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ON THE SOLUTION OF THE CONSTRAINED MULTIOBJECTIVE CONTROL PROBLEM WITH THE RECEDING HORIZON APPROACH

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This paper deals with a multiobjective control problem for nonlinear discrete time systems. The problem consists of finding a control strategy which minimizes a number of performance indexes subject to state and control constraints. A solution to this problem through the Receding Horizon approach is proposed. Under standard assumptions, it is shown that the resulting control law guarantees closed-loop stability. The proposed method is also used to provide a robustly stabilizing solution to the problem of simultaneously minimizing a set of H_∞ cost functions for a class of systems subject to bounded disturbances and/or parameter uncertainties. Numeric examples are reported to highlight the stabilizing action of the proposed control laws.

Keywords: multiobjective optimization, receding horizon control, robust control, stability
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1. INTRODUCTION

Many control design problems require to minimize a number of often conflicting performance measures. These are known as multiobjective control problems and have been tackled in different ways in recent years, see e. g. [6, 11] where linear unconstrained systems have been considered. The case of nonlinear systems with constraints has been treated in [4], where a solution based on the closed-loop optimal control of stationary discrete time systems in the infinite horizon multiobjective worst case has been given by taking into account the maximum value of a bounded disturbance term affecting the state dynamics.

In order to deal with constrained nonlinear systems, in this paper a method based on the Receding Horizon (RH) approach is proposed. It is shown that, under quite standard assumptions in nonlinear Model Predictive Control (MPC) based on the RH paradigm, see e. g. [7], the resulting control law guarantees the asymptotic stability of the origin of the state space. The method proposed here is also used to provide a solution to the problem of simultaneously minimizing a set of H_∞ performance indexes for a class of systems subject to bounded disturbances and/or parameter uncertainties.

The paper is organized as follows. In Section 2 the adopted notation and some basic definitions are reported. The multiobjective optimization problem is formulated and solved for nominal unperturbed systems in Section 3, while Section 4 deals with the control problem for systems subject to bounded disturbances. The paper ends with some concluding remarks.

2. NOTATION AND BASIC DEFINITIONS

The Euclidean norm is denoted by $\|\cdot\|$. The symbol $\lambda_i(T)$ denotes the ith eigenvalue of the matrix T, while the maximum eigenvalue of T is $\lambda_M(T)$. A diagonal matrix T with diagonal elements $t_1, t_2, \ldots, t_n$ is written as $\mathrm{diag}\{t_1, t_2, \ldots, t_n\}$. A function $\alpha(\cdot)$: $R_{\geq 0} \to R_{\geq 0}$ is a $\mathcal{K}$ function if it is continuous, positive definite and strictly increasing. A function $\beta(\cdot)$ is a $\mathcal{K}_\infty$ function if it is a $\mathcal{K}$ function and $\beta(s) \to \infty$ as $s \to \infty$.

3. PROBLEM FORMULATION AND SOLUTION

Consider the nonlinear, discrete, time-invariant system described by

$$x(k+1) = f\big(x(k), u(k)\big) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control variable and $f(\cdot,\cdot)$ is a C^1 function of its arguments with $f(0,0) = 0$. The state and control variables are required to fulfill the following constraints

$$x \in X, \quad u \in U \tag{2}$$

where X and U are compact sets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, containing the origin as an interior point.

For system (1) assume to know an "auxiliary" stabilizing control law

$$u = \kappa_f(x) \tag{3}$$

and let $X_f \subseteq X$ be a known positively invariant set for the closed-loop system (1), (3) such that

$$\begin{cases} \kappa_f(x) \in U, & \forall\, x \in X_f \\ f\big(x, \kappa_f(x)\big) \in X_f, & \forall\, x \in X_f. \end{cases} \tag{4}$$

Moreover, at any time instant k, let $u_{[k,k+N_c-1]} = [u(k)\ u(k+1)\ \cdots\ u(k+N_c-1)]$ be the sequence of current and future control moves over the *control horizon* N_c, N_c being a positive integer, and define by the integer N_p, $N_p \geq N_c$, the adopted *prediction horizon*. Then, the multiobjective control problem considered here consists of minimizing, at any time instant k and with respect to $u_{[k,k+N_c-1]}$, the r cost functions

$$J_i\big(x(k), u(\cdot), N_c, N_p\big) = \sum_{j=k}^{k+N_p-1} \varphi_i\big(x(j), u(j)\big) + V_{f_i}\big(x(k+N_p)\big), \quad i = 1, \ldots, r \tag{5}$$

subject to:

(i) the system dynamics (1),

(ii) the state and the control constraints (2),

(iii) the auxiliary control law

$$u(t) = \kappa_f(x(t)), \quad t \in [k + N_c \ldots k + N_p - 1], \tag{6}$$

(iv) the terminal state constraint

$$x(k + N_p) \in X_f. \tag{7}$$

In (5), $\alpha_{\varphi_i}(\|x\|) \le \varphi_i(x, u)$, $i = 1, \ldots, r$, $\alpha_{\varphi_i}(\cdot)$ being $\mathcal{K}_\infty$ functions, while the terminal costs V_{f_i} have to be selected as positive functions, with $V_{f_i}(0) = 0$, satisfying the following conditions

$$\begin{aligned} &V_{f_i}\big(f(x, \kappa_f(x)\big) - V_{f_i}(x) \le\ -\varphi_i\big(x, \kappa_f(x)\big), && \forall\, x \in X_f, \quad i = 1, \ldots, r \\ &V_{f_i}(x) \le \beta_{V_{f_i}}(\|x\|), && \forall\, x \in X_f, \quad i = 1, \ldots, r \end{aligned} \tag{8}$$

where also $\beta_{V_{f_i}}$ are $\mathcal{K}_\infty$ functions.

