KYBERNETIKA — VOLUME *44* (2008), NUMBER 6, PAGES 777 – 794

DYNAMIC DEPENDENCE ORDERING FOR ARCHIMEDEAN COPULAS AND DISTORTED COPULAS

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This paper proposes a general framework to compare the strength of the dependence in survival models, as time changes, i. e. given remaining lifetimes $\boldsymbol{X}$, to compare the dependence of $\boldsymbol{X}$ given $\boldsymbol{X} > t$, and $\boldsymbol{X}$ given $\boldsymbol{X} > s$, where $s > t$. More precisely, analytical results will be obtained in the case the survival copula of $\boldsymbol{X}$ is either Archimedean or a distorted copula. The case of a frailty based model will also be discussed in details.

Keywords: Archimedean copulas, Cox model, dependence, distorted copulas, ordering

AMS Subject Classification: 60A05, 60E15, 62H05

1. INTRODUCTION

In the context of insurance (and reinsurance) of large claims, [14] pointed out that "*in case of heavy tailed random variables, apart from the fact that the coefficient of correlation may not be defined, its main disadvantage is that it does not capture very well possible dependence in the tails*". In finance and yield curve modeling, [20] observed that "*dependence in the center of the distribution may be treated separately from the dependence in the distribution tails*", and that symmetric as well as asymmetric tail dependence should be considered.

Hence, the mathematical formulation is that risk managers need to assess whether random vector $\boldsymbol{X}$ given $\boldsymbol{X} > \boldsymbol{x}_1$ is more or less dependent than $\boldsymbol{X}$ given $\boldsymbol{X} > \boldsymbol{x}_2$, when $\boldsymbol{x}_1 > \boldsymbol{x}_2$. This problem can easily be related to the comparison of survival models: is $\boldsymbol{X}$ given $\boldsymbol{X} > t_1$ is more or less dependent than $\boldsymbol{X}$ given $\boldsymbol{X} > t_2$, when $t_1 > t_2$, i. e. do we have more or less dependence as time elapses ?

1.1. Copulas, Archimedean copulas, and distorted copulas

Definition 1.1. A d-dimensional copula is a d-dimensional distribution function restricted to $[0, 1]^d$ with standard uniform margins, for a non-negative integer $d \geq 2$.

For example, the function $C^{\perp}(u_1, \ldots, u_d) = u_1 \times \ldots \times u_d$ is a copula, called *independent* or *product* copula. C is a copula of the random vector $\boldsymbol{X}$ if

$$\Pr(X_1 \leq x_1, \ldots, X_d \leq x_d) = C(\Pr(X_1 \leq x_1), \ldots, \Pr(X_d \leq x_d)).$$

The existence of a copula C such that this equality holds is insured by Sklar's theorem (see [31] or [28]). Further, $C^\star$ is called a *survival* copula of random vector $\boldsymbol{X}$ if

$$\Pr(X_1 > x_1, \ldots, X_d > x_d) = C^\star(\Pr(X_1 > x_1), \ldots, \Pr(X_d > x_d)).$$

Remark 1.1. From this definition, we see that we can conveniently study exceeding properties ($\boldsymbol{X} > \boldsymbol{x}$) using the *survival* copula of $\boldsymbol{X}$, $C^\star$, and that for bounding properties ($\boldsymbol{X} \leq \boldsymbol{x}$), the use of C will be more convenient. Hence, for convenience in the first part of this paper we will derive properties on $\boldsymbol{X}$ given $\boldsymbol{X} \leq \boldsymbol{x}$ assuming that C satisfies some properties (e. g. Archimedean). Then in order to derive properties on $\boldsymbol{X}$ given $\boldsymbol{X} > t$ (residual lifetimes), some properties on $C^\star$ will be assumed.

Note that a random vector $\boldsymbol{X}$ has independent components if and only if $C^\perp$ is a copula of $\boldsymbol{X}$ (or equivalently a survival copula).

Definition 1.2. Let ϕ denote a decreasing function $(0,1] \to [0,\infty]$ such that $\phi(1) = 0$, and such that ϕ^{-1} is d-monotone, i. e. for all $k = 0, 1, \ldots, d$, $(-1)^k[\phi^{-1}]^{(k)}(t) \geq 0$ for all t. Define the inverse (or *quasi-inverse* if $\phi(0) < \infty$) as

$$\phi^{-1}(t) = \begin{cases} \phi^{-1}(t) & \text{for } 0 \leq t \leq \phi(0) \\ 0 & \text{for } \phi(0) < t < \infty. \end{cases}$$

The function

$$C(u_1, \ldots, u_n) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_d)), u_1, \ldots, u_n \in [0,1],$$

is a copula, called an Archimedean copula, with generator ϕ.

The proof that those conditions are necessary and sufficient to define a proper copula in dimension d can be found in [6] or [26]. Let Φ_d denote the set of Archimedean generators in dimension d. Note that ϕ and $c \cdot \phi$ (where c is a positive constant) yield the same copula, and conversely, two Archimedean copulas are equal if their generators are equal up to a multiplicative constant. If $\phi(t) \to \infty$ when $t \to 0$, the generator will be said to be strict.

Example 1.1. The independent copula $C^\perp$ is an Archimedean copula, with generator $\phi(t) = -\log t$. The upper Fréchet–Hoeffding copula, defined as the minimum componentwise, $M(\boldsymbol{u}) = \min\{u_1, \ldots, u_d\}$, is not Archimedean (but can be obtained as the limit of some Archimedean copulas).

Example 1.2. A large subclass of Archimedean copula in dimension d is the class of Archimedean copulas obtained using the *frailty approach*. Those copulas are obtained when ϕ is the inverse of the Laplace transform of a positive random variable (i. e. a completely monotone function taking value 1 in 0). Consider random variables $X_1, \ldots, X_d$ conditionally independent, given a latent factor Θ, a positive random variable, such that $\Pr(X_i \le x_i|\Theta) = G_i(x)^{\Theta}$ where G_i denotes a baseline distribution function. The joint distribution function of $\boldsymbol{X}$ is given by

$$\begin{aligned} F_{\boldsymbol{X}}(x_1,\ldots,x_d) &= \mathrm{E}\left(\Pr\left(X_1 \le x_1,\ldots,X_d \le X_d|\Theta\right)\right) \\ &= \mathrm{E}\left(\prod_{i=1}^{d} \Pr\left(X_i \le x_i|\Theta\right)\right) = \mathrm{E}\left(\prod_{i=1}^{d} G_i(x_i)^{\Theta}\right) \\ &= \mathrm{E}\left(\prod_{i=1}^{d} \exp\left[-\Theta\left(-\log G_i(x_i)\right)\right]\right) = \psi\left(-\sum_{i=1}^{d} \log G_i(x_i)\right), \end{aligned}$$

where ψ is the Laplace transform of the distribution of Θ, i. e. $\psi(t) = \mathrm{E}(\exp(-t\Theta))$. Because the marginal distributions are given respectively by

$$F_i(x_i) = \Pr(X_i \le x_i) = \psi\left(-\log G_i(x_i)\right),$$

the copula of $\boldsymbol{X}$ is

$$C(\boldsymbol{u}) = F_{\boldsymbol{X}}\left(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)\right) = \psi\left(\psi^{-1}(u) + \cdots + \psi^{-1}(u_d)\right)$$

This copula is an Archimedean copula with generator $\phi = \psi^{-1}$ (see e. g. [7, 29, 33], or [2] for more details).

[17] extended the concept of Archimedean copulas introducing the *multivariate probability integral transformation* ([32] called this the *distorted copula*, while [23] or [11] called this the *transformed copula*). Consider a copula C. Let h be a continuous strictly concave increasing function $[0,1] \to [0,1]$ satisfying $h(0) = 0$ and $h(1) = 1$, such that

$$\mathcal{D}_h(C)(u_1,\ldots,u_d) = h^{-1}\left(C\left(h(u_1),\ldots,h(u_d)\right)\right), 0 \le u_i \le 1$$

is a copula. Those functions will be called *distortion functions*.

