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THE EXISTENCE OF STATES ON EVERY ARCHIMEDEAN ATOMIC LATTICE EFFECT ALGEBRA WITH AT MOST FIVE BLOCKS

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Effect algebras are very natural logical structures as carriers of probabilities and states. They were introduced for modeling of sets of propositions, properties, questions, or events with fuzziness, uncertainty or unsharpness. Nevertheless, there are effect algebras without any state, and questions about the existence (for non-modular) are still unanswered. We show that every Archimedean atomic lattice effect algebra with at most five blocks (maximal MV-subalgebras) has at least one state, which can be obtained by "State Smearing Theorem" from a state on its sharp elements.

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0. INTRODUCTION

Generalizations of Boolean algebras including noncompatible pairs of elements are orthomodular lattices [10], while generalizations including unsharp elements are MV-algebras [4]. Lattice effect algebras are common generalizations of both these cases, hence they may contain noncompatible pairs as well as unsharp elements. On the other hand, the subset of all sharp elements in every lattice effect algebra E is an orthomodular lattice [9] and every maximal subset of pairwise compatible elements of E is an MV-algebra (MV-effect algebra) called a block and E is a union of its blocks [15].

In spite of the fact that on every MV algebra there exists a state [8] there are lattice effect algebras without any state (probability) [16]. The question, for which maximal positive integer n every lattice effect algebra with at most n blocks has a state, is still open. In this paper we are going to show that on every atomic Archimedean lattice (e. g., on every finite and on every complete atomic) effect algebra E with at most five blocks there exists a state. Note that the existence of states on all complete atomic modular lattice effect algebras was proved in [18]. For non-modular cases the existence is known only for lattice effect algebras with two blocks.

For the convenience of the reader we remind some necessary definitions and basic facts in Section 1. In Section 2 we prove statements which can be used for all orthomodular lattices with finitely many blocks. In Sections 3, 4 and 5 we prove the existence of an (o)-continuous two-valued state on every atomic orthomodular lattice with at most five blocks. In Section 6 we prove the main result of this paper: the existence of states on every Archimedean atomic lattice effect algebras with at most five blocks (even, more generally, the set of all sharp elements of which has at most five blocks). To prove this, our main tool is the "State Smearing Theorem" for (o)-continuous states on sharp elements of complete atomic lattice effect algebras [17] and a theorem on the MacNeille completions of Archimedean block-finite lattice effect algebras [14].

1. BASIC DEFINITIONS AND KNOWN FACTS

Effect algebras as generalizations of Hilbert space effects interpreted as the unsharp quantum events were introduced by D. J. Foulis and M. K. Bennett [5].

Definition 1.1. A partial algebra $(E; \oplus, 0, 1)$ is called an effect algebra if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,

(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,

(Eiii) for every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b = 1$ (we put $a' = b$),

(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. In every effect algebra E we can define the partial order $\leq$ by putting

$$a \leq b \text{ and } b \ominus a = c \text{ iff } a \oplus c \text{ is defined and } a \oplus c = b \text{, we set } c = b \ominus a\,.$$

If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (*a complete lattice effect algebra*). If, moreover, E is modular or distributive lattice then E is called *modular* or *distributive effect algebra*.

A set $Q \subseteq E$ is called a *sub-effect algebra* of the effect algebra E if

(i) $1 \in Q$

(ii) if out of elements $a, b, c \in E$ with $a \oplus b = c$ two are in Q, then $a, b, c \in Q$.

Note that lattice effect algebras generalize orthomodular lattices [10] (including Boolean algebras) if we assume the existence of unsharp elements $x \in E$, meaning that $x \wedge x' \neq 0$. On the other hand the *set* $S(E) = \{x \in E \mid x \wedge x' = 0\}$ *of all sharp elements* of a lattice effect algebra E is an orthomodular lattice [9]. In this sense a lattice effect algebra is a "smeared" orthomodular lattice. An orthomodular lattice

L can be organized into a lattice effect algebra by setting $a \oplus b = a \vee b$ for every pair $a, b \in L$ such that $a \leq b^{\perp}$. This is the original idea of G. Boole, who supposed that $a + b$ denote the logical disjunction of a and b when the logical conjunction $ab = 0$. And this is all that is needed for probability theory on Boolean algebras. If $ab = 0$ then $P(a + b) = P(a) + P(b)$, where P is a probability measure (hence $+$ can be partially defined).