The problem stated can now be reformulated as follows:

Multiobjective Optimization Control Problem (MOCP)

$$\begin{cases} \min\limits_{u_{[k,k+N_c-1]}} \ \varepsilon \\ \qquad\qquad \left[\begin{array}{c} J_1\big(x(k), u(\cdot), N_c, N_p\big) \le \varepsilon \\ \vdots \\ J_r\big(x(k), u(\cdot), N_c, N_p\big) \le \varepsilon \end{array} \right. \end{cases} \tag{9}$$

subject to (1), (2), (6), (7).

Let $u^o_{[k,k+N_c-1]}$ be the optimal solution of *MOCP* at time k, define the *RH* control law $\kappa^{RH}(x) = u^o_{[k,k]}(x)$, where $u^o_{[k,k]}(x)$ is the first entry of $u^o_{[k,k+N_c-1]}$, and apply at any time instant the *RH* control law

$$u = \kappa^{RH}(x). \tag{10}$$

Denoting by $X^{RH}(N_c, N_p)$ the set of states such that a feasible solution of *MOCP* exists, the following stability result can be stated.

Theorem 1. The origin of the closed-loop system (1), (10) is an asymptotically stable equilibrium point with positively invariant set $X^{RH}(N_c, N_p)$.

Proof. Let $x(k) = \hat{x} \in X^{RH}(N_c, N_p)$ be the current state at the generic time instant k and

$$\xi = f\big(\hat{x}, \kappa^{RH}(\hat{x})\big) \tag{11}$$

its value at the next step due to the first entry of the optimal sequence $u^o_{[k,k+N_c-1]}$ computed at time k. The control sequence $\tilde{u}_{[k,k+N_c]} = \big[u^o_{[k,k+N_c-1]}\ \kappa_f\big(x(k + N_c)\big)\big]$

is a feasible suboptimal sequence for the control problem at k with control horizon N_c+1 and prediction horizon N_p+1.

Now, let $\tilde{u}_{[k+1,k+N_c]}$ be the sequence obtained from $\tilde{u}_{[k,k+N_c]}$ by removing its first entry $\tilde{u}_{[k,k]}$ and define

$$\begin{cases} h = \arg\max_{1\le i\le r} J_i(\hat{x}, u^o_{[k,k+N_c-1]}, N_c, N_p) \\ j = \arg\max_{1\le i\le r} J_i(\xi, \tilde{u}_{[k+1,k+N_c]}, N_c, N_p) \\ V(\hat{x}, N_c, N_p) = J_h(\hat{x}, u^o_{[k,k+N_c-1]}, N_c, N_p). \end{cases} \tag{12}$$

Notice that, for any $x \in X^{RH}(N_c, N_p)$, the first inequality in (8) allows one to obtain the following

$$\begin{aligned} V(\xi, N_c, N_p) &\le J_j(\xi, \tilde{u}_{[k+1,k+N_c]}, N_c, N_p) \\ &= J_j(\hat{x}, u^o_{[k,k+N_c-1]}, N_c, N_p) - \varphi_j\big(\hat{x}, \kappa^{RH}(\hat{x})\big) \\ &\quad + \varphi_j\Big(x(k+N_p), \kappa_f\big(x(k+N_p)\big)\Big) \\ &\quad - V_{f_j}\big(x(k+N_p)\big) + V_{f_j}\big(f(x(k+N_p), \kappa_f(x(k+N_p)))\big) \\ &\le V(\hat{x}, N_c, N_p) - \varphi_j\big(\hat{x}, \kappa^{RH}(\hat{x})\big) \end{aligned} \tag{13}$$

so that, in view of (12) and the hypothesis concerning the $\varphi_i(x,u)$ functions, for any $x \in X^{RH}(N_c, N_p)$,

$$\begin{cases} V(x, N_c, N_p) \ge \min_{1\le i\le r} \varphi_i(x, \kappa^{RH}(x)) \ge \min_{1\le i\le r} \alpha_{\varphi_i}(\|x\|) = \overline{\alpha}_\varphi(\|x\|) \\ \Delta V(x, N_c, N_p) = V(\xi, N_c, N_p) - V(x, N_c, N_p) \le -\overline{\alpha}_\varphi(\|x\|) \end{cases} \tag{14}$$

where $\overline{\alpha}_\varphi(\|x\|)$ is a $\mathcal{K}_\infty$ function. Moreover, letting

$$l = \arg\max_{1\le i\le r} J_i(\hat{x}, \tilde{u}_{[k,k+N_c]}, N_c+1, N_p+1)$$

one has

$$\begin{aligned} J_l(\hat{x}, \tilde{u}_{[k,k+N_c]}, N_c+1, N_p+1) &\le J_h(\hat{x}, u^o_{[k,k+N_c-1]}, N_c, N_p) - V_{f_l}\big(x(k+N_p)\big) \\ &+ V_{f_l}\big(f(x(k+N_p), \kappa_f(x(k+N_p)))\big) + \varphi_l\Big(x(k+N_p), \kappa_f\big(x(k+N_p)\big)\Big) \end{aligned}$$