Example 1.3. A classical example is obtained when h is a power function, and when the power is the inverse of an integer, $h_n(x) = x^{1/n}$, i. e.

$$\mathcal{D}_{h_n}(C)(u,v) = C^n(u^{1/n}, v^{1/n}), 0 \le u, v \le 1 \text{ and } n \in \mathbb{N}.$$

Then this copula is the survival copula of the componentwise maxima: the copula of $(\max\{X_1,\ldots,X_n\}, \max\{Y_1,\ldots,Y_n\})$ is $\mathcal{D}_{h_n}(C)$, where $\{(X_1,Y_1),\ldots,(X_n,Y_n)\}$ is an i.i.d. sample, and the (X_i,Y_i)'s have copula C.

Example 1.4. Let ϕ denote a convex decreasing function on $(0,1]$ such that $\phi(1)=0$, and define $C(u,v) = \phi^{-1}(\phi(u) + \phi(v)) = \mathcal{D}_{\exp[-\phi]}(C^{\perp})$. This function is a copula, called Archimedean copula (see [25] and [15]), and function ϕ is a generator of that copula.

In the bivariate case (Examples 1.3 and 1.4), h need not be differentiable, and concavity is a sufficient condition. Unfortunately, in higher dimension, it is much difficult to characterize the set of distortion function which might generate a copula.

Let $\mathcal{H}_d$ denote the set of continuous strictly increasing functions $[0,1] \to [0,1]$ such that $h(0) = 0$ and $h(1) = 1$, for all $h \in \mathcal{H}_d$ and $C \in \mathcal{C}$,

$$\mathcal{D}_h(C)(u_1, \dots, u_d) = h^{-1}(C(h(u_1), \dots, h(u_d))), 0 \le u_i \le 1$$

is a copula, called distorted copula. $\mathcal{H}_d$-copulas will be functions $\mathcal{D}_h(C)$ for some distortion function h and some copula C. The d-monotonicity of function $\mathcal{D}_h(C)$ (in order to define a proper copula function) is obtained when $h \in \mathcal{H}_d$, i.e. h is continuous, with $h(0) = 0$ and $h(1) = 1$, and such that $h^{(k)}(x) \le 0$ for all $x \in (0,1)$ and $k = 2, 3, \dots, d$ (from Theorem 2.6 and 4.4 in [27]).

As a corollary, note that if $\phi \in \Phi_d$, then $h(x) = \exp(-\phi(x))$ belongs to $\mathcal{H}_d$. Further, observe that for $h, h' \in \mathcal{H}_d$,

$$\mathcal{D}_{h \circ h'}(C)(u_1, \dots, u_d) = (\mathcal{D}_h \circ \mathcal{D}_{h'})(C)(u_1, \dots, u_d), 0 \le u_i \le 1.$$

1.2. Outline of the paper

The goal of this paper is to answer the question mentioned above: is $\boldsymbol{X}$ given $\boldsymbol{X} > \boldsymbol{x}_1$ more or less dependent than $\boldsymbol{X}$ given $\boldsymbol{X} > \boldsymbol{x}_2$, in the case the surival copula of $\boldsymbol{X}$ is Archimedean. Hence, Section 2 will study properties of $\boldsymbol{X}$ given $\boldsymbol{X} \le \boldsymbol{x}$, when the copula of $\boldsymbol{X}$ is Archimedean, and give details in the case $\boldsymbol{X}$ admits a frailty representation. In Section 3, analogous properties will be derived in the case the survival copula of $\boldsymbol{X}$ is a distorted copula, and we will extend the frailty model to that case. And finally, in the case of Archimedean copulas, a characterization of Archimedean copulas which are more and more dependent (in tails, or as time elapses in aging models) will be given in Section 4.

2. RIGHT CENSORING OF ARCHIMEDEAN COPULAS

Let C be a copula and let $\boldsymbol{U}$ be a random vector with joint distribution function C. Let $\boldsymbol{u} \in (0,1]^d$ be such that $C(\boldsymbol{u}) > 0$. The *lower tail dependence copula* of C at level $\boldsymbol{u}$ is defined as the copula, denoted $C_{\boldsymbol{u}}$, of the joint distribution of $\boldsymbol{U}$ conditionally on the event $\{\boldsymbol{U} \le \boldsymbol{u}\} = \{U_1 \le u_1, \dots, U_d \le u_d\}$. Formally,

$$C_{\boldsymbol{u}}(x_1, \dots, x_d) = \frac{C(x'_1, \dots, x'_d)}{C(\boldsymbol{u})}$$

where $0 \le x'_i \le u_i$ are the solutions to the equations

$$\frac{C(u_1, \dots, u_{i-1}, x'_i, u_{i+1}, \dots, u_d)}{C(\boldsymbol{u})} = x_i,$$

(see Definition 3.1 in [21] or Definition 2.2 in [22] when $\boldsymbol{u} = u \cdot \mathbf{1}$, or [12] and Definition 2.5 in [3] in a more general context).

If C is a strict Archimedean copula with generator ϕ (i.e. $\phi(0) = \infty$), then the lower tail dependence copula relative to C at level $\boldsymbol{u}$ is given by the strict Archimedean copula with generator ϕ_u defined by

$$\phi_{\boldsymbol{u}}(t) = \phi(t \cdot C(\boldsymbol{u})) - \phi(C(\boldsymbol{u})), \quad 0 \le t \le 1, \tag{1}$$

where $C(\boldsymbol{u}) = \phi^{-1}[\phi(u_1) + \cdots + \phi(u_d)]$ (Proposition 3.2 in [21]). Note that tail properties of Archimedean copulas, based on this conditional copulas, have recently been intensively studied (see e. g. [4] and [5])

Example 2.1. Gumbel copulas have generator $\phi(t) = [-\ln t]^{\theta}$ where $\theta \geq 1$. For any $\boldsymbol{u} \in (0,1]^d$, the corresponding conditional copula has generator

$$\phi_{\boldsymbol{u}}(t) = \left[M^{1/\theta} - \ln t\right]^{\theta} - M \text{ where}$$
$$M = [-\ln u_1]^{\theta} + \cdots + [-\ln u_d]^{\theta}.$$

Example 2.2. Clayton copulas C have generator $\phi(t) = t^{-\theta} - 1$ where $\theta > 0$. Hence,

$$\begin{aligned} \phi_{\boldsymbol{u}}(t) &= [t \cdot C(\boldsymbol{u})]^{-\theta} - 1 - \phi(C(\boldsymbol{u})) \\ &= t^{-\theta} \cdot C(\boldsymbol{u})^{-\theta} - 1 - [C(\boldsymbol{u})^{-\theta} - 1] = C(\boldsymbol{u})^{-\theta} \cdot [t^{-\theta} - 1], \end{aligned}$$

hence $\phi_{\boldsymbol{u}}(t) = C(\boldsymbol{u})^{-\theta} \cdot \phi(t)$. Since the generator of an Archimedean copula is unique up to a multiplicative constant, $\phi_{\boldsymbol{u}}$ is also the generator of Clayton copula, with parameter θ.

Note that this stability of the class can be obtained in the subclass of Archimedean copulas with a factor representation, obtained using the frailty approach.

Example 2.3. Gumbel copulas could be obtained when factor Θ has a stable distribution, i. e. its Laplace transform equal to $\psi(t) = \exp\left[-t^{1/\theta}\right]$. Furthermore, Clayton copulas are obtained when the heterogeneity factor Θ has a Laplace transform equal to $\psi(t) = [1-t]^{-1/\theta}$. The heterogeneity distribution is a Gamma distribution with degrees of freedom $1/\theta$.