In next we will write $a \oplus b$ instead of $a \vee b$ for elements a, b of an orthomodular lattice L whenever $a \leq b'$, hence $b \ominus a$ instead of $a' \wedge b$ whenever $a \leq b$. Then $(L; \oplus, 0, 1)$ is called a *lattice effect algebra derived from the orthomodular lattice* L.

Definition 1.2. Let E be an effect algebra. A map $\omega : E \to [0,1]$ is called a state on E if $\omega(0) = 0$, $\omega(1) = 1$ and $\omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \oplus y$ exists in E.

It is easy to check that the notion of a state ω on an orthomodular lattice L coincides with the notion of a state on its derived effect algebra L. It is because $x \leq y'$ iff $x \oplus y$ exists in L, hence $\omega(x \vee y) = \omega(x \oplus y) = \omega(x) + \omega(y)$ whenever $x \leq y'$.

Recall that elements x and y of a lattice effect algebra are called *compatible* (written $x \leftrightarrow y$) if $x \vee y = x \oplus (y \ominus (x \wedge y))$ (see [11]). For $x \in E$ and $Y \subseteq E$ we write $x \leftrightarrow Y$ iff $x \leftrightarrow y$ for all $y \in Y$. If every two elements are compatible then E is called an MV-effect algebra. In fact, every MV-effect algebra can be organized into an MV-algebra if we extend the partial $\oplus$ into a total operation by setting $x + y = x \oplus (x' \wedge y)$ for all $x, y \in E$ (also conversely, restricting total $+$ into partial $\oplus$ for only $x, y \in E$ with $x \leq y'$ we obtain MV-effect algebra).

In [15] it was proved that every lattice effect algebra is a set-theoretical union of MV-effect algebras called blocks. *Blocks* of E are maximal subsets of pairwise compatible elements of E. By Zorn's Lemma, every subset of pairwise compatible elements of E is contained in a maximal one. Further, blocks are sub-lattices and sub-effect algebras of E and hence maximal sub-MV-effect algebras of E. If the number of blocks of E is finite then E is called *block-finite*. Moreover, for elements x, y of an orthomodular lattice L we have $x \leftrightarrow y$ (resp. xCy see [10]) iff $x \leftrightarrow y$ in the derived effect algebra L and consequently their blocks coincide, as well.

An element a of an effect algebra E is an *atom* if $0 \leq b < a$ implies $b = 0$ and E is called *atomic* if for every nonzero element $x \in E$ there is an atom a of E with $a \leq x$. If E is a lattice effect algebra then for $x \in E$ and an atom a of E we have $a \leftrightarrow x$ iff $a \leq x$ or $a \leq x'$. It follows that if a is an atom of a block M of E then a is also an atom of E. On the other hand if E is atomic then, in general, every block in E need not be atomic [1].

For an element x of an effect algebra E we write $\operatorname{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\operatorname{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. An effect algebra E is *Archimedean* if $\operatorname{ord}(x) < \infty$ for all $x \in E$. We can show that every complete effect algebra is Archimedean (see [14]).

Lemma 1.3. Let $(E; \oplus, 0, 1)$ be an Archimedean atomic lattice effect algebra. Then

(i) (Riečanová [17], Theorem 3.3) To every nonzero element $x \in E$ there are mutually distinct atoms $a_\alpha \in E$ and positive integers k_α, $\alpha \in \mathcal{E}$ such that

$$x = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\} = \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\},$$

under which $x \in S(E)$ iff $k_\alpha = n_{a_\alpha} = \mathrm{ord}\,(a_\alpha)$ for all $\alpha \in \mathcal{E}$.

(ii) (Mosná [12], Theorem 8) A block M of E is atomic iff there exists a maximal pairwise compatible set A of atoms of E such that $A \subseteq M$ and if M_1 is a block of E with $A \subseteq M_1$ then $M = M_1$. Moreover, for $x \in E$ it holds $x \in M$ iff $x \leftrightarrow a$ for all $a \in A$.