and, recalling the conditions (8)

$$\begin{aligned} &V(\hat{x}, N_c+1, N_p+1) \le J_l(\hat{x}, \tilde{u}_{[k,k+N_c]}, N_c+1, N_p+1) \\ \le\quad &J_h(\hat{x}, u^o_{[k,k+N_c-1]}, N_c, N_p) = V(\hat{x}, N_c, N_p) \end{aligned}$$

so that

$$V(x, N_c+1, N_p+1) \le V(x, N_c, N_p), \quad \forall\, x \in X^{RH}(N_c, N_p). \tag{15}$$

Now it is proven that, for $N_p \ge N_c$, also the following monotonicity property holds

$$V(x, N_c, N_p+1) \le V(x, N_c, N_p), \quad \forall\, x \in X^{RH}(N_c, N_p). \tag{16}$$

To this end, let $u^o_{[k,k+N_c-1],N_{p+1}}$ be the optimal control sequence at time k with control and prediction horizons N_c and N_p+1 respectively,

$$h' = \arg\max_{1\le i\le r} J_i(\hat{x}, u^o_{[k,k+N_c-1],N_{p+1}}, N_c, N_p+1)$$

and $V(x,N_c,N_p{+}1) = J_{h'}(\hat{x}, u^o_{[k,k+N_c-1],N_{p+1}}, N_c, N_{p+1})$. Now recall that $u^o_{[k,k+N_c-1]}$ is the optimal control sequence computed at time k by considering the same control horizon N_c and the prediction horizon N_p and that

$$h = \arg\max_{1\le i\le r} J_i(\hat{x}, u^o_{[k,k+N_c-1]}, N_c, N_p).$$

It then follows that

$$\begin{aligned} & V(\hat{x}, N_c, N_p+1) \le J_{h'}(\hat{x}, u^o_{[k,k+N_c-1]}, N_c, N_p+1) \\ = \;& J_{h'}(\hat{x}, u^o_{[k,k+N_c-1]}, N_c, N_p) - V_{f_{h'}}\big(x(k+N_p)\big) \\ & \quad + V_{f_{h'}}\big(f(x(k+N_p), \kappa_f(x(k+N_p)))\big) + \varphi_{h'}\Big(x(k+N_p), \kappa_f\big(x(k+N_p)\big)\Big) \\ \le \;& J_h(\hat{x}, u^o_{[k,k+N_c-1]}, N_c, N_p) = V(\hat{x}, N_c, N_p). \end{aligned}$$

Finally, since for any $x \in X_f$

$$V(x,0,0) \le \max_{1\le i\le r} V_{f_i}(x) \le \max_{1\le i\le r} \beta_{V_{f_i}}(\|x\|) = \overline{\beta}_{V_f}(\|x\|) \tag{17}$$

where $\overline{\beta}_{V_f}(\|x\|)$ is a $\mathcal{K}_\infty$ function, from (15) and (16) it follows that in $x \in X_f$

$$V(x,N_c,N_p) \le V(x,N_c,N_p{-}1) \le \ldots \le V(x,N_c,N_c) \le \ldots \le V(x,0,0) \le \overline{\beta}_{V_f}(\|x\|). \tag{18}$$

In conclusion, from (14) and (18), $V(x,N_c,N_p)$ is a Lyapunov function for the closed-loop system, see [5, 9], with the receding horizon predictive control law (10) and the result follows. □

Remark 1. Many different choices of the auxiliary control law (3), of the terminal set X_f and of the terminal costs V_{f_i} have been proposed in the literature in order to guarantee the closed-loop stability of standard *RH* control laws, i. e. such that conditions (8) are satisfied, see e. g. [1, 3]. Among them, for systems (1) whose linearization at the origin is stabilizable and for quadratic cost functions $\varphi_i(x,u) = \|x\|^2_{Q_i} + \|u\|^2_{R_i}$, $i = 1,\ldots,r$, a possible way to compute the auxiliary control law (3), the terminal costs V_{f_i}, $i = 1,\ldots,r$, and the terminal set X_f can be summarized in the following steps:

1. compute the linearization of system (1) at the origin, i. e.

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0,u=0}, \qquad B = \left.\frac{\partial f}{\partial u}\right|_{x=0,u=0};$$

2. compute a stabilizing control law $u = -Kx$ for the linearized system and solve the Lyapunov equation

$$(A - BK)'P(A - BK) - P = -\alpha I \tag{19}$$

where I is the identity matrix of appropriate size and $\alpha > 1$;

3. compute the neighborhood of the origin

$$X_f = \{x \,|x'Px - c < 0\} \tag{20}$$

c being a positive constant, where the following two conditions are simultaneously satisfied

$$u = -Kx \in U \tag{21}$$

$$f(x, Kx)'Pf(x, Kx) - x'Px < 0 \tag{22}$$

Eq. (22) guarantees that there exists a positive value γ such that

$$f(x, Kx)'Pf(x, Kx) - x'Px < -\gamma x'x$$

4. define $\lambda_{Mi} = \lambda_M\,(Q_i + K'R_iK)$, $i = 1, \ldots, r$;

5. set
$$V_{f_i}(x) = \frac{\lambda_{Mi}}{\gamma}x'Px, \quad i = 1, \ldots, r. \tag{23}$$

It is easy to see that this choice satisfies conditions (21), (22) inside X_f.