Theorem 2.4. Consider $\boldsymbol{X}$ with Archimedean copula, having a factor representation, and let ψ denote the Laplace transform of the heterogeneity factor Θ. Let $\boldsymbol{u} \in (0,1]^d$, then $\boldsymbol{X}$ given $\boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$ (in the pointwise sense, i. e. $X_1 \leq F_1{}^{-1}(u_1), \ldots, X_d \leq F_d{}^{-1}(u_d)$) is an Archimedean copula with a factor representation, where the factor has Laplace transform

$$\psi_{\boldsymbol{u}}(t) = \frac{\psi\left(t + \psi^{-1}\left(C(\boldsymbol{u})\right)\right)}{C(\boldsymbol{u})}.$$

Proof. Note that $\boldsymbol{X}$ given $\boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$ will be said to have an Archimedean copula with a factor representation if all the components are independent, given a positive factor Θ', and if marginal distribution functions can be written as $G_i'(x_i)^{\Theta'}$.

Consider a random vector $\boldsymbol{Y}$ such that $\boldsymbol{Y} \overset{\mathcal{L}}{=} \boldsymbol{X} | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$. The joint distribution function of $\boldsymbol{Y}$, denoted F', is

$$\begin{aligned} F'(\boldsymbol{x}) &= \Pr(\boldsymbol{Y} \leq \boldsymbol{x}) = \Pr(\boldsymbol{X} \leq \boldsymbol{x} | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})) \\ &= \frac{\Pr(\boldsymbol{X} \leq \boldsymbol{x})}{\Pr(\boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u}))} \text{ on } (-\infty, F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})], \\ &= \frac{\psi(\psi^{-1}(F_1(x_1)) + \ldots \psi^{-1}(F_d(x_d)))}{C(\boldsymbol{u})} \\ &= \frac{\psi(-\log G_1(x_1) - \ldots . - \log G_d(x_d))}{C(\boldsymbol{u})}, \end{aligned}$$

since $F_i(x_i) = \psi(-\log G_i(x_i))$. Hence, from this relationship one gets that the marginal distribution of $\boldsymbol{Y}$ is

$$\begin{aligned} F_i'(x_i) &= \lim_{x_j \to F_j{}^{-1}(u_j), j \neq i} F(\boldsymbol{x}) \\ &= \frac{\psi(-\log(G_i(x_i))) + \psi^{-1}(u_1) + \ldots + \psi^{-1}(u_{i-1}) + \psi^{-1}(u_{i+1}) + \ldots + \psi^{-1}(u_d)}{C(\boldsymbol{u})} \\ &= \frac{\psi\left([-\log(G_i(x_i))) - \psi^{-1}(u_i)] + \psi^{-1}(u_1) + \ldots + \psi^{-1}(u_d)\right)}{\psi\left(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_d)\right)}. \end{aligned}$$

Recall (see [13]) that if ψ is the Laplace transform of random variable Z, so that $\psi(t) = \mathrm{E}(\exp(-tZ))$, where Z has distribution function F_Z, then ϕ defined as $\phi(t) = \psi(t+c)/\psi(c)$ is the Laplace transform of some random variable Z' with cumulative distribution function $F_{Z'}(t) = \exp(-ct) F_Z(t)$.

Hence, the marginal distribution function of Y_i can be written

$$F_i'(x_i) = \psi_{\boldsymbol{u}}([-\log(G_i(x_i)) - \psi^{-1}(u_i)]),$$

where $\psi_{\boldsymbol{u}}$ is the Laplace transform defined as

$$\psi_{\boldsymbol{u}}(t) = \frac{\psi\left(t + \psi^{-1}(u_1) + \ldots + \psi^{-1}(u_d)\right)}{\psi\left(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_d)\right)} = \frac{\psi(t + \psi^{-1}(C(\boldsymbol{u})))}{C(\boldsymbol{u})}.$$

Set further $G_i'(x_i) = \exp(\log(G_i(x_i)) + \psi(u_i))$ on $(-\infty, F_i{}^{-1}(u_i)]$. One gets easily that G_i' is an increasing function, with $G_i'(x_i) \to 0$ as $x_i \to -\infty$ and $G_i'(F_i{}^{-1}(u_i)) = \exp(0) = 1$. Hence, G_i' is a cumulative distribution function.

As at now, we have that there exists a random variable Θ' with Laplace transform $\psi_{\boldsymbol{u}}$, such that $\Pr(Y_i \leq x_i | \Theta') = G_i(x_i)^{\Theta'}$ for all $i \in \{1, \ldots, d\}$. Let us prove that given Θ', the components of $\boldsymbol{Y}$ are independent.

On the one hand, we have obtained that the joint distribution function of $\boldsymbol{Y}$ is

$$F'(\boldsymbol{x}) = \frac{\psi(-\log G_1(x_1) - \ldots . - \log G_d(x_d))}{C(\boldsymbol{u})}.$$

From the expression of ψ', note that this expression becomes

$$F'(\boldsymbol{x}) = \psi_{\boldsymbol{u}}(-\log G_1(x_1) - \ldots . - \log G_d(x_d) - \psi^{-1}(u_1) - \ldots - \psi^{-1}(u_d)).$$

On the other hand,

$$\begin{aligned}
& \mathrm{E}\left(\Pr\left(Y_1 \le x_1|\Theta'\right)\cdot\ldots\cdot\Pr\left(Y_d \le x_d|\Theta'\right)\right) \\
=\; & \mathrm{E}\left(G_1'(x_1)^{\Theta'}\cdot\ldots\cdot G_d'(x_d)^{\Theta'}\right) \\
=\; & \mathrm{E}\left(\exp[-\Theta'(-\log G_1'(x_1))]\cdot\ldots\cdot\exp[-\Theta'(-\log G_d'(x_d))]\right) \\
=\; & \mathrm{E}\left(\exp[-\Theta'(-[\log G_1(x_1)+\psi^{-1}(u_1)])]\cdot\ldots\cdot\exp[-\Theta'(-[\log G_d(x_d)+\psi^{-1}(u_d)])]\right) \\
=\; & \psi_{\boldsymbol{u}}\left(-\log(G_1(x_1))-\psi^{-1}(u_1)-\ldots-\log(G_d(x_d))-\psi^{-1}(u_d)\right),
\end{aligned}$$

and therefore, one gets that

$$\begin{aligned}
\mathrm{E}\left(\Pr\left(Y_1 \le x_1|\Theta'\right)\cdot\ldots\cdot\Pr\left(Y_d \le x_d|\Theta'\right)\right) &= F'(\boldsymbol{x}) \\
&= \mathrm{E}\left(\Pr\left(Y_1 \le x_1,\ldots,Y_d \le x_d|\Theta'\right)\right),
\end{aligned}$$

i. e. given Θ', the components of $\boldsymbol{Y}$ are independent.

In order to conclude, let us just observe that Θ' is a positive random variable, since

$$\Pr(\Theta' < 0) = \lim_{t\to\infty}\psi_{\boldsymbol{u}}(t) = \lim_{t\to\infty}\psi(t) = 0,$$

since Θ is a positive variable. Finally, the conditional vector $\boldsymbol{X}$ given $\boldsymbol{X} \le F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$ will be said to have an Archimedean copula with a factor representation. This finishes the proof of Theorem 2.4. □

Example 2.5. If Θ has a Gamma distribution, with shape parameter $\kappa > 0$ and scale parameter $\alpha > 0$, then its Laplace transform is $(1-\alpha\, t)^{-\kappa}$ for $t < 1/\alpha$. Thus,

$$\begin{aligned}
\psi_{\boldsymbol{u}}(t) &= \frac{\psi\left(t+\psi^{-1}\left(C(\boldsymbol{u})\right)\right)}{C(\boldsymbol{u})} = \frac{(1-\alpha(t-[C(\boldsymbol{u})^{\kappa}-1]/\alpha))^{-\kappa}}{C(\boldsymbol{u})} \\
&= \frac{(C(\boldsymbol{u})^{\kappa}-\alpha t)^{-\kappa}}{C(\boldsymbol{u})} = \left(1-[\alpha C(\boldsymbol{u})^{-\kappa}]t\right)^{-\kappa}
\end{aligned}$$

which is still a Gamma distribution with the same shape parameter κ.