(iii) (Mosná [12], Theorem 8) $E = \bigcup\{M \subseteq E \mid M \text{ an atomic block of } E\}$.

(iv) (Pulmannová, Riečanová [13]) Every block of an atomic block-finite orthomodular lattice is atomic.

Clearly, a lattice effect algebra E is an orthomodular lattice iff $\mathrm{ord}\,(x) = 1$ for every $x \in E$.

2. BLOCK–FINITE ORTHOMODULAR LATTICES AND LATTICE EFFECT ALGEBRAS

Recall that a lattice effect algebra E (orthomodular lattice E) is called *block-finite* if the number of blocks of E is finite. A lattice effect algebra is called a *horizontal sum* of the family $\{E_i \mid i \in I\}$ of sub-effect algebras E_i of E if $E_k \cap E_l = \{0,1\}$ for all $k \neq l$, $k, l \in I$. It follows that for all $x \in E_k$, $y \in E_l$, $k \neq l$ we have $x \wedge y = 0$ and $x \vee y = 1$. A *direct product* $E = \prod\{E_\varkappa \mid \varkappa \in H\}$ of effect algebras (orthomodular lattices) $E_\varkappa$, $\varkappa \in H$, we mean a Cartesian product with "componentwise" defined $\oplus$, 0 and 1 $(\vee, \wedge, ', 0, 1)$.

Finally, recall that a state ω on a lattice effect algebra E is called *(o)-continuous* if for every net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of E such that $x_\alpha \uparrow x$ (meaning that $x_{\alpha_1} \leq x_{\alpha_2}$ for every $\alpha_1 \leq \alpha_2$, $\alpha_1, \alpha_2 \in \mathcal{E}$ and $x = \bigvee\{x_\alpha \mid \alpha \in \mathcal{E}\}$ $\omega(x) = \sup\{\omega(x_\alpha) \mid \alpha \in \mathcal{E}\}$ holds.

Theorem 2.1. Let L be a block-finite atomic orthomodular lattice. Let $\omega : L \to \{0,1\}$ be a two valued state on L such that for every block B of L there exists an atom $b \in B$ such that $\omega(b) = 1$. Then ω is (o)-continuous.

Proof. Assume that $x_\alpha \in L$, $x \in L$, $\alpha \in \mathcal{E}$ and $x_\alpha \uparrow x$. Let $\omega(x) = 1$. Since the number of blocks of L is finite there exists a block B of L and a cofinal subset $\mathcal{B} \subseteq \mathcal{E}$ such that for every $\beta \in \mathcal{B}$ we have $x_\beta \in B$. The last follows from the fact that L is a set-theoretical union of its blocks. Because every block is closed with respect to all suprema existing in L and $x = \bigvee\{x_\beta \mid \beta \in \mathcal{B}\}$ we obtain that $x \in B$. Let b be an atom of B such that $\omega(b) = 1$. Then $b \leq x$, because otherwise $x \leq b'$ and $\omega(b') = 1$, a contradiction. $\square$

Lemma 2.2. (Bruns [3], p. 966) Every orthomodular lattice with two blocks is isomorphic with the direct product $B \times L_1$ where B is a Boolean algebra and L_1 is the horizontal sum of Boolean algebras A_1 and A_2.

Corollary 2.3. On every atomic orthomodular lattice with two blocks there exists an (o)-continuous two-valued state ω.

Lemma 2.4. Let L be an atomic orthomodular lattice with two blocks B_1 and B_2 and $C(L) \neq \{0,1\}$. Then there exists an atom a of L such that $a \in B_1 \cap B_2$.