3.1. Simulation example

The system under control is the model of a head box of a paper machine, first presented in [2] and later considered in [10], and described by

$$\begin{aligned} x(k+1) =& Ax(k) + Bu(k) \\ =& \begin{bmatrix} 0.99 & -0.0088 \\ 0.81 & 0.771 \end{bmatrix} x(k) + \begin{bmatrix} 0.899 & -0.0046 \\ 19.39 & 0.88 \end{bmatrix} u(k). \end{aligned} \tag{24}$$

The two state variables correspond to the stock level and the total pressure, respectively, while the inputs are the deviations in the stock flow and in the air flow.

For this system it is possible to compute the stabilizing Linear Quadratic control law

$$u(k) = -Kx(k) = -\begin{bmatrix} 0.4946 & 0.0185 \\ -9.9509 & 0.4664 \end{bmatrix} x(k). \tag{25}$$

Letting the input constraints be given by

$$|u_i| \leq 5, \quad i = 1, 2 \tag{26}$$

by means of the procedure described in Remark 1 it is possible to calculate the terminal set X_f, reported in Figure 1 and defined by (20) with $c = 0.4$, and the matrix

$$P = \begin{bmatrix} 1.4672 & -0.0168 \\ -0.0168 & 1.1008 \end{bmatrix} \tag{27}$$

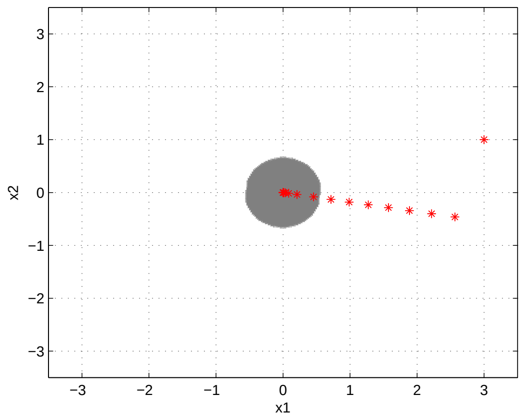

Fig. 1. Example 1: terminal region and state trajectory.

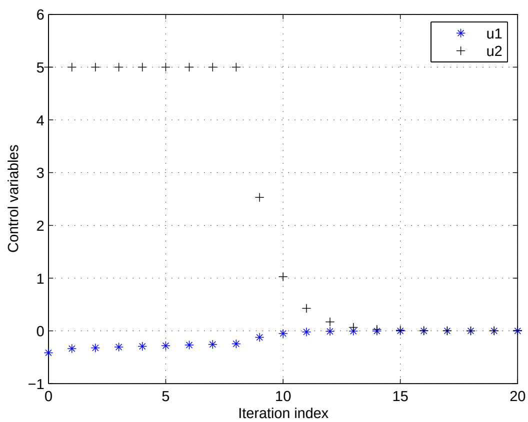

Fig. 2. Example 1: control variables.

obtained from (19) with $\alpha = 1.1$.

The optimization problem consists of minimizing three quadratic performance indexes, defined as follows

$$J_i\big(x(k), u(\cdot), N\big) = \sum_{j=k}^{k+N-1} \big(x(j)'Q_i x(j) + u(j)'R_i u(j)\big) + V_{f_i}\big(x(k+N)\big)$$

where $N_p = N_c = N = 10$, $R_1 = R_2 = R_3 = 0.01 I_2$ and

$$Q_1 = \operatorname{diag}\{2.3,\ 2.3\}\,, \quad Q_2 = \operatorname{diag}\{4,\ 1\}\,, \quad Q_3 = \operatorname{diag}\{1.,\ 2.8\}$$

which apparently pose different weights on the transients of the state variables. The functions $V_{f_i}(\cdot)$ are computed as in the procedure described in Remark 1, i. e. from

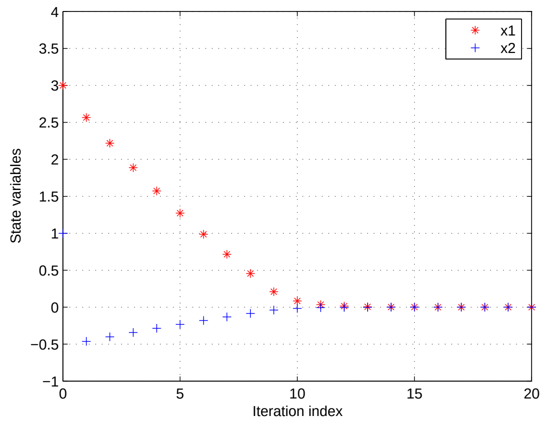

Fig. 3. Example 1: state variables.

