valued function satisfying condition (3), and the following conditions hold whenever $i \in \{1, 2, \ldots, n-1\}$, $\boldsymbol{u}' \in \mathbb{I}^{n-1}$ and $u_i < v_i \leq 1$:

  (i) $|a(\boldsymbol{u}')| \leq \min\left(D(\boldsymbol{u}'), 1 - D(\boldsymbol{u}')\right)$.

  (ii) If $\Delta a_i(\boldsymbol{u}') < 0$, then $-\Delta a_i(\boldsymbol{u}') \leq \Delta D_i(\boldsymbol{u}')$.

  (iii) If $\Delta a_i(\boldsymbol{u}') > \Delta D_i(\boldsymbol{u}')$, then $(\Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))^2 \leq 4(v_i - u_i)\Delta a_i(\boldsymbol{u}')$.

P r o o f. From Lemma 2 and Corollary 3, a function C is an n-quasi-copula with quadratic sections in the last variable if, and only if, C is a function of the form (2) – where D is an $(n-1)$-quasi-copula and a is a real-valued function satisfying condition (3) – such that conditions (Q1) and (Q2) hold. Thus, if we assume the necessary conditions provided by Corollary 3, we only need to prove that C satisfies conditions (Q1) and (Q2) if, and only if, conditions (i), (ii) and (iii) hold whenever $i \in \{1, 2, \ldots, n-1\}$, $\boldsymbol{u}' \in \mathbb{I}^{n-1}$ and $u_i < v_i \leq 1$.

Since the Lipschitz condition (Q2) is equivalent to all the analogous Lipschitz conditions in each variable together, then C satisfies both conditions (Q1) and (Q2) if, and only if, the following condition holds for each $i = 1, 2, \ldots, n$:

$$0 \leq \Delta C_i(\boldsymbol{u}) \leq v_i - u_i \quad \text{for all } \boldsymbol{u} \in \mathbb{I}^n \text{ and } v_i \in \mathbb{I} \text{ such that } u_i < v_i. \qquad (4)$$

Observe that $\Delta C_n(\boldsymbol{u}) = (v_n - u_n)\left[D(\boldsymbol{u}') + (1 - v_n - u_n)a(\boldsymbol{u}')\right]$ and, for each $i = 1, 2, \ldots, n-1$, $\Delta C_i(\boldsymbol{u}) = u_n \Delta D_i(\boldsymbol{u}') + u_n(1 - u_n)\Delta a_i(\boldsymbol{u}')$. Thus, C satisfies condition (4) for the case $i = n$ if, and only if, $0 \leq D(\boldsymbol{u}') + (1 - v_n - u_n)a(\boldsymbol{u}') \leq 1$ whenever $\boldsymbol{u}' \in \mathbb{I}^{n-1}$ and $0 \leq u_n < v_n \leq 1$, which is equivalent to requiring that condition (i) holds for every $\boldsymbol{u}' \in \mathbb{I}^{n-1}$. And C satisfies condition (4) for every $i = 1, 2, \ldots, n-1$ if, and only if,

$$0 \leq u_n \Delta D_i(\boldsymbol{u}') + u_n(1 - u_n)\Delta a_i(\boldsymbol{u}') \leq v_i - u_i \tag{5}$$

whenever $(\boldsymbol{u}', u_n) \in \mathbb{I}^n$ and $u_i < v_i \leq 1$. Observe that condition (5) holds trivially when $u_n = 0$; and it also holds when $u_n = 1$, since $\Delta D_i(\boldsymbol{u}') \leq v_i - u_i$. Let $\boldsymbol{u}' \in \mathbb{I}^{n-1}$, $u_n \in (0,1)$ and $v_i \in \mathbb{I}$ be such that $u_i < v_i$. We consider three cases:

(I) If $\Delta a_i(\boldsymbol{u}') < 0$, then $u_n \Delta D_i(\boldsymbol{u}') + u_n(1 - u_n)\Delta a_i(\boldsymbol{u}') \leq u_n \Delta D_i(\boldsymbol{u}') \leq v_i - u_i$, i.e., the second inequality in (5) holds; thus, condition (5) holds for every $u_n \in \mathbb{I}$ if, and only if, $-(1 - u_n)\Delta a_i(\boldsymbol{u}') \leq \Delta D_i(\boldsymbol{u}')$ for every $u_n \in (0,1)$, i.e., $-\Delta a_i(\boldsymbol{u}') \leq \Delta D_i(\boldsymbol{u}')$.