P r o o f. Since $C(L) \neq \{0,1\}$, L is not a horizontal sum of B_1 and B_2. More precisely, there exists $c \in C(L)$, $c \notin \{0,1\}$ such that $L \cong [0,c] \times [0,c']$, where $[0,c] = B$ is a nontrivial atomic Boolean algebra and $[0,c']$ is a horizontal sum of Boolean algebras A_1, A_2. It follows that every atom of B is an atom of $B_1 \cong B \times A_1$ and also of $B_2 \cong B \times A_2$. This proves that there exists an atom a of L such that $a \in B_1 \cap B_2$, because atoms of blocks are also atoms of L. □

Theorem 2.5. Let an atomic orthomodular lattice with exactly n blocks be a set-theoretical union of a Boolean algebra B and an orthomodular lattice L_1 with exactly $(n-1)$ blocks, under which $B \cap L_1 = [c,1] \cup [0,c']$, $c \neq 0$. Then

(i) if $A = [c,1] \cup [0,c']$, $x \in B \setminus A$ and $y \in L_1 \setminus A$ then $x \not\leq y'$,

(ii) if $a \in B$ is an atom of B with $0 < a < c$ then for every $y \in L_1$ either $y \in A$ or $a \wedge y = 0$.

(iii) Every (o)-continuous two-valued state ω existing on L_1 can be extended to an (o)-continuous two-valued state on L.

P r o o f. (i) Assume that $x \leq y'$. Then $x \oplus y$ exists in L. If $x \oplus y \in B$ then $y = (x \oplus y) \ominus x \in B$, as $x \in B$ and hence $y \in B \cap (L_1 \setminus A) = \emptyset$, a contradiction. If $x \oplus y \in L_1$ then $x = (x \oplus y) \ominus y \in L_1$, which gives $x \in L_1 \cap (B \setminus A) = \emptyset$, a contradiction. This proves that $x \not\leq y'$.

(ii) Assume that c is an atom of B, then $B = [c,1] \cup [0,c']$, as B is a Boolean algebra. Moreover, then $B = A \subseteq L_1 = B_1 \cup B_2 \cup \cdots \cup B_{n-1}$ and hence there exists $k \in \{1,2,\ldots,n-1\}$ such that $c \in B_k$. Because B is a block of L, every atom of B is also an atom of L and hence c is an atom of B_k. This implies that $B_k = [c,1] \cup [0,c'] = B$, which contradicts to the number of blocks of L. Hence there exists an atom a of B with $0 < a < c$.

Assume now that $y \in L_1 \setminus A$ and $a \leq y = (y')'$. Then there exists $a \oplus y'$ and because $a \in B \setminus A$ we obtain by (i) that $y' \notin L_1 \setminus A$. It follows that $y' \in A$ and hence $y \in A$, a contradiction. This proves (ii).

(iii) Let us assume first that $\omega(c) = 1$. Let $a \in B$ be an atom of B such that $0 < a < c$. We have shown in (ii) that such atom of B exists. Define a map $\hat{\omega} : L \to \{0,1\}$ by the conditions: $\hat{\omega}|L_1 = \omega$ and for all $x \in B$ let $\hat{\omega}(x) = 1$ if $a \leq x$, otherwise $\hat{\omega}(x) = 0$. Then $\hat{\omega}|A = \omega|A$ as evidently $\omega(c') = 0$ and $\omega(x) = 1$ for all $x \in [c, 1]$. Moreover, $\hat{\omega}|B$ is a state on B and using (i) we obtain that for all $y \in L_1 \setminus A$ and $x \in B \setminus A$ we have $x \not\leq y'$. This proves that $\hat{\omega}$ is a state on L.

Assume now that $\omega(c) = 0$ and hence $\omega(c') = 1$. Then by the (o)-continuity of ω there exists an atom p of L_1, $p \leq c'$, such that $\omega(p) = 1$, since otherwise $\omega(c') = 0$. Since $p \in A \subseteq B$ we can set for all $x \in B$, $\hat{\omega}(x) = 1$ if $p \leq x$, otherwise $\hat{\omega}(x) = 0$. Then $\hat{\omega}$ is a state on B such that $\hat{\omega}|A = \omega|A$ as A is a Boolean algebra. Now, using (i) we obtain that $\hat{\omega}$ is a state on L.