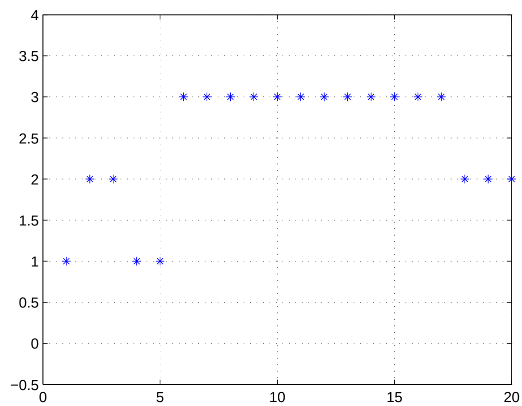

Fig. 4. Example 1: index of the maximum cost function at any iteration.

(23) with $\gamma = 1$. Finally, the vector of the initial states is $x(0) = x_0 = [3\ 1]'$, which is outside the terminal region X_f.

By applying the algorithm previously described, the transients of the control variables shown in Figure 2 have been computed and the transients of the state variables reported in Figure 3 have been obtained. These figures highlight that the predictive control law stabilizes the system (24) in about 10 iterations notwithstanding the presence of the control saturations, which are active in the initial time instants. The state trajectories are also reported in Figure 1. Finally, Figure 4 shows, at any iteration, the index of the cost function corresponding to the active constraint in the optimization problem (9).

4. SYSTEMS WITH DISTURBANCES

In this section, the *RH* approach to the multiobjective control problem is extended to simultaneously minimize a set of H_∞ cost functions for systems subject to a class of bounded disturbances. To this end, consider the perturbed model

$$\begin{cases} x(k+1) = f\big(x(k), u(k), w(k)\big) \\ \varphi_\infty\big(x(k), u(k)\big) = \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \end{cases} \tag{28}$$

where the state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$ variables must satisfy the constraints (2), $f(\cdot,\cdot,\cdot)$ is a C^1 function of its arguments, with $f(0,0,0) = 0$, and w is a disturbance term.

It is well known that for perturbed systems the *RH* approach calls for the solution of a closed-loop min-max optimization problem, where optimization must be performed with respect to control policies, instead of sequences, so as to guarantee that the effect of the disturbance term is compensated for any choice made by the "nature", see e. g. [8]. Then, let $Q_i \in \mathbb{R}^{n+m,n+m}$, $i = 1,\dots,r$, be a set of positive definite matrices,

$$\lambda_i = \min_{j=1,\dots,n+m} \lambda_j(Q_i) \tag{29}$$

$$\lambda_m = \min_{i=1,\dots,r} \lambda_i \tag{30}$$

and assume that the disturbance w belongs to the class W_{γ_Δ} of admissible disturbances such that

$$\|w(k)\|^2 \le \lambda_m \gamma_\Delta^2 \|\varphi_\infty(x(k), u(k))\|^2, \quad k \ge 0 \tag{31}$$

where γ_Δ^2 is a positive constant. The space of admissible disturbances satisfying (31) will be denoted by W_{γ_Δ}.

For the perturbed system (28) assume to know an "auxiliary" stabilizing control law

$$u = \kappa_f(x) \tag{32}$$

such that the closed-loop system (28), (32) has an attenuation level γ, with $\gamma\gamma_\Delta < 1$, in an associated positively invariant set X_f where

$$\begin{cases} \kappa_f(x) \in U, & \forall\, x \in X_f \\ f(x, \kappa_f(x), w) \in X_f, & \forall\, x \in X_f, \forall\, w \in W_{\gamma_\Delta}. \end{cases} \tag{33}$$

Define at time k the sequence of control laws by $\kappa_{[k,k+N_c-1]} = \big[\kappa_{0,k}\big(x(k)\big) \dots \dots \kappa_{N_c-1,k}\big(x(k+N_c-1)\big)\big]$ and by $w_{[k,k+N_p-1]} = [w(k) \dots w(k+N_p-1)]$ the future admissible disturbance samples over the prediction horizon. Then for system (28) and for any admissible disturbance it is possible to state the following control problem

$$\min_{\kappa_{[k,k+N_c-1]}} \max_{w_{[k,k+N_p-1]}} J_{\infty_i}\big(x(k), \kappa_{[k,k+N_c-1]}, w_{[k,k+N_p-1]}, N_c, N_p\big), \quad i = 1,\dots,r \tag{34}$$

where, for γ such that $\gamma^2\gamma_\Delta^2 < 1$,

$$J_{\infty_i}(x(k), \kappa_{[k,k+N_c-1]}, w_{[k,k+N_p-1]}, N_c, N_p) = \sum_{j=k}^{k+N_p-1} \left(\|\varphi_\infty(x(j), u(j))\|_{Q_i}^2 - \gamma^2 \|w(j)\|^2 \right) + V_{f_{\infty_i}}(x(k+N_p))$$

and subject to:

(i) the system dynamics (28),

(ii) the state and the control constraints (2),

(iii) the control law

$$\begin{cases} u(t) = \kappa_{t-k}(x(t)), & t \in [k \dots k+N_c-1] \\ u(t) = \kappa_f(x(t)), & t \in [k+N_c \dots k+N_p-1], \end{cases} \tag{35}$$