(II) If $0 \leq \Delta a_i(\boldsymbol{u}') \leq \Delta D_i(\boldsymbol{u}')$, then the first inequality in (5) holds trivially; and the second one also holds, since $u_n \Delta D_i(\boldsymbol{u}') + u_n(1 - u_n)\Delta a_i(\boldsymbol{u}') \leq u_n(2 - u_n)\Delta D_i(\boldsymbol{u}') \leq \Delta D_i(\boldsymbol{u}') \leq v_i - u_i$.

(III) If $\Delta a_i(\boldsymbol{u}') > \Delta D_i(\boldsymbol{u}')$, then the first inequality in (5) holds trivially as in case (II); and the second inequality holds for every $u_n \in (0,1)$ if, and only if, the function $h(u_n) = (\Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))u_n - \Delta a_i(\boldsymbol{u}')u_n^2$ is less than or equal to $v_i - u_i$ for every $u_n \in (0,1)$. Observe that h is a concave quadratic function with vertex in $u_{n_0} = (\Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))/(2\Delta a_i(\boldsymbol{u}'))$. Since u_{n_0} is in $(0,1)$ and $h(u_{n_0}) = (\Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))^2/(4\Delta a_i(\boldsymbol{u}'))$, we can conclude that, for this case, condition (5) is satisfied for every $u_n \in (0,1)$ if, and only if, $(\Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))^2 \leq 4(v_i - u_i)\Delta a_i(\boldsymbol{u}')$, which completes the proof. $\qquad\square$

As a consequence of the previous theorem, we will be able to construct families of proper n-quasi-copulas with quadratic sections in the last variable – and similarly in the other variables – whenever $n \geq 3$: see Section 4. However, we cannot find proper bivariate quasi-copulas with quadratic sections, as the following result shows.

Corollary 5. Every bivariate quasi-copula with quadratic sections in one variable is a copula.

P r o o f. Suppose, without loss of generality, that C is a bivariate quasi-copula with quadratic sections in the second variable. Thus, from Theorem 4 we have that $C(x, y) = xy + a(x)y(1 - y)$ for some real function a such that $a(0) = a(1) = 0$. Let $y, y' \in \mathbb{I}$ be such that $y < y'$. From conditions (ii) and (iii) of Theorem 4 we have that $|a(y') - a(y)| \leq y' - y$ whenever $a(y') - a(y) < 0$, and that $|a(y') - a(y)| = a(y') - a(y) > y' - y$ does not hold; so $|a(y') - a(y)| \leq y' - y$ always holds. Hence, from Theorem 2.2 in [18], we conclude that C is a copula. $\qquad\square$

The following corollary constructs a family of n-quasi-copulas with quadratic sections in the last variable from a non-trivial n-quasi-copula of that type.

Corollary 6. Let C be an n-quasi-copula of the form (2) such that a is not the null function, i.e., C is a non-trivial n-quasi-copula with quadratic sections in the last variable. Then the function C_λ, with $\lambda \in (0, 1)$, defined by

$$C_\lambda(\boldsymbol{u}) = D(\boldsymbol{u}') \cdot u_n + \lambda \cdot a(\boldsymbol{u}') \cdot u_n(1 - u_n) \quad \text{for all } \boldsymbol{u} \in \mathbb{I}^n, \tag{6}$$

is also an n-quasi-copula with quadratic sections in the last variable.