Finally, $\hat{\omega}$ is (o)-continuous on L, by Theorem 2.1, as clearly $\hat{\omega}$ satisfies its assumptions. Really, for every block D of L we have: If $D = B$ then either there exists an atom a of B with $0 < a < c$ and $\hat{\omega}(a) = 1$, or an atom $p \leq c'$ with $\hat{\omega}(p) = 1$. If D is a block of L_1 then $\hat{\omega}|D = \omega|D$ is an (o)-continuous two-valued state on D and thus there exists an atom $q \in D$ such that $\omega(q) = 1 = \hat{\omega}(q)$, otherwise by the (o)-continuity of ω on D we obtain $\omega(1) = 0$, a contradiction. □

3. ATOMIC ORTHOMODULAR LATTICES WITH THREE BLOCKS

A block B of an orthomodular lattice L with exactly three blocks A, B, C is called a *middle block* if $A \cup B$ and $B \cup C$ are subalgebras of L, $A \cap C \subseteq B$ and $A \cap B$ and $B \cap C$ are unequal to $C(L) = A \cap B \cap C$ ([3] and [10], p. 306). If $C(L) = \{0, 1\}$ then L has either a middle block, or L is a horizontal sum of its blocks, or L is a horizontal sum of a Boolean algebra B and an orthomodular lattice L_1 with two blocks ([3], [10], p. 306).

Theorem 3.1. Let L be an irreducible atomic orthomodular lattice with exactly three blocks. Then there exists an (o)-continuous two-valued state ω on L.

Proof. By [13] every block of L is atomic. Hence there exist exactly three maximal pairwise compatible sets of atoms $\mathcal{A}_A$, $\mathcal{A}_B$ and $\mathcal{A}_C$ being sets of all atoms of blocks A, B and C respectively [12].

Assume first that B is a middle block. Then by Lemma 2.4 we have $\mathcal{A}_A \cap \mathcal{A}_B \neq \emptyset$, $\mathcal{A}_B \cap \mathcal{A}_C \neq \emptyset$ and $\mathcal{A}_A \cap \mathcal{A}_C = \emptyset$ because $A \cap B \neq \{0, 1\}$, $B \cap C \neq \{0, 1\}$ and $A \cap C = A \cap B \cap C = C(L) = \{0, 1\}$. It follows by the maximality of $\mathcal{A}_A$, $\mathcal{A}_B$ and $\mathcal{A}_C$ (Lemma 1.3, (ii)) that $\mathcal{A}_A \not\subseteq \mathcal{A}_B \cup \mathcal{A}_C$, $\mathcal{A}_B \not\subseteq \mathcal{A}_A \cup \mathcal{A}_C$ and $\mathcal{A}_C \not\subseteq \mathcal{A}_A \cup \mathcal{A}_B$. Let $a \in \mathcal{A}_A \setminus (\mathcal{A}_B \cup \mathcal{A}_C)$, $b \in \mathcal{A}_B \setminus (\mathcal{A}_A \cup \mathcal{A}_C)$ and $c \in \mathcal{A}_C \setminus (\mathcal{A}_A \cup \mathcal{A}_B)$. Then every atom $p \in \{a, b, c\}$ is in exactly one block $M \in \{A, B, C\}$ and for every such M we have $M \cap \{a, b, c\} \neq \emptyset$.

Let us define a map $\omega : L \to \{0, 1\}$ by conditions: $\omega(x) = 1$ if $[0, x] \cap \{a, b, c\} \neq \emptyset$ and $\omega(x) = 0$ if $[0, x] \cap \{a, b, c\} = \emptyset$. Then evidently $\omega(0) = 0$ and $\omega(1) = 1$. If $x, y \in L$ with $x \leq y'$ then there exists a block M of L with $\{x, y, x \oplus y\} \subseteq M$. If $\omega(x \oplus y) = 0$ then evidently $\omega(x) = \omega(y) = 0$, because $x, y \leq x \oplus y$. Let $\omega(x \oplus y) = 1$. Then there exists $p \in [0, x \oplus y] \cap \{a, b, c\}$. Let $q \in M \cap \{a, b, c\}$. Then $q \leq x \oplus y$, because otherwise $p \neq q$ and $p \leq x \oplus y \leq q'$ as $x \oplus y \leftrightarrow q$, which contradicts to the fact that for no block $M_1 \in \{A, B, C\}$, $\{p, q\} \subseteq M_1$ holds. This proves that $p = q \in M$ and hence $p = p \wedge (x \oplus y) = (p \wedge x) \oplus (p \wedge y)$, which gives that exactly one of $p \wedge x$ and $p \wedge y$ equals 0. Thus $\omega(x \oplus y) = \omega(x) + \omega(y)$, which gives that ω is a two-valued state on L. Because for every block M of L there exists an atom $p \in M$ with $\omega(p) = 1$ we obtain, using Theorem 2.1, that ω is (o)-continuous on L.