(iv) the terminal state constraint

$$x(k+N_p) \in X_f. \tag{36}$$

Also in this case, assume that the terminal costs have to be chosen so that, for any $x \in X_f$, for any $w \in W_{\gamma_\Delta}$ and for any $i = 1, \dots, r$,

$$\begin{gathered} V_{f_{\infty_i}}(x) \le \beta_{V_{f_{\infty_i}}}(\|x\|) \\ V_{f_{\infty_i}}\big(f(x, \kappa_f(x), w\big) - V_{f_{\infty_i}}(x) \le -\|\varphi_\infty\big(x, \kappa_f(x)\big)\|_{Q_i}^2 + \gamma^2\|w\|^2 \end{gathered} \tag{37}$$

where $\beta_{V_{f_{\infty_i}}}$ are $\mathcal{K}_\infty$ functions. Now, the original control problem (34) can be reformulated in the following

Multiobjective Robust Optimization Control Problem (MROCP)

$$\begin{cases} \min\limits_{\kappa_{[k,k+N_c-1]}} \max\limits_{w_{[k,k+N_p-1]}} \varepsilon \\ \qquad \left[\begin{array}{l} J_{\infty_1} \le \varepsilon \\ \vdots \\ J_{\infty_r} \le \varepsilon \end{array} \right. \end{cases} \tag{38}$$

subject to (2), (28), (32), (35), (36).

As in the previous section, let $\kappa^o_{[k,k+N_c-1]}$ and $w^o_{[k,k+N_p-1]}$ be the optimal (minimizing) control and (maximizing) disturbance sequences, respectively, of *MROCP* at time k. According to the *Receding Horizon* principle, define $\kappa^{RH}(x) = \kappa^o_{0,k}(x)$ and apply the control law

$$u = \kappa^{RH}(x). \tag{39}$$

Denoting by $X^{RH}(N_c, N_p)$ the set of states such that a feasible solution of *MROCP* exists, the following stability result can be stated.

Theorem 2. If $\gamma\gamma_\Delta < 1$, then for any $w \in W_{\gamma_\Delta}$ the origin of the closed-loop system (28), (39) is an asymptotically stable equilibrium point with region of attraction $X^{RH}(N_c, N_p)$.

Proof. Let $x(k) = \hat{x} \in X^{RH}(N_c, N_p)$ and $w(k) = \hat{w} \in W_{\gamma_\Delta}$ be the current state and the current disturbance sample, respectively, at the generic time instant k. Moreover, denote by

$$\xi = f\big(\hat{x}, \kappa^{RH}(\hat{x}), \hat{w}\big) \tag{40}$$

the value of the state at the next step due to the first entry of the optimal sequence $\kappa^o_{[k,k+N_c-1]}$.

Now define

$$V(\hat{x}, N_c, N_p) = \max_{i=1,\dots,r} \Big\{ J_{\infty_i}(\hat{x}, \kappa^o_{[k,k+N_c-1]}, w^o_{[k,k+N_p-1]}, N_c, N_p) \Big\} \tag{41}$$

and observe that $\tilde{\kappa}_{[k,k+N_c]} = \big[\kappa^o_{[k,k+N_c-1]}\ \kappa_f\big(x(k+N_c)\big)\big]$ is an admissible suboptimal sequence for the control problem at time k concerning a control and a prediction horizon of length $N_c + 1$ and $N_p + 1$, respectively.

Then

$$\begin{aligned} & J_{\infty_i}(\hat{x}, \tilde{\kappa}_{[k,k+N_c]}, w_{[k,k+N_p]}, N_c+1, N_p+1) \\ = \ & J_{\infty_i}(\hat{x}, \kappa^o_{[k,k+N_c-1]}, w_{[k,k+N_p-1]}, N_c, N_p) \\ & + \|\varphi_\infty(x(k+N_p), \kappa_f(x(k+N_p))\|^2_{Q_i} - \gamma^2 \|w(k+N_p)\|^2 \\ & + V_{f_{\infty_i}}(f(x(k+N_p), \kappa_f(x(k+N_p)), w(k+N_p)) - V_{f_{\infty_i}}(x(k+N_p)) \end{aligned}$$

and, by recalling (37)

$$J_{\infty_i}(\hat{x}, \tilde{\kappa}_{[k,k+N_c]}, w_{[k,k+N_p]}, N_c+1, N_p+1) \le J_{\infty_i}(\hat{x}, \kappa^o_{[k,k+N_c-1]}, w_{[k,k+N_p-1]}, N_c, N_p).$$

As such

$$\begin{aligned} & V(\hat{x}, N_c+1, N_p+1) \\ \le \ & \max\nolimits_{i=1,\dots,r} \Big\{ \max\nolimits_{w_{[k,k+N_p-1]}} \Big\{ J_{\infty_i}(\hat{x}, \kappa^o_{[k,k+N_c-1]}, w_{[k,k+N_p-1]}, N_c, N_p) \Big\} \Big\} \\ = \ & V(\hat{x}, N_c, N_p) \end{aligned} \tag{42}$$

for any $x \in X^{RH}(N_c, N_p)$.