P r o o f. We only have to prove that the function $a_\lambda = \lambda \cdot a$ satisfies conditions (i), (ii) and (iii) of Theorem 4 for every $i \in \{1, \ldots, n-1\}$, $\boldsymbol{u}' \in \mathbb{I}^{n-1}$ and $v_i \in \mathbb{I}$ such that $u_i < v_i$. Since a satisfies condition (i), it is immediate that a_λ also satisfies that condition. Suppose that $(\Delta a_\lambda)_i(\boldsymbol{u}') = \lambda \Delta a_i(\boldsymbol{u}') < 0$. Then $\Delta a_i(\boldsymbol{u}') < 0$, whence $-\lambda \Delta a_i(\boldsymbol{u}') \leq -\Delta a_i(\boldsymbol{u}') \leq \Delta D_i(\boldsymbol{u}')$, i.e., a_λ satisfies condition (ii). Finally, suppose that $(\Delta a_\lambda)_i(\boldsymbol{u}') > \Delta D_i(\boldsymbol{u}')$. So we also have that $\Delta a_i(\boldsymbol{u}') > \Delta D_i(\boldsymbol{u}')$, whence $(\Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))^2 \leq 4(v_i - u_i)\Delta a_i(\boldsymbol{u}')$. We need to prove that $(\lambda \Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))^2 \leq 4(v_i - u_i)\lambda \Delta a_i(\boldsymbol{u}')$. From our hypotheses, we have $4(v_i - u_i)\lambda \Delta a_i(\boldsymbol{u}') \geq \lambda(\Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))^2$. So it suffices to prove that $\lambda(\Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))^2 \geq (\lambda \Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))^2$. This inequality is equivalent to the following $\lambda(\Delta a_i(\boldsymbol{u}'))^2 \geq (\Delta D_i(\boldsymbol{u}'))^2$, which is trivially satisfied since $\Delta a_i(\boldsymbol{u}') > \lambda \Delta a_i(\boldsymbol{u}') > \Delta D_i(\boldsymbol{u}')$. □

The following two corollaries provide sufficient conditions – simpler than those of Theorem 4 – to obtain n-quasi-copulas with quadratic sections in the last variable.

Corollary 7. Let C be a function of the form (2), where D is an $(n-1)$-quasi-copula and a is a real-valued function satisfying condition (3). Suppose that for every $i = 1, 2, \ldots, n-1$, $\boldsymbol{u}' \in \mathbb{I}^{n-1}$ and $v_i \in \mathbb{I}$ such that $u_i < v_i$, we have that conditions (i) and (ii) of Theorem 4 and the following condition (iv) hold:

 (iv) If $\Delta a_i(\boldsymbol{u}') > \Delta D_i(\boldsymbol{u}')$, then $\Delta a_i(\boldsymbol{u}') \leq v_i - u_i$.

Then C is an n-quasi-copula with quadratic sections in the last variable.

P r o o f. Let $i \in \{1, 2, \ldots, n-1\}$, $\boldsymbol{u}' \in \mathbb{I}^{n-1}$ and $v_i \in \mathbb{I}$ be such that $u_i < v_i$ and $\Delta a_i(\boldsymbol{u}') > \Delta D_i(\boldsymbol{u}')$. Then $(\Delta a_i(\boldsymbol{u}') + \Delta D_i(\boldsymbol{u}'))^2 = (\Delta a_i(\boldsymbol{u}'))^2 + (\Delta D_i(\boldsymbol{u}'))^2 + 2\Delta a_i(\boldsymbol{u}')\Delta D_i(\boldsymbol{u}') < (\Delta a_i(\boldsymbol{u}'))^2 + (\Delta a_i(\boldsymbol{u}'))^2 + 2\Delta a_i(\boldsymbol{u}')\Delta a_i(\boldsymbol{u}') = 4(\Delta a_i(\boldsymbol{u}'))^2 \leq 4(v_i - u_i)\Delta a_i(\boldsymbol{u}')$, whence the proof follows. □

An application of Corollary 7 can be found in Example 2 of Section 4.

Corollary 8. Let C be a function of the form (2), where D is an $(n-1)$-quasi-copula and a is a real-valued function satisfying condition (3). Suppose that the following condition holds:

 (v) $|a(\boldsymbol{v}') - a(\boldsymbol{u}')| \leq D(\boldsymbol{v}') - D(\boldsymbol{u}')$ for all $\boldsymbol{u}', \boldsymbol{v}' \in \mathbb{I}^{n-1}$ such that $\boldsymbol{u}' \leq \boldsymbol{v}'$.

Then C is an n-quasi-copula with quadratic sections in the last variable.

P r o o f. Let $i \in \{1, 2, \ldots, n-1\}$, $\boldsymbol{u}' \in \mathbb{I}^{n-1}$ and $v_i \in \mathbb{I}$ such that $u_i < v_i$. Taking $\boldsymbol{v}' = (u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_n)$ in condition (v) we have $|\Delta a_i(\boldsymbol{u}')| \leq \Delta D_i(\boldsymbol{u}')$, whence conditions (ii) and (iii) of Theorem 4 hold trivially. Taking $u_1 = 0$ in (v) we obtain that $|a(\boldsymbol{v}')| \leq D(\boldsymbol{v}')$ for all $\boldsymbol{v}' \in \mathbb{I}^{n-1}$; and taking $\boldsymbol{v}' = \boldsymbol{1}_{n-1}$ in (v) we obtain $|a(\boldsymbol{u}')| \leq 1 - D(\boldsymbol{u}')$, whence condition (i) of Theorem 4 holds and the proof is completed. □

An application of Corollary 8 can be found in Example 1 of Section 4.