3. $\mathcal{H}$-COPULAS AND LATENT FACTOR MODELS

We have introduced earlier the class of $\mathcal{H}_d$-copulas, defined as

$$\mathcal{D}_h(C)(u_1,\ldots,u_d) = h^{-1}(C(h(u_1),\ldots,h(u_d))), \quad 0 \le u_i \le 1,$$

where C is a copula, and $h \in \mathcal{H}_d$ is a d-distortion function. As noticed earlier, copulas $\mathcal{D}_h(C^{\perp})$ are Archimedean copulas. An idea can be to focus on the factor interpretation of Archimedean copulas, and to extend it in the non-independent case.

Assume that there exists a positive random variable Θ, such that, conditionally on Θ, random vector $\boldsymbol{X} = (X_1,\ldots,X_d)$ has copula C, which does not depend on Θ. Assume moreover that C is in extreme value copula, or max-stable copula (see e. g. [19]): $C\left(x_1^h,\ldots,x_d^h\right) = C^h\left(x_1,\ldots,x_d\right)$ for all $h \ge 0$. The following result holds,

Lemma 3.1. Let Θ be a random variable with Laplace transform ψ, and consider a random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ such that $\boldsymbol{X}$ given Θ has copula C, an extreme value copula. Assume that, for all $i = 1, \ldots, d$, $\Pr(X_i \le x_i|\Theta) = G_i(x_i)^\Theta$ where the G_i's are distribution functions. Then $\boldsymbol{X}$ has copula

$$C_{\boldsymbol{X}}(x_1, \ldots, x_d) = \psi\left(-\log\left(C\left(\exp\left[-\psi^{-1}(x_1)\right], \ldots, \exp\left[-\psi^{-1}(x_d)\right]\right)\right)\right),$$

whose copula is of the form $\mathcal{D}_h(C)$ with $h(\cdot) = \exp\left[-\psi^{-1}(\cdot)\right]$.

Proof. Let $\boldsymbol{X}$ be a random vector such that $\boldsymbol{X}$ given Θ has copula C and $\Pr(X_i \le x_i|\Theta) = G_i(x_i)^\Theta$, $i = 1, \ldots, d$. Then, the (unconditional) joint distribution function of $\boldsymbol{X}$ is given by

$$\begin{aligned} F(\boldsymbol{x}) &= \mathrm{E}\left(\Pr(X_1 \le x_1, \ldots, X_d \le x_d|\Theta)\right) = \mathrm{E}\left(C\left(\Pr(X_1 \le x_i|\Theta), \ldots, \Pr(X_d \le x_d|\Theta)\right)\right) \\ &= \mathrm{E}\left(C\left(G_1(x_1)^\Theta, \ldots, G_d(x_d)^\Theta\right)\right) = \mathrm{E}\left(C^\Theta\left(G_1(x_1), \ldots, G_d(x_d)\right)\right) \\ &= \psi\left(-\log C\left(G_1(x_1), \ldots, G_d(x_d)\right)\right), \end{aligned}$$

where ψ is the Laplace transform of the distribution of Θ, i. e. $\psi(t) = \mathrm{E}(\exp(-t\Theta))$. Because C is an extreme value copula,

$$C\left(G_1(x_1)^\Theta, \ldots, G_d(x_d)^\Theta\right) = C^\Theta\left(G_1(x_1), \ldots, G_d(x_d)\right).$$

One gets finally that the unconditional marginal distribution functions are $F_i(x_i) = \psi(-\log G_i(x_i))$, and therefore

$$C_{\boldsymbol{X}}(x_1, \ldots, x_d) = \psi\left(-\log\left(C\left(\exp\left[-\psi^{-1}(x)\right], \exp\left[-\psi^{-1}(y)\right]\right)\right)\right).$$

Note that since ψ^{-1} is completely monotone, then h belongs to $\mathcal{H}_d$. This finishes the proof of Lemma 3.1. □

We will see with the Theorem below that, in the case where the copula of $\boldsymbol{X}$ is an $\mathcal{H}_d$-copula, the stability of exchangeable Archimedean copulas with a factor representation can be extended to $\mathcal{H}_d$-copula, with additional assumptions.

Theorem 3.2. Let $\boldsymbol{X}$ be a random vector with an $\mathcal{H}_d$-copula with a factor representation, let ψ denote the Laplace transform of the heterogeneity factor Θ, C denote the underlying copula, and G_i's the marginal distributions.

(1) Let $\boldsymbol{u} \in (0,1]^d$, then, the copula of $\boldsymbol{X}$ given $\boldsymbol{X} \le F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$ is

$$C_{\boldsymbol{X},\boldsymbol{u}}(\boldsymbol{x}) = \psi_{\boldsymbol{u}}\left(-\log\left(C_{\boldsymbol{u}}\left(\exp\left[-\psi_{\boldsymbol{u}}{}^{-1}(x_1)\right], \ldots, \exp\left[-\psi_{\boldsymbol{u}}{}^{-1}(x_d)\right]\right)\right)\right) = \mathcal{D}_{h_{\boldsymbol{u}}}(C_{\boldsymbol{u}})(\boldsymbol{x}),$$

where $h_{\boldsymbol{u}}(\cdot) = \exp\left[-\psi_{\boldsymbol{u}}{}^{-1}(\cdot)\right]$, and where

- $\psi_{\boldsymbol{u}}$ is the Laplace transform defined as $\psi_{\boldsymbol{u}}(t) = \psi(t+\alpha)/\psi(\alpha)$ where $\alpha = -\log(C(\boldsymbol{u}^*))$, $\boldsymbol{u}_i^* = \exp\left[-\psi^{-1}(u_i)\right]$ for all $i = 1, \ldots, d$. Hence, $\psi_{\boldsymbol{u}}$ is the Laplace transform of Θ given $\boldsymbol{X} \le F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$,

- $\Pr\left(X_i \leq x_i | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u}), \Theta\right) = G_i'(x_i)^{\Theta}$ for all $i = 1, \ldots, d$, where

$$G_i'(x_i) = \frac{C(u_1^*, u_2^*, \ldots, G_i(x_i), \ldots, u_d^*)}{C(u_1^*, u_2^*, \ldots, u_i^*, \ldots, u_d^*)},$$

- and $C_{\boldsymbol{u}}$ is the following copula

$$C_{\boldsymbol{u}}(\boldsymbol{x}) = \frac{C\left(G_1\left(G_1'^{-1}(x_1)\right), \ldots, G_d\left(G_d'^{-1}(x_d)\right)\right)}{C\left(G_1\left(F_1{}^{-1}(u_1)\right), \ldots, G_d\left(F_d{}^{-1}(u_d)\right)\right)}.$$

(2) Furthermore, the copula of $\boldsymbol{X}$ given $\boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$ is an $\mathcal{H}_d$-copula with a factor representation if and only if $C_{\boldsymbol{u}}$ is an extreme value copula.