Also in this case, a monotonicity property with respect to the prediction horizon N_p, and for a constant control horizon N_c, can be proven as follows. Let $\kappa^o_{[k,k+N_c-1],N_p+1}$ and $w^o_{[k,k+N_p],N_p+1}$ be the optimal solutions of the optimization problem at time k with control and prediction horizons N_c and N_p+1, respectively. Then, letting

$$h = \arg \max_{1 \le i \le r} J_{\infty_i}(\hat{x}, \kappa^o_{[k,k+N_c-1],N_p+1}, w^o_{[k,k+N_p],N_p+1}, N_c, N_p+1)$$

one has

$$\begin{aligned}
&J_{\infty_h}(\hat{x}, \kappa^o_{[k,k+N_c-1],N_p+1}, w^o_{[k,k+N_p],N_p+1}, N_c, N_p+1)\\
&\le J_{\infty_h}(\hat{x}, \kappa^o_{[k,k+N_c-1]}, w^o_{[k,k+N_p],N_p+1}, N_c, N_p+1)\\
&= J_{\infty_h}(\hat{x}, \kappa^o_{[k,k+N_c-1]}, w^o_{[k,k+N_p-1],N_p+1}, N_c, N_p)\\
&\qquad + \|\varphi_\infty(x(k+N_p), \kappa_f(x(k+N_p))\|^2_{Q_h} - \gamma^2\|w^o(k+N_p)\|^2\\
&\qquad + V_{f_{\infty_h}}(f(x(k+N_p), \kappa_f(x(k+N_p)), w^o(k+N_p))) - V_{f_{\infty_h}}(x(k+N_p))\\
&\le J_{\infty_h}(\hat{x}, \kappa^o_{[k,k+N_c-1]}, w^o_{[k,k+N_p-1],N_p+1}, N_c, N_p)\\
&\le \max_{i=1,\dots,r}\left\{\max_{w_{[k,k+N_p-1]}}\left\{J_{\infty_i}(\hat{x}, \kappa^o_{[k,k+N_c-1]}, w_{[k,k+N_p-1]}, N_c, N_p)\right\}\right\} = V(\hat{x}, N_c, N_p).
\end{aligned}$$

Therefore

$$V(\hat{x}, N_c, N_p+1) \le V(\hat{x}, N_c, N_p) \tag{43}$$

for any $x \in X^{RH}(N_c, N_p)$.

From the above relations, it also follows that for any $x \in X_f$ one has

$$\begin{aligned}
V(x, N_c, N_p+1) &\le V(x, N_c, N_p) \le \ldots \le V(x, N_c, N_c) \le \ldots\\
&\le V(x, 0, 0) \le \max_{i=1,\dots,r} \beta_{V_{f_i}}(\|x\|).
\end{aligned} \tag{44}$$

Now note that, denoting by $\bar{\kappa}_{[k+1,k+N_c-1]}$ the control sequence obtained from $\kappa^o_{[k,k+N_c-1]}$ by removing its first entry, for any $\hat{w}$ one has

$$\begin{aligned}
V(\hat{x}, N_c, N_p) \ge \max_{i=1,\dots,r}\Big\{&J_{\infty_i}(\xi, \bar{\kappa}_{[k+1,k+N_c-1]}, w_{[k+1,k+N_p-1]}, N_c-1, N_p-1\\
&+\|\varphi_\infty(\hat{x}, \kappa^{RH}(\hat{x}))\|^2_{Q_i} - \gamma^2\|\hat{w}\|^2\Big\}
\end{aligned}$$

and, in view of (31) and (43), it follows that for any $x \in X^{RH}(N_c, N_p)$ one has

$$V(\hat{x}, N_c, N_p) \ge V(\xi, N_c-1, N_p-1) + \min_{i=1,\dots,r}(\|\varphi_\infty(\hat{x}, \kappa^{RH}(\hat{x}))\|^2_{Q_i} - \gamma^2\|\hat{w}\|^2) \tag{45}$$

$$\ge V(\xi, N_c, N_p) + \vartheta(\|\hat{x}\|) \tag{46}$$

where $\vartheta(\|\hat{x}\|)$ is a $\mathcal{K}_\infty$ function.

Finally, it is simple to verify that, for any $x \in X^{RH}(N_c, N_p)$,

$$V(x, N_c, N_p) \ge \sigma(\|x\|) \tag{47}$$

where $\sigma(\|x\|)$ is a suitable $\mathcal{K}_\infty$ function. In conclusion, from (44), (45) and (47), $V(x, N_c, N_p)$ is a Lyapunov function for the considered system, see [5, 9], and, as a consequence, the origin is robustly asymptotically stable in $x \in X^{RH}(N_c, N_p)$.

Remark 2. In principle, *MROCP* calls for the solution of a computationally difficult infinite dimensional optimization problem. To this regard, note that in the design phase it is possible to resort to finite dimensional parametrizations of the control policies, see the simulation example reported below.

Remark 3. A possible way to compute the robust auxiliary control law $\kappa_f(x)$ together with an associated positively invariant terminal set has been proposed in [8] for a wide class of nonlinear systems.

4.1. Simulation example

The receding horizon approach previously described is now applied to the simulation example already considered in [8]. The system under control is a cart moving on a plane, which is attached to the wall via both a spring and a damper with an uncertain damping coefficient h. Denoting by x_1 and x_2 the position and the speed of the cart, the discretized equations of this system are

$$\begin{cases} x_1(k+1) = x_1(k) + T_c x_2(k) \\ x_2(k+1) = x_2(k) - T_c \frac{k_0}{M} e^{-x_1(k)} x_1(k) - T_c \frac{\bar{h}}{M} x_2(k) + T_c \frac{u(k)}{M} + T_c w(k) \end{cases} \tag{48}$$

where $M = 1$ is the mass of the cart, $k_0 = 0.33$ is the spring coefficient, $\overline{h} = 1$ is the nominal value of the uncertain parameter h, and the adopted sampling period is $T_c = 0.4$. Moreover, $w(k) = -\frac{\Delta h_d}{M} x_2(k)$, where $|\Delta h_d| < 0.5$ is the uncertain parameter affecting the system dynamics.