From its proof, it is clear that the sufficient condition in Corollary 8 is stronger than that of Corollary 7. Both sufficient conditions are not necessary for the function C given by (2) to be a quasi-copula. For instance, Example 2 in the next section proves this fact for the strongest condition: specifically, we show that certain function C is a quasi-copula by using Corollary 7, but that function C does not satisfy the hypotheses of Corollary 8.

Finally, recall that condition (v) in Corollary 8 is a necessary condition for the function C given by (2) to be a copula (see [22]).

4. EXAMPLES

In this section we provide some examples of families of proper multivariate quasi-copulas with quadratic sections in the last variable. To obtain quasi-copulas with quadratic sections in other variables, it suffices to permute the variables. The following example provides one of those families for every $n \geq 3$.

Example 1. Let $n \geq 3$. If $D = \Pi^{n-1}$ – i.e., the product $(n-1)$-copula, which is given by $\Pi^{n-1}(\boldsymbol{u}') = u_1 \cdots u_{n-1}$ – and a is the function defined by $a(\boldsymbol{u}') = (1 - \max_{1 \leq i \leq n-1} u_i)\Pi^{n-1}(\boldsymbol{u}')$ for every $\boldsymbol{u}' \in \mathbb{I}^{n-1}$, then the function C defined by (2) can be expressed as follows:

$$C(\boldsymbol{u}) = \Pi^n(\boldsymbol{u}) \left[1 + (1 - u_n) \left(1 - \max_{1 \leq i \leq n-1} u_i \right) \right] \quad \text{for all } \boldsymbol{u} \in \mathbb{I}^n. \qquad (7)$$

Now we prove that C is an n-quasi-copula. The function a satisfies immediately condition (3). And C satisfies condition (v) in Corollary 8 if, and only if,

$$\left| \Pi^{n-1}(\boldsymbol{v}') \left(1 - \max_{1 \leq i \leq n-1} v_i \right) - \Pi^{n-1}(\boldsymbol{u}') \left(1 - \max_{1 \leq i \leq n-1} u_i \right) \right| \leq \Pi^{n-1}(\boldsymbol{v}') - \Pi^{n-1}(\boldsymbol{u}')$$
$$(8)$$

for every $\boldsymbol{u}', \boldsymbol{v}' \in \mathbb{I}^{n-1}$ such that $\boldsymbol{u}' \leq \boldsymbol{v}'$. The inequality (8) is equivalent to the following condition:

$$0 \leq \Pi^{n-1}(\boldsymbol{v}') \cdot \max_{1 \leq i \leq n-1} v_i - \Pi^{n-1}(\boldsymbol{u}') \cdot \max_{1 \leq i \leq n-1} u_i \leq 2(\Pi^{n-1}(\boldsymbol{v}') - \Pi^{n-1}(\boldsymbol{u}')).$$

Since the first inequality in this expression holds trivially, we can conclude that C
satisfies condition (v) if, and only if,

$$\Pi^{n-1}(\boldsymbol{u}')\left(2 - \max_{1\leq i\leq n-1} u_i\right) \leq \Pi^{n-1}(\boldsymbol{v}')\left(2 - \max_{1\leq i\leq n-1} v_i\right)$$

whenever $\boldsymbol{0}_{n-1} \leq \boldsymbol{u}' \leq \boldsymbol{v}' \leq \boldsymbol{1}_{n-1}$. This inequality is trivially satisfied when $u_i = 0$
for some $i = 1, 2, \ldots, n-1$. So we can assume $u_i > 0$ for all $i = 1, 2, \ldots, n-1$. Since
$\max_{1\leq i\leq n-1} v_i = v_k$ for some $k = 1, 2, \ldots, n-1$ and the function $f(x) = x(2-x)$ is
increasing on $\mathbb{I}$, we have