P r o o f. (1) Let $C_{\boldsymbol{X}}$ be the copula of $\boldsymbol{X}$, that is

$$C_{\boldsymbol{X}}(u_1, \ldots, u_d) = \psi\left(-\log\left(C\left(\exp\left[-\psi^{-1}(u_1)\right], \ldots, \exp\left[-\psi^{-1}(u_d)\right]\right)\right)\right).$$

(i) The marginal distribution of X_i given $\boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$, and given $\Theta = \theta$ is

$$\begin{aligned} &\Pr\left(X_i \leq x_i | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u}), \Theta = \theta\right) \\ = &\frac{\Pr\left(X_1 \leq F_1{}^{-1}(u_1), \ldots, X_i \leq x_i, X_{i+1} \leq F_1{}^{-1}(u_{i+1}), \ldots, X_d \leq F_d{}^{-1}(u_d) | \Theta = \theta\right)}{\Pr\left(X_1 \leq F_1{}^{-1}(u_1), \ldots, X_i \leq F_i{}^{-1}(u_i), X_{i+1} \leq F_1{}^{-1}(u_{i+1}), \ldots, X_d \leq F_d{}^{-1}(u_d) | \Theta = \theta\right)} \\ = &\frac{C\left(\Pr\left(X_1 \leq F_1{}^{-1}(u_1) | \Theta = \theta\right), \ldots, \Pr\left(X_i \leq x_i | \Theta = \theta\right), \ldots, \Pr\left(X_d \leq F_d{}^{-1}(u_d) | \Theta = \theta\right)\right)}{C\left(\Pr\left(X_1 \leq F_1{}^{-1}(u_1) | \Theta = \theta\right), \ldots, \Pr\left(X_i \leq F_i{}^{-1}(u_i) | \Theta = \theta\right), \ldots, \Pr\left(X_d \leq F_d{}^{-1}(u_d) | \Theta = \theta\right)\right)} \end{aligned}$$

since C is the copula of $\boldsymbol{X}$ given Θ, i. e.

$$\Pr\left(X_1 \leq x_1, \ldots, X_d \leq x_d | \Theta = \theta\right) = C\left(\Pr\left(X_1 \leq x_1 | \Theta = \theta\right), \ldots, \Pr\left(X_d \leq x_d\right) | \Theta = \theta\right).$$

Hence,

$$\begin{aligned} &\Pr\left(X_i \leq x_i | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u}), \Theta = \theta\right) \\ = &\frac{C(G_1(F_1{}^{-1}(u_1))^{\theta}, \ldots, G_i(x_i)^{\theta}, \ldots, G_d(F_d{}^{-1}(u_d))^{\theta})}{C(G_1(F_1{}^{-1}(u_1))^{\theta}, \ldots, G_i(F_i{}^{-1}(u_i))^{\theta}, \ldots, G_d(F_d{}^{-1}(u_d))^{\theta})} \end{aligned}$$

because C is an extreme value copula. Since $F_j(x_j) = \psi(-\log G_j(x_j))$, set $u_j^* = G_j\left(F_j{}^{-1}(u_j)\right) = \exp\left[-\psi^{-1}(u_j)\right]$ for all $j = 1, \ldots, d$. The marginal distribution satisfies

$$\Pr\left(X_i \leq x_i | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u}), \Theta = \theta\right) = \left(\frac{C(u_1^*, \ldots, u_{i-1}^*, G_i(x_i), u_{i+1}^*, \ldots, u_d^*)}{C(u_1^*, \ldots, u_{i-1}^*, u_i^*, u_{i+1}^*, \ldots, u_d^*)}\right)^{\theta}.$$

One can get easily that

$$G_i^*(x_i) = \frac{C(u_1^*, \ldots, u_{i-1}^*, G_i(x_i), u_{i+1}^*, \ldots, u_d^*)}{C(u_1^*, \ldots, u_{i-1}^*, u_i^*, u_{i+1}^*, \ldots, u_d^*)}$$

is (univariate) distribution function, since C and G_i are both increasing, and moreover $G_i^*\left(F_i{}^{-1}(u_i)\right) = u_i^*$.

(ii) The joint distribution function of $\boldsymbol{X}$ given $\boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$ is

$$\begin{aligned}\Pr\left(\boldsymbol{X} \leq \boldsymbol{x} | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})\right) &= \frac{\Pr(\boldsymbol{X} \leq \boldsymbol{x})}{\Pr\left(\boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})\right)} = \frac{\mathrm{E}\left(\Pr(\boldsymbol{X} \leq \boldsymbol{x}|\Theta)\right)}{C(\boldsymbol{u})} \\ &= \frac{\mathrm{E}\left(C\left(G_1(x_1)^{\Theta}, \dots, G_d(x_d)^{\Theta}\right)\right)}{C(\boldsymbol{u})} \\ &= \frac{\mathrm{E}\left(C\left(G_1(x_1), \dots, G_d(x_d)\right)\right)^{\Theta}}{C(\boldsymbol{u})}\end{aligned}$$

From the expression of copula $C_{\boldsymbol{X}}$,

$$\begin{aligned}C_{\boldsymbol{X}}(\boldsymbol{u}) &= \psi\left(-\log\left(C\left(\exp\left[-\psi^{-1}(u_1)\right], \dots, \exp\left[-\psi^{-1}(u_d)\right]\right)\right)\right) \\ &= \psi\left(-\log\left(C\left(u_1^*, \dots, u_d^*\right)\right)\right),\end{aligned}$$

one gets

$$\begin{aligned}\Pr\left(\boldsymbol{X} \leq \boldsymbol{x} | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})\right) &= \frac{\psi(-\log C(G_1(x_1), \dots, G_d(x_d)))}{\psi(-\log C(u_1^*, \dots, u_d^*))} \\ &= \frac{\psi[-\log C(u_1^*, \dots, u_d^*) - \alpha] + \alpha}{\psi(\alpha)}\end{aligned}$$

where $\alpha = -\log\left(C\left(u_1^*, \dots, u_d^*\right)\right)$. Set $\psi_{\boldsymbol{u}}(t) = \psi(t+\alpha)/\psi(\alpha)$. From this expression, $\psi_{\boldsymbol{u}}$ is also a Laplace transform. Furthermore, the expression above could be written

$$\Pr\left(\boldsymbol{X} \leq \boldsymbol{x} | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})\right) = \psi_{\boldsymbol{u}}\left(-\log \frac{C\left(G_1(x_1), \dots, G_d(x_d)\right)}{C\left(u_1^*, \dots, u_d^*\right)}\right).$$

We can then write the conditional marginal distribution function as

$$\begin{aligned}\Pr\left(X_i \leq x_i | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})\right) &= \psi_{\boldsymbol{u}}\left(-\log \frac{C\left(u_1^*, \dots, G_i(x_i), \dots, u_d^*\right)}{C\left(u_1^*, \dots, u_d^*\right)}\right) \\ &= \psi_{\boldsymbol{u}}(-\log G_i^*(x_i)),\end{aligned}$$

i. e.,

$$\Pr\left(X_i \leq x_i | \boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})\right) = \mathrm{E}\left(G_i^*(x_i)^{\Theta}\right),$$

where Θ has Laplace transform $\psi_{\boldsymbol{u}}$.

(iii) Note that $h_{\boldsymbol{u}}(\cdot) = \exp\left[-\psi_{\boldsymbol{u}}{}^{-1}(\cdot)\right]$ also belongs to $\mathcal{H}_d$ since $\psi_{\boldsymbol{u}}$ is completely monotone.

(iv) Let $C_{\boldsymbol{u}}$ be the functional defined on $[0,1]^d$ by

$$C_{\boldsymbol{u}}(x_1, \dots, x_d) = \frac{C\left(G_1\left(G_1^{*-1}(x_1)\right), \dots, G_d\left(G_d^{*-1}(x_d)\right)\right)}{C\left(G_1\left(F_1{}^{-1}(u_1)\right), \dots, G_d\left(F_d{}^{-1}(u_d)\right)\right)}.$$

Because C is d-increasing (C is a copula) and the G_i's are increasing, $C_{\boldsymbol{u}}$ is d-increasing. Furthermore,

$$\begin{aligned} & C_{\boldsymbol{u}}\left(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_d\right) \\ = & \frac{C\left(G_1\left(G_1^{*-1}\left(x_0\right)\right),\ldots,G_i\left(G_i^{*-1}\left(0\right)\right),\ldots,G_d\left(G_d^{*-1}\left(x_d\right)\right)\right)}{C\left(G_1\left(F_1{}^{-1}\left(u_1\right)\right),\ldots,G_d\left(F_d{}^{-1}\left(u_d\right)\right)\right)} = 0, \end{aligned}$$

and

$$\begin{aligned} & C_{\boldsymbol{u}}\left(1,\ldots,1,x_i,1,\ldots,1\right) \\ = & \frac{C\left(G_1\left(G_1^{*-1}\left(1\right)\right),\ldots,G_i\left(G_i^{*-1}\left(x_i\right)\right),\ldots,G_d\left(G_d^{*-1}\left(1\right)\right)\right)}{C\left(G_1\left(F_1{}^{-1}\left(u_1\right)\right),\ldots,G_d\left(F_d{}^{-1}\left(u_d\right)\right)\right)} \\ = & \frac{C\left(u_1^*,\ldots,u_{i-1}^*,G_i\left(G_i^{*-1}\left(x_i\right)\right),u_{i+1}^*,\ldots,u_d^*\right)}{C\left(G_1\left(F_1{}^{-1}\left(u_1\right)\right),\ldots,G_d\left(F_d{}^{-1}\left(u_d\right)\right)\right)}. \end{aligned}$$

Thus, $C_{\boldsymbol{u}}\left(1,\ldots,1,G_i^*\left(x_i\right),1,\ldots,1\right) = G_i^*\left(x_i\right)$, that is, since G_i^* is bijective on $[0,1]$, for all z_i in $[0,1]$, $C_{\boldsymbol{u}}\left(1,\ldots,1,z_i,1,\ldots,1\right) = z_i$. So, finally, $C_{\boldsymbol{u}}$ is a copula.