In the cases when L is a horizontal sum of its blocks or L is a horizontal sum of an orthomodular lattice L with two blocks and a Boolean algebra B the existence of an (o)-continuous two-valued state ω is obvious. □

Note that if L in Theorem 3.1 is not irreducible, i. e., $C(L) \neq \{0, 1\}$, then it is isomorphic to a direct product of nontrivial atomic Boolean algebra B and an irreducible orthomodular lattice L_1 with exactly three blocks. Hence we obtain:

Theorem 3.2. On every atomic orthomodular lattice with exactly three blocks there exists an (o)-continuous two-valued state.

4. ATOMIC ORTHOMODULAR LATTICES WITH FOUR BLOCKS

Lemma 4.1. (Bruns [3], p. 977) Every orthomodular lattice L with exactly four blocks is either the direct product of two orthomodular lattices with two blocks each or can be obtained by pasting a Boolean algebra B and an orthomodular lattice L_1 with exactly three blocks along a segment $[c, 1] \cup [0, c'] = A$, meaning that $c \neq 0$, $B \cap L_1 = A$ and $B \cup L_1 = L$.

Theorem 4.2. Let L be an atomic orthomodular lattice with exactly 4 blocks. Then there exists an (o)-continuous two-valued state ω on L.

Proof. (1) Let L be the direct product of orthomodular lattices L_1 and L_2 with two blocks each. Let ω be an (o)-continuous two-valued state ω on L_1. For every $x \in L$ there are unique $u \in L_1$, $v \in L_2$ such that $x = u \oplus v$. We set $\hat{\omega}(x) = \omega(u)$. Because the operations $0, 1, \vee, \wedge, ', \oplus$ are defined "componentwise" on $L_1 \times L_2$ we can easily show that $\hat{\omega}$ is an (o)-continuous two-valued state on L.

(2) If L can be obtained by pasting a Boolean algebra B and an orthomodular lattice L_1 with exactly three blocks around a segment $[c, 1] \cup [0, c']$ then there exists an (o)-continuous two-valued state $\hat{\omega}$ on L by Theorems 2.5 and 3.1. □

5. ATOMIC ORTHOMODULAR LATTICES WITH FIVE BLOCKS

Definition 5.1. (Bruns [3])

(i) For blocks B_1, B_2 of an orthomodular lattices L define $B_1 \approx B_2$ iff $B_1 \neq B_2$, $B_1 \cup B_2$ is a subalgebra of L and $B_1 \cap B_2 \neq C(L)$.

(ii) A strong link in L is an unordered pair of blocks B_1, B_2 of L satisfying $B_1 \approx B_2$.

Lemma 5.2. (Bruns [3]) Let L be an atomic orthomodular lattice with exactly five blocks. Then:

(i) either $L = B_0 \cup B_1 \cup B_2 \cup B_3 \cup B_4$ such that $L = B_0 \approx B_1 \approx B_2 \approx B_3 \approx B_4$ holds and there are no other strong links.

(ii) or L can be obtained by pasting of an orthomodular lattice L_1 with exactly four blocks and a Boolean algebra B along a segment $A = [c, 1] \cup [0, c']$, $c \neq 0$ meaning that $L = B \cup L_1$ and $B \cap L_1 = A$.

Theorem 5.3. Let L be an atomic orthomodular lattice with exactly five blocks. Then there exists an (o)-continuous state on L.

Proof. (1) Assume first that L satisfies condition (ii) of Lemma 5.2. Then by Theorem 2.5, there exists an (o)-continuous two-valued state $\hat{\omega}$ on L which extends an (o)-continuous two-valued state ω on L_1 existing by Theorem 4.2.

(2) Assume now that L satisfies condition (i) of Lemma 5.2 and that $C(L) = \{0, 1\}$. Then by ([3], (8.1) and (8.2)), for every i (modulo 5) we have:

(a) The only unions of two blocks which are subalgebras of L are $B_i \cup B_{i+1}$.