$$\begin{aligned}
\Pi^{n-1}(\boldsymbol{u}')\left(2 - \max_{1\leq i\leq n-1} u_i\right) &\leq \Pi^{n-1}(\boldsymbol{u}')(2-u_k) = \left(\Pi^{n-1}(\boldsymbol{u}')/u_k\right) u_k(2-u_k) \\
&\leq \left(\Pi^{n-1}(\boldsymbol{v}')/v_k\right) v_k(2-v_k) = \Pi^{n-1}(\boldsymbol{v}')\left(2-v_k\right) \\
&= \Pi^{n-1}(\boldsymbol{v}')\left(2 - \max_{1\leq i\leq n-1} v_i\right).
\end{aligned}$$

So condition (v) holds and, from Corollary 8, we have that the function C defined by
(7) is an n-quasi-copula. Now, by using Corollary 6, we obtain a family $\{C_\lambda \, : \, \lambda \in \mathbb{I}\}$
of n-quasi-copulas with quadratic sections in the last variable, namely:

$$C_\lambda(\boldsymbol{u}) = \Pi^n(\boldsymbol{u})\left[1 + \lambda(1-u_n)\left(1 - \max_{1\leq i\leq n-1} u_i\right)\right] \quad \text{for all } \boldsymbol{u} \in \mathbb{I}^n.$$

If $\lambda = 0$, then C_λ is the product copula Π^n. Now we prove that C_λ is a proper
n-quasi-copula for every $\lambda \in (0,1]$. Let $\lambda \in (0,1]$ and $n \geq 3$. From Theo-
rem 1, we only need to prove that there exists an $(n-1)$-box $J' \subset \mathbb{I}^{n-1}$ such that
$|V_{\lambda a}(J')| > V_{\Pi^{n-1}}(J')$. Let $r \in (0,1)$ and let $J = [r,1]^{n-1}$. Then, $|V_{\lambda a}(J)| =$
$|(-1)^{n-1}\lambda a(\boldsymbol{r}_{n-1})| = \lambda(1-r)r^{n-1}$ and $V_{\Pi^{n-1}}(J) = (1-r)^{n-1}$. So $|V_{\lambda a}(J)| >$
$V_{\Pi^{n-1}}(J)$ if, and only if, $\lambda > (1-r)^{n-2}/r^{n-1}$. Since $\lim_{r\to 1}(1-r)^{n-2}/r^{n-1} = 0$, we
can conclude that $|V_{\lambda a}([r,1]^{n-1})| > V_{\Pi^{n-1}}([r,1]^{n-1})$ for some $r \in (0,1)$, as desired.

Example 1 – and the next Example 2 as well – shows that an n-quasi-copula of
the form (2) can be proper even though the $(n-1)$-quasi-copula D is not proper –
i.e., D is a copula. On the other hand, if D is a proper $(n-1)$-quasi-copula, then
every n-quasi-copula C of the form (2) is proper since its margin $C_{12\ldots n-1} = D$ is a
proper $(n-1)$-quasi-copula.

In the following example we introduce a family of proper 3-quasi-copulas. This
example illustrates the usefulness of Corollary 7.

Example 2. Let $D = M$ (M is the pointwise upper bound 2-copula, i.e., $M(x,y) = \min(x,y)$ for every $(x,y) \in \mathbb{I}^2$). Let a be the function defined on $\mathbb{I}^2$ by

$$a(x,y) = \begin{cases} 2xy & \text{if } \max(x,y) \leq 1/2, \\ 1 - \min(x,y) & \text{if } \min(x,y) \geq 1/2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

And let C be the function defined by (2), i. e.,

$$C(x, y, z) = zM(x, y) + z(1 - z)a(x, y) \quad \text{for all } (x, y, z) \in \mathbb{I}^3. \tag{9}$$

Observe that C does not satisfy condition (v) of Corollary 8: if $0 < x < y < y' \leq 1/2$, then $|a(x, y') - a(x, y)| = 2x(y' - y) > 0 = M(x, y') - M(x, y)$. This fact also implies that C is not a copula.