System (48) can be written in the form

$$x(k+1) = f_1(x(k)) + F_2 u(k) + F_3 w(k) \tag{49}$$

and, according to the results reported in [8], the auxiliary control law can be computed by solving, with respect to the positive definite matrix P, the H_∞ Riccati equation

$$-P + F_1' P F_1 + I - F_1' P \begin{bmatrix} F_2 & F_3 \end{bmatrix} R^{-1} \begin{bmatrix} F_2 & F_3 \end{bmatrix}' P F_1 = 0 \tag{50}$$

where

$$F_1 = \partial f_1 / \partial x|_{x=0}, \quad R = \begin{bmatrix} F_2' P F_2 + I & F_2' P F_3 \\ F_3' P F_2 & F_3' P F_3 - \gamma^2 I \end{bmatrix} \tag{51}$$

and by setting

$$\kappa_f(x) = -\begin{bmatrix} 1 & 0 \end{bmatrix} R^{-1} \begin{bmatrix} F_2 & F_3 \end{bmatrix}' P f_1(x). \tag{52}$$

In the design phase, it has been set $\gamma = 1$, while two H_∞ cost functions have been defined by setting $Q_1 = \text{diag}\{1,1,2\}$, $Q_2 = \text{diag}\{2,2,1\}$, $V_{f_{\infty_1}}(x) = V_{f_{\infty_2}}(x) = 100x'Px$. The adopted terminal set

$$X_f = \{x | \, x'Px - 0.001 < 0\} \tag{53}$$

guarantees that the decreasing conditions (37) are satisfied.

The considered control policies are $\kappa_i(x) = \alpha_i \kappa_f(x) + \beta_i x_1^2 + \gamma_i x_2^2$, where the parameters α_i, β_i and γ_i to be optimized at any time instant over the control horizon are limited as follows: $0.5 \leq \alpha_i \leq 1.8$, $|\beta_i| \leq 0.5$, $|\gamma_i| \leq 0.5$, while the adopted control constraint is $|u| \leq 1.5$. Different prediction ($N_p = 20$) and control ($N_c = 5$) horizons have been used, so reducing to 15 the total number of parameters α_i, β_i and γ_i to be iteratively determined. Finally, at any time instant the maximization with respect to the future disturbance sequence has been performed under the further constraint $\Delta h_d(k+i) = \Delta h_d(k), i = 1, \ldots, N_p$.

The simulation results obtained starting from the initial state $x(0) = [3 \ \ 0]'$ and with $\Delta h_d = 0.2$ are reported in Figures 5 and 6. It is apparent that the control constraint is active in the initial time instants.

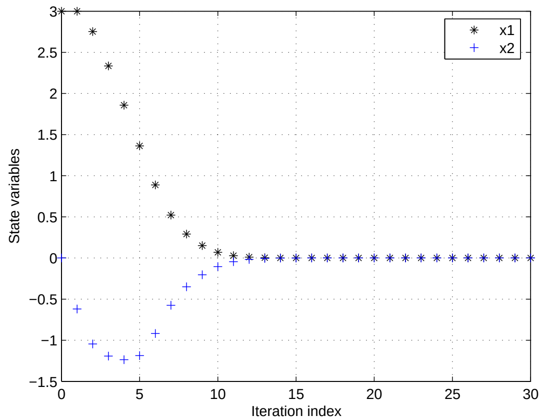

Fig. 5. State variables of Example 2.

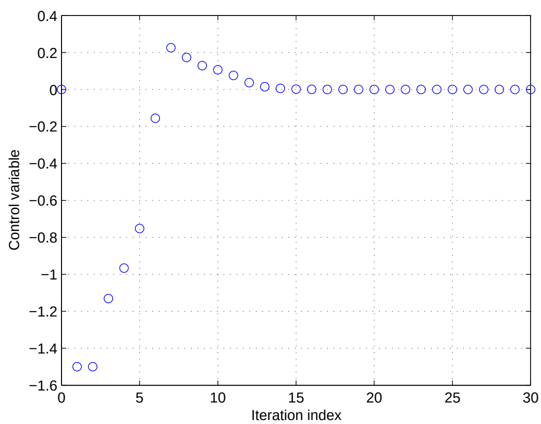

Fig. 6. Control variable of Example 2.

5. CONCLUSION

In this paper the multiobjective control problem for discrete-time, constrained, nonlinear systems has been solved by resorting to the *Receding Horizon* approach. The proposed method has also been used to provide a solution to the problem of contemporarily minimizing a set of H_∞ performance indexes for systems subject to a class of bounded disturbances. Two simulation examples have been reported in order to witness the validity of the proposed theory. The first example concerns a deterministic linear discrete time model of a flow-box, while the second one deals with the perturbed nonlinear model of a cart.

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