(b) $B_i \cap B_{i+2} = C(L)$.

(c) $B_i \cap B_{i+1} \not\subseteq B_{i+2}$, $B_{i+1} \cap B_{i+2} \not\subseteq B_i$.

(d) The union of three or four blocks of L is never a subalgebra of L.

By Lemma 2.4 there exist atoms a, b of L such that $a \in B_0 \cap B_1$ and $b \in B_2 \cap B_3$. It follows that $a \notin B_2, B_3, B_4$ because $B_0 \cap B_2 = B_3 \cap B_0 = B_4 \cap B_1 = \{0, 1\}$ and $b \notin B_0, B_1, B_4$ because $B_0 \cap B_2 = B_1 \cap B_3 = B_2 \cap B_4 = \{0, 1\}$. Assume that the sets of all atoms of B_0, B_1, B_2, B_3 and B_4 are A_{B_0}, A_{B_1}, A_{B_2}, A_{B_3} and A_{B_4}, respectively, and that $A_{B_4} \subseteq A_{B_0} \cup A_{B_1} \cup A_{B_2} \cup A_{B_3}$. Then $A_{B_4} = A_{B_4} \cap (A_{B_0} \cup A_{B_3})$ because $A_{B_4} \cap A_{B_1} = A_{B_4} \cap A_{B_2} = \emptyset$ as $B_4 \cap B_1 = B_4 \cap B_2 = \{0, 1\}$. Because $A_{B_4} \subseteq A_{B_0}$ resp. $A_{B_4} \subseteq A_{B_3}$ contradicts the maximality of sets of all atoms in blocks we obtain that there exist atoms $p \in A_{B_4} \cap A_{B_0}$ and $q \in A_{B_3} \cap A_{B_4}$ and hence $p \leftrightarrow q$ which gives $p \leq q'$. It follows that $p \vee q' = q' < 1$, which contradicts to the fact that $B_3 \cap B_0 = \{0, 1\}$ and hence for all nonzero $x \in B_3$, $y \in B_0$ we have $x \vee y = 1$. This proves that there exists an atom c of L such that $c \in A_{B_4} \setminus (A_{B_0} \cup A_{B_1} \cup A_{B_2} \cup A_{B_3})$.

Let $F = \{a, b, c\}$ and define $\hat{\omega}(x) = 1$ if $F \cap [0, x] \neq \emptyset$ and $\hat{\omega}(x) = 0$ if $F \cap [0, x] = \emptyset$. Then evidently $\hat{\omega}(0) = 0$ and $\hat{\omega}(1) = 1$. Assume that $x, y \in L$ are such that $x \leq y'$. If $[0, x \oplus y] \cap F = \emptyset$ then $\hat{\omega}(x \oplus y) = 0 = \hat{\omega}(x) + \hat{\omega}(y)$ because $x, y \leq x \oplus y$. If $[0, x \oplus y] \cap F \neq \emptyset$ then there exists $p \in F$ such that $p \leq x \oplus y$. Moreover, there exists a block $M \in \{B_0, B_1, B_2, B_3, B_4\}$ with $\{x, y, x \oplus y\} \subseteq M$. Further there exists $q \in M \cap F$, as we can easily see. If $q \not\leq x \oplus y$ then because $q, x \oplus y \in M$ we have $x \oplus y \leq q'$ which implies that $p \leq x \oplus y \leq q'$. The last contradicts the fact that no pair of elements $p, q \in F$ is in the same block (i. e., $p \not\leftrightarrow q$). This proves that $q \leq x \oplus y$ and hence $q = (q \wedge x) \oplus (q \wedge y)$ since $\{q, x, y, x \oplus y\} \subseteq M$, and M is a Boolean algebra. It follows that exactly one of $q \wedge x$ and $q \wedge y$ equals q. Thus $\hat{\omega}(x \oplus y) = 1 = \hat{\omega}(x) + \hat{\omega}(y)$.

Finally, $\hat{\omega}$ is (o)-continuous on L by Theorem 2.1, because $M \cap F \neq \emptyset$ for every block M of L and $\hat{\omega}(p) = 1$ for every $p \in F$.