Next we show that C satisfies the hypotheses of Corollary 7 to prove that C is a 3-quasi-copula. The function a satisfies trivially condition (3). Moreover, since $|a| = a$ and

$$\min\left(M(x, y), 1 - M(x, y)\right) = \begin{cases} \min(x, y) & \text{if } \min(x, y) \leq 1/2, \\ 1 - \min(x, y) & \text{otherwise,} \end{cases}$$

it is immediate that condition (i) of Theorem 4 holds. Observe that the functions a and M are symmetric, i. e., $a(x, y) = a(y, x)$ and $M(x, y) = M(y, x)$ for all $(x, y) \in \mathbb{I}^2$. Thus, in order to prove the remaining two conditions in the hypotheses of Corollary 7, it suffices to consider one of the variables, for instance the second one. Hence we only need to prove that, for every $x, y, y' \in \mathbb{I}$ such that $y < y'$, the following conditions hold:

$$\text{if } a(x, y') - a(x, y) < 0, \text{ then } a(x, y) - a(x, y') \leq M(x, y') - M(x, y); \tag{10}$$

$$\text{if } a(x, y') - a(x, y) > M(x, y') - M(x, y), \text{ then } a(x, y') - a(x, y) \leq y' - y. \tag{11}$$

However, instead of condition (11), we will prove a stronger – and simpler – condition, namely:

$$\text{if } a(x, y') - a(x, y) > 0, \text{ then } a(x, y') - a(x, y) \leq y' - y. \tag{12}$$

After straightforward computations, it is easy to obtain that

$$a(x, y') - a(x, y) = \begin{cases} 2x(y' - y) & \text{if } \max(x, y') \leq 1/2, \\ x(1 - 2y) & \text{if } \max(x, y) \leq 1/2 < y', \\ 0 & \text{if } \max(x, 1/2) < y, \\ y' - y & \text{if } y' \leq 1/2 < x, \\ 1 - y' - y & \text{if } y \leq 1/2 < y' \leq x, \\ 1 - x - y & \text{if } y \leq 1/2 < x < y', \\ y - y' & \text{if } 1/2 < y < y' \leq x, \\ y - x & \text{if } 1/2 < y \leq x < y'. \end{cases}$$

Thus, the cases where $a(x, y') - a(x, y) < 0$ and $a(x, y') - a(x, y) > 0$ are respectively shown in the following expressions:

$$
\begin{cases}
1 - y' - y & \text{if } y \leq 1/2 \leq 1 - y < y' \leq x, \\
1 - x - y & \text{if } y \leq 1/2 \leq 1 - y < x < y', \\
y - y' & \text{if } 1/2 < y < y' \leq x, \\
y - x & \text{if } 1/2 < y < x < y',
\end{cases}
$$

and

$$
\begin{cases}
2x(y' - y) & \text{if } 0 < x \leq 1/2, y' \leq 1/2, \\
x(1 - 2y) & \text{if } 0 < x \leq 1/2, y < 1/2 < y', \\
y' - y & \text{if } y' \leq 1/2 < x, \\
1 - y' - y & \text{if } y < 1 - y' < 1/2 < y' \leq x, \\
1 - x - y & \text{if } y < 1 - x < 1/2 < x < y'.
\end{cases}
$$

Since

$$
M(x, y') - M(x, y) = \begin{cases}
y' - y & \text{if } y \leq 1/2 \leq 1 - y < y' \leq x \text{ or } 1/2 < y < y' \leq x, \\
x - y & \text{if } y \leq 1/2 \leq 1 - y < x < y' \text{ or } 1/2 < y < x < y',
\end{cases}
$$

it is immediate that condition (10) holds. And with respect to the five cases where $a(x, y') - a(x, y) > 0$, it is easy to check that $a(x, y') - a(x, y) \leq y' - y$: for instance, if $0 < x \leq 1/2$ and $y < 1/2 < y'$, then $a(x, y') - a(x, y) = x(1 - 2y) \leq (1 - 2y)/2 < y' - y$. Hence condition (12) holds, and the function C defined by (9) is a proper 3-quasi-copula. Finally, from Corollary 6, we obtain a family $\{C_\lambda : \lambda \in \mathbb{I}\}$ of 3-quasi-copulas with quadratic sections in the last variable, namely:

$$
C_\lambda(x, y, z) = zM(x, y) + \lambda z(1 - z)a(x, y) \text{ for all } (x, y, z) \in \mathbb{I}^3.
$$

An analogous argument to that used for the 3-quasi-copula $C = C_1$ proves that C_λ is a proper 3-quasi-copula for every $\lambda \in (0, 1)$ (but C_0 is a 3-copula).

R E F E R E N C E S

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