(v) Using the results obtained above, one gets that the copula of $\boldsymbol{X}$ given $\boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u})$ is $C_{\boldsymbol{X},\boldsymbol{u}}$ defined as

$$\begin{aligned} & C_{\boldsymbol{X},\boldsymbol{u}}(x_1,\ldots,x_d) \\ = & \psi_{\boldsymbol{u}}\left(-\log\left(C_{\boldsymbol{u}}\left(\exp\left[-\psi_{\boldsymbol{u}}^{-1}\left(x_1\right)\right],\exp\left[-\psi_{\boldsymbol{u}}{}^{-1}\left(x_d\right)\right]\right)\right)\right) = \mathcal{D}_{h_{\boldsymbol{u}}}(C_{\boldsymbol{u}})(x_1,\ldots,x_d). \end{aligned}$$

which is the analogous of the result of Proposition (3.1).

(2) Assume that $\boldsymbol{X} = (X_1,\ldots,X_d)$ has an $\mathcal{H}_d$-copula. Using the notions of the beginning of the prof, let $C_{\boldsymbol{u}}$ denote the copula of $\boldsymbol{X}$ given $\boldsymbol{X} \leq F_{\boldsymbol{X}}{}^{-1}\left(\boldsymbol{u}\right))$ and given Θ. Then, for all $\theta \geq 0$

$$\begin{aligned} C_{\boldsymbol{u}}\left(\boldsymbol{x}\right)^\theta &= \frac{C\left(G_1\left(G_1^{*-1}\left(x_1\right)\right),\ldots,G_d\left(G_d^{*-1}\left(x_d\right)\right)\right)^\theta}{C\left(G_1\left(F_1{}^{-1}\left(u_1\right)\right),\ldots,G_d\left(F_d{}^{-1}\left(u_d\right)\right)\right)^\theta} \\ &= \frac{C\left(G_1\left(G_1^{*-1}\left(x_1\right)\right)^\theta,\ldots,G_d\left(G_d^{*-1}\left(x_d\right)\right)^\theta\right)}{C\left(G_1\left(F_1{}^{-1}\left(u_1\right)\right)^\theta,\ldots,G_d\left(F_d{}^{-1}\left(u_d\right)\right)^\theta\right)} \\ &= \frac{C\left(\Pr\left(X_1 \leq G_1^{*-1}\left(x_1\right)|\Theta=\theta\right),\ldots,\Pr\left(X_d \leq G_d^{*-1}\left(x_d\right)|\Theta=\theta\right)\right)}{C\left(\Pr\left(X_1 \leq F_1{}^{-1}\left(u_1\right)|\Theta=\theta\right),\ldots,\Pr\left(X_d \leq F_d{}^{-1}\left(u_d\right)|\Theta=\theta\right)\right)} \\ &= \frac{\Pr\left(X_1 \leq G_1^{*-1}\left(x_1\right),\ldots,X_d \leq G_d^{*-1}\left(x_d\right)|\Theta=\theta\right)}{C\left(u_1^*,\ldots,u_d^*\right)}. \end{aligned}$$

Note that the numerator could be written

$$\begin{aligned} & \Pr\left(\boldsymbol{X} \leq G^{*-1}\left(\boldsymbol{x}\right)|\Theta=\theta\right) \\ = & \Pr\left(\boldsymbol{X} \leq G^{*-1}\left(\boldsymbol{x}\right)|\boldsymbol{X} \leq F^{-1}\left(\boldsymbol{u}\right),\Theta=\theta\right)\cdot\Pr\left(\boldsymbol{X} \leq F^{-1}\left(\boldsymbol{u}\right)|\Theta=\theta\right) \\ = & \Pr\left(\boldsymbol{X} \leq G^{*-1}\left(\boldsymbol{x}\right)|\boldsymbol{X} \leq F^{-1}\left(\boldsymbol{u}\right),\Theta=\theta\right)\cdot C(\boldsymbol{u}^*), \end{aligned}$$

and therefore

$$C_{\boldsymbol{u}}(\boldsymbol{x})^{\theta} = \Pr\left(\boldsymbol{X} \le G^{*-1}(\boldsymbol{x}) \,|\, \boldsymbol{X} \le F^{-1}(\boldsymbol{u}), \Theta = \theta\right).$$

From this expression, using the fact that $C_{\boldsymbol{u}}$ is the copula of $\boldsymbol{X} \le G^{*-1}(\boldsymbol{x})$ and $\boldsymbol{X} \le F^{-1}(\boldsymbol{u})$ and $\Theta = \theta$, we get

$$\begin{aligned}
&\Pr\left(\boldsymbol{X} \le G^{*-1}(\boldsymbol{x}) \,|\, \boldsymbol{X} \le F^{-1}(\boldsymbol{u}), \Theta = \theta\right) \\
&= C_{\boldsymbol{u}}\left(\Pr\left(X_1 \le G_1^{*-1}(x_1) \,|\, \boldsymbol{X} \le F^{-1}(\boldsymbol{u}), \Theta=\theta\right), \dots\right. \\
&\qquad \left.\dots, \Pr\left(X_d \le G_d^{*-1}(x_d) \,|\, \boldsymbol{X} \le F^{-1}(\boldsymbol{u}), \Theta=\theta\right)\right) \\
&= C_{\boldsymbol{u}}\left(\Pr\left(X_1 \le G_1^{*-1}(x_1) \,|\, \boldsymbol{X} \le F^{-1}(\boldsymbol{u})\right)^{\theta}, \dots, \Pr\left(X_d \le G_d^{*-1}(x_d) \,|\, \boldsymbol{X} \le F^{-1}(\boldsymbol{u})\right)^{\theta}\right) \\
&= C_{\boldsymbol{u}}(x_1^{\theta}, \dots, x_d^{\theta}).
\end{aligned}$$

Hence, for all $\theta \ge 0$, $C_{\boldsymbol{u}}(\boldsymbol{x})^{\theta} = C_{\boldsymbol{u}}(\boldsymbol{x}^{\theta})$ and therefore, $C_{\boldsymbol{u}}$ is an extreme value copula.

Conversely, assume that $C_{\boldsymbol{u}}$ is an extreme value copula. The conditional joint distribution of $\boldsymbol{X}$ given $\boldsymbol{X} \le F^{-1}(\boldsymbol{u})$, and $\Theta = \theta$ is

$$\begin{aligned}
&\Pr\left(\boldsymbol{X} \le \boldsymbol{x} | \boldsymbol{X} \le F^{-1}(\boldsymbol{u}), \Theta = \theta\right) &(2)\\
&= \frac{\Pr(\boldsymbol{X} \le \boldsymbol{x} | \Theta\theta)}{\Pr(\boldsymbol{X} \le F^{-1}(\boldsymbol{u}), \Theta = \theta)} \\
&= \frac{C(\Pr(X_1 \le x_1 | \Theta = \theta), \dots, \Pr(X_d \le x_d | \Theta = \theta))}{C(\Pr(X_1 \le F_1{}^{-1}(u_1) | \Theta = \theta), \dots, \Pr(X_d \le F_d{}^{-1}(u_d) | \Theta = \theta))} \\
&= \frac{C\left(G_1(x_1)^{\theta}, \dots, G_d(x_d)^{\theta}\right)}{C\left(G_1(F_1{}^{-1}(u_1))^{\theta}, \dots, G_d(F_d{}^{-1}(u_d))^{\theta}\right)} \\
&= \left[\frac{C(G_1(x_1), \dots, G_d(x_d))}{C(G_1(F_1{}^{-1}(u_1)), \dots, G_d(F_d{}^{-1}(u_d)))}\right]^{\theta} \\
&= C_{\boldsymbol{u}}(G_1^*(x_1), \dots, G_d^*(x_d))^{\theta} = C^*\left(G_1^*(x_1)^{\theta}, \dots, G_d^*(x_d)^{\theta}\right) &(3)\\
&= C_{\boldsymbol{u}}\left(\Pr(X_1 \le x_1 | \boldsymbol{X} \le F^{-1}(\boldsymbol{u}), \Theta=\theta), \dots, \Pr(X_d \le x_d | \boldsymbol{X} \le F^{-1}(\boldsymbol{u}), \Theta=\theta)\right), &(4)
\end{aligned}$$

because $C_{\boldsymbol{u}}$ is an extreme value copula. So finally, $C_{\boldsymbol{u}}$ is the copula of $\boldsymbol{X}$ given $\boldsymbol{X} \le F_{\boldsymbol{X}}{}^{-1}(\boldsymbol{u}))$ and given Θ. This finishes the proof of Theorem 3.2. □

4. COMPARING TAILS OF ARCHIMEDEAN COPULAS

This idea of comparing dependence structure as time elapses can be found in [8]. Here, a characterization based on the Archimedean generator is given.

From Theorem 3.2, one can notice that the generator of the conditional copula is the same on a given level curve of the copula C : if $C(\boldsymbol{u}_1) = C(\boldsymbol{u}_2)$, then $C_{\boldsymbol{u}_1} = C_{\boldsymbol{u}_2}$. Since C is continuous, for all $\boldsymbol{u} \in (0,1]^d$, there is $t \in (0,1]$ such that $C(\boldsymbol{u}) = C(t \cdot \boldsymbol{1})$. Hence, for convenience, instead of comparing $C_{\boldsymbol{u}_1} = C_{\boldsymbol{u}_2}$, we simply have to compare C_{t_1} and C_{t_2} (for appropriate t_i's such that $C(\boldsymbol{u}_i) = C(t_i \cdot \boldsymbol{1})$).

When studying the evolution of the conditional copula on the diagonal, one can expect a dependence structure which is all the more positively dependent as t decreases, or similarly, all the less dependent. In the first case, if $0 < t_2 \leq t_1 \leq 1$, $C_{t_1} \preceq C_{t_2}$, in the sense that $C_{t_1}(\boldsymbol{u}) \leq C_{t_2}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $(0,1]^d$, which is the lower orthant-ordering (see [24]).

[16] proved a so-called Cooper's Theorem in dimension 2, stating that if $\phi_1 \circ \phi_2^{-1}$ is subadditive, then $C_1(u,v) \leq C_2(u,v)$ where C_i is the Archimedean copula induced by ϕ_i. Recall that function f is subadditive if and only if $f(x+y) \leq f(x)+f(y)$ for all x, y. Actually, this result also holds in higher dimension, since f is subadditive if and only if

$$f(x_1 + \ldots + x_d) \leq f(x_1) + \ldots + f(x_d), \text{ for all } x_1, \ldots, x_d.$$

A sufficient condition for this result to hold is when ϕ_1/ϕ_2 is increasing (from [9] or [10]), and no condition on the dimension are necessary here.

Proposition 4.1. Let t_1 and t_2 such that $0 < t_2 \leq t_1 \leq 1$, and let C be an Archimedean copula with generator ϕ. Let

$$f_{12}(x) = \phi\left(\frac{C(t_1 \cdot \mathbf{1})}{C(t_2 \cdot \mathbf{1})}\phi^{-1}\left(x + \phi\left(C(t_2 \cdot \mathbf{1})\right)\right)\right) - \phi\left(C(t_1 \cdot \mathbf{1})\right)$$

$$f_{21}(x) = \phi\left(\frac{C(t_2 \cdot \mathbf{1})}{C(t_1 \cdot \mathbf{1})}\phi^{-1}\left(x + \phi\left(C(t_1 \cdot \mathbf{1})\right)\right)\right) - \phi\left(C(t_2 \cdot \mathbf{1})\right),$$

Then

- $C_{t_2}(\boldsymbol{u}) \leq C_{t_1}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $[0,1]^d$ if and only if $f_{21}(x)$is sudadditive,
- $C_{t_2}(\boldsymbol{u}) \geq C_{t_1}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $[0,1]^d$ if and only if $f_{12}(x)$is sudadditive.

P r o o f. As shown in Theorem 3.1 in [16], if C_1 and C_2 are two Archimedean copulas with generator ϕ_1 and ϕ_2, then $C_2 \preceq C_1$ if and only if $\phi_2 \circ {\phi_1}^{-1}$ is subadditive, that is

$$\phi_2 \circ {\phi_1}^{-1}(x+y) \leq \phi_2 \circ {\phi_1}^{-1}(x) + \phi_2 \circ {\phi_1}^{-1}(y) \text{ for all } x, y \geq 0$$

In the case of conditional copulas, $\phi_2(x) = \phi(C(t_2 \cdot \mathbf{1})x) - \phi(C(t_2 \cdot \mathbf{1}))$ and $\phi_1(x) = \phi(C(t_1 \cdot \mathbf{1})x) - \phi(C(t_1 \cdot \mathbf{1}))$, and so, $C_{t_2} \preceq C_{t_1}$ if and only if $f_{21}(x)$is sudadditive, where

$$f_{21}(x) = \phi\left(\frac{C(t_2 \cdot \mathbf{1})}{C(t_1 \cdot \mathbf{1})}\phi^{-1}\left(x + \phi\left(C(t_1 \cdot \mathbf{1})\right)\right)\right) - \phi\left(C(t_2 \cdot \mathbf{1})\right).$$

One gets analogous results for f_{12}.

This finishes the proof of Proposition 4.1. □

Example 4.2. The case of Clayton copulas could be seen as a limiting case, in the sense that $\phi(t) = t^{-\theta} - 1$ and so, f_{12} is linear, i. e.

$$f_{12}(x) = ax + b \text{ where } a = C(t_1 \cdot \mathbf{1})^{\theta} / C(t_2 \cdot \mathbf{1})^{\theta}.$$

We obtain here the particular case mentioned in Lemma 5.5.8. in [30].

In the case were ϕ is twice differentiable, a sufficient condition for uniform ordering of conditional copula is the following.

Proposition 4.3. If ϕ is twice differentiable, set $\psi(x) = \log -\phi''(x)$,

(i) If ψ is concave on $]0,1]$, then $C_{t_1}(\boldsymbol{u}) \leq C_{t_2}(\boldsymbol{u})$ in $[0,1]^d$, for all $0 < t_2 \leq t_1 \leq 1$.

(ii) Similarly, if $\psi(x)$ is convex on $]0,1]$, then $C_{t_2}(\boldsymbol{u}) \geq C_{t_1}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $[0,1]^d$, for all $0 < t_2 \leq t_1 \leq 1$.

Proof. (i) Let $0 \leq t_2 \leq t_1 \leq 1$, and $\beta = C(t_2 \cdot \mathbf{1})$, $\gamma = C(t_1 \cdot \mathbf{1})$ and $\alpha = \gamma/\beta, \alpha \leq 1$. Let $f(x) = \phi\left(\alpha\phi^{-1}(x + \phi(\beta))\right) - \phi(\gamma)$, then

$$f'(x) = \frac{\alpha}{\phi'\left(\phi^{-1}(x+\phi(\beta))\right)} \phi'\left(\alpha\phi^{-1}(x + \phi(\beta))\right)$$

$$f''(x) = \alpha \frac{\alpha\phi''\left(\alpha\phi^{-1}(x+\phi(\beta))\right)\phi'\left(\phi^{-1}(x+\phi(\beta))\right) - \phi'\left(\alpha\phi^{-1}(x+\phi(\beta))\right)\phi''\left(\phi^{-1}(x+\phi(\beta))\right)}{\phi'\left(\phi^{-1}(x+\phi(\beta))\right)^3}$$

Because ϕ is a generator of an Archimedean copula, ϕ is positive, and ϕ' is negative. So, finally, $f''_{12}(x)$ is negative if and only if
$\alpha\phi''\left(\alpha\phi^{-1}(x + \phi(\beta))\right) \cdot \phi'\left(\phi^{-1}(x + \phi(\beta))\right) - \phi'\left(\alpha\phi^{-1}(x + \phi(\beta))\right) \cdot \phi''\left(\phi^{-1}(x + \phi(\beta))\right) \geq 0$
for all x, that is $\alpha\phi''(\alpha y) \cdot \phi'(y) - \phi'(\alpha y) \cdot \phi''(y) \geq 0$ for all y, or, dividing by $\phi'(y) \cdot \phi'(\alpha y)$,

$$\frac{\alpha\phi''(\alpha y)}{\phi'(\alpha y)} - \frac{\phi''(y)}{\phi'(y)} \geq 0 \text{ or } \frac{-\alpha\phi''(\alpha y)}{-\phi'(\alpha y)} \geq \frac{-\phi''(y)}{-\phi'(y)} \text{ for all } y,\ \alpha \leq 1.$$

Because $\alpha\phi''(\alpha y) = (\phi'(\alpha y))'$ and $\phi''(y) = (\phi'(y))'$, let $g(t) = D\log -D\phi(t) = D\psi(t)$, then $f''_{12}(x)$ is negative if and only if $g(\alpha y) \geq g(y)$ for all y and $\alpha \leq 1$, that is g is decreasing, or ψ is concave. In this case, f is concave, and, furthermore, $f(0) = 0$. From Lemma 4.4.3 in Nelsen [28] one gets that f is subadditive.

(ii) Same proof holds: $f''_{21}(x)$ is negative if and only if $g(\alpha y) \geq g(y)$ for all y and $\alpha \geq 1$, that is g is increasing, or ψ is convex.

This finishes the proof of Proposition 4.3. □

Example 4.4. Let C be a Ali–Mikhail–Haq copula (from [1]), with generator $\phi(x) = \log(1 - \theta(1 - x)) - \log x$. Then

$$\phi'(x) = \frac{\theta}{1 - \theta(1 - x)} - \frac{1}{x} \text{ and } \psi(x) = \log\left(\frac{1}{x} - \frac{\theta}{1 - \theta(1 - x)}\right)$$

One gets that

$$\psi''(x) = \frac{-2(1-\theta)}{\phi'(x)^2}\left[\frac{3\theta^2x^2 + 3\theta(1-\theta)x + (1-\theta)^2}{x^3(1-\theta(1-x))^3}\right]$$

which is positive. So finally, ψ is a concave function on $[0,1]$, and so $C_{t_2}(\boldsymbol{u}) \leq C_{t_1}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $[0,1]^d$, for all $0 < t_2 \leq t_1 \leq 1$: $\boldsymbol{X}$ given $X_i \leq F_i(t)$ for all $i = 1, \ldots, d$ is less and less positively dependent, as t decreases toward 0.

Example 4.5. Let C be the copula given by (4.2.19) in Nelsen [28], that is with generator $\phi(x) = \exp(\theta/x) - \exp(\theta)$. Then, for all t_1 and t_2 such that $0 < t_2 \leq t_1 \leq 1$, and let $C_i = \theta/\log[2\exp(\theta/t_i) - \exp(\theta)]$ where $i = 1, 2$. One gets

$$f_{12}(x) = \exp\left(\frac{\log[2\exp(\theta/t_1) - \exp(\theta)]}{\log[2\exp(\theta/t_2) - \exp(\theta)]}\log(x + 2\exp(\theta/t_2) - \exp(\theta))\right) - 2\exp(\theta/t_1) + \exp(\theta)$$

After derivating two times with respect to x, one gets $f''_{12}(x) \geq 0$ and $f_{12}(x)$ is concave. Hence, because $f_{12}(0) = 0$ and $f_{12}(x)$ is convex, then $f_{12}(x)$ is subadditive. $C_{t_2}(\boldsymbol{u}) \geq C_{t_1}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $[0,1]^d$, for all $0 < t_2 \leq t_1 \leq 1$: $\boldsymbol{X}$ given $X_i \leq F_i(t)$ for all $i = 1, \ldots, d$ is more and more positively dependent, as t decreases toward 0.

One can notice that this case is an application of Proposition 4.3:

$$\psi(x) = \log -\phi'(t) = \frac{\theta}{x} + \log\theta - 2\log x$$

is a convex function on $[0,1]$, and so $C_{t_2}(x,y) \geq C_{t_1}(x,y)$ for all x, y in $[0,1] \times [0,1]$, for all $0 < t_2 \leq t_1 \leq 1$.

Example 4.6. Let C be a copula in the Gumbel–Barnett family (cf. [18]), that is $\phi(x) = \log(1 - \theta\log x)$. Then

$$\phi'(x) = \frac{-\theta}{x(1-\theta\log x)} \text{ and } \psi(x) = \log\theta - \log x - \log(1-\theta\log x),$$

which is a convex function on $[0,1]$, and so $C_{t_2}(\boldsymbol{u}) \geq C_{t_1}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $[0,1]^d$, for all $0 < t_2 \leq t_1 \leq 1$. In that case $\boldsymbol{X}$ given $X_i \leq F_i(t)$ for all $i = 1, \ldots, d$ is more and more positively dependent as t decreases toward 0 should be understood as $\boldsymbol{X}$ given $X_i \leq F_i(t)$ for all $i = 1, \ldots, d$ is less and less negatively dependent as t decreases toward 0. This is a direct implication of the fact that the conditional copula of a Gumbel–Barnett copula remains in this family, with a smaller parameter.

Example 4.7. Let C be a Frank copula, with generator

$$\phi(x) = -\log[(\exp(-\theta x) - 1)/(\exp(-\theta) - 1)],$$

then

$$\phi'(t) = \frac{\theta\exp(-\theta x)}{\exp(-\theta x) - 1} \text{ and } \psi(t) = \log\theta - \theta x - \log(1 - \exp(-\theta x)),$$

which satisfies $\psi''(x) = -\theta^2\exp(-\theta x)/[\exp(-\theta x) - 1]^2 \leq 0$: ψ is concave, and so $C_{t_2}(\boldsymbol{u}) \leq C_{t_1}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $[0,1]^d$, for all $0 < t_2 \leq t_1 \leq 1$.

Example 4.8. Let C be a Gumbel copula, with generator $\phi(x) = (-\log x)^\theta$, $\theta \geq 1$, then

$$\phi'(x) = -\theta(-\log x)^{\theta-1}/x, \text{ and } \psi(x) = \log\theta - \log x + (\theta - 1)\log(\log[-x])$$

This function being twice differentiable, one gets

$$\psi''(x) = \frac{(\log x)^2 - [\theta-1]\log x - [\theta-1]}{x^2[\log x]^2} = \frac{h(\log x)}{x^2[\log x]^2},$$

where $h(y) = y^2 - [\theta-1]y - [\theta-1]$: this polynomial has two (real) roots, and one is negative. So finally, $\psi''(x) \leq 0$ on $]0, x_0]$ and $\psi''(x) \geq 0$ on $[x_0, 1]$ for some x_0: ψ is neither concave nor convex.

ACKNOWLEDGEMENTS

The author wants to thank the editor and the two anonymous referees for stimulating comments and remarks, as well as participants of the 2008 International Workshop on Applied Probability.

(Received May 5, 2008.)

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