Assume now that $C(L) \neq \{0, 1\}$. Then there exists an element $z \in C(L)$ such that $L \cong [0, c] \times [0, c']$ and $[0, c] = B$ is a Boolean algebra and $[0, c']$ is an irreducible orthomodular lattice L_1 with exactly five blocks. Since B and L_1 are atomic the existence of an (o)-continuous two-valued state on L is obvious. □

6. ARCHIMEDEAN ATOMIC LATTICE EFFECT ALGEBRAS WITH AT MOST FIVE BLOCKS

The notion of sharp elements of effect algebras was introduced by S. P. Gudder ([6, 7]). *The set* $S(E) = \{x \in E \mid x \wedge x' = 0\}$ *of all sharp elements* is an orthomodular lattice, a sub-effect algebra and a full sub-lattice of E (meaning that $S(E)$ is closed with respect to all suprema and infima existing in E), [9].

Theorem 6.1. Let $(E; \oplus, 0, 1)$ be an Archimedean atomic lattice effect algebra with at most n blocks. Then

(i) the orthomodular lattice $S(E)$ has at most n blocks,

(ii) $S(E)$ is atomic and $p \in S(E)$ is an atom of $S(E)$ iff there exists an atom a of E with $n_a a = p$ and there is no atom b of E with $n_b b < n_a a$

Proof. (i) Let $B \subseteq S(E)$ be a block of $S(E)$. Since B is a maximal pairwise compatible set of elements of $S(E)$, there exists a block M of E with $B \subseteq M$ and in view of the maximality of B in $S(E)$, we have $B = M \cap S(E)$. Further, if B_1 and B_2 are different blocks of $S(E)$ then there exist different blocks M_1 and M_2 of E such that $B_1 \subseteq M_1$ and $B_2 \subseteq M_2$, because $M_1 = M_2$ implies that $B_1 = M_1 \cap S(E) = M_2 \cap S(E) = B_2$, a contradiction. This proves that $S(E)$ has at most n blocks.

(ii) Since E is Archimedean and atomic, to every nonzero $x \in E$ there exists a set $\{a_\alpha \mid \alpha \in \mathcal{E}\}$ of atoms of E such that $x = \bigoplus\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\} = \bigvee\{k_\alpha a_\alpha \mid \alpha \in \mathcal{E}\}$, under which $x \in S(E)$ iff $k_\alpha = \operatorname{ord}(a_\alpha) = n_{a_\alpha}$ for all $\alpha \in \mathcal{E}$ ([17], Theorem 3.3). It follows that $p \in S(E)$ is an atom of $S(E)$ iff there exists an atom a of E with $n_a a = p$ and for no atom b of E, $n_b b < n_a a$ holds. Assume to the contrary that there are atoms $a_1, a_2, a_3, \ldots$ of E such that $n_{a_1} a_1 > n_{a_2} a_2 > n_{a_3} a_3 > \ldots$. Then $a_k \not\leftrightarrow a_l$ for all $k \neq l$, which contradicts the assumption that E is block-finite. Really, if there exist $l < k$ such that $a_k \leftrightarrow a_l$ then $a_k \leq a_l'$ and, because $n_{a_k} a_k < n_{a_l} a_l$, we have $n_{a_l} \neq 1$. This implies that $a_k \oplus a_l = a_k \vee a_l \leq a_l'$, which gives that $a_k \oplus 2a_l = (a_k \vee a_l) \oplus a_l = (a_k \oplus a_l) \vee 2a_l = a_k \vee 2a_l$. By induction $a_k \oplus n_{a_l} a_l$ exists, hence $n_{a_k} a_k < n_{a_l} a_l < a_k'$, which contradicts to $n_{a_k} = \operatorname{ord}(a_k)$. □

Theorem 6.2. Let $(E; \oplus, 0, 1)$ be an Archimedean atomic lattice effect algebra with at most five blocks. Then there exists a state on E.

Proof. In [14] it was proved that every block-finite Archimedean atomic lattice effect algebra E can be embedded as a subeffect algebra into a complete lattice effect algebra $\hat{E}$. In fact, $\hat{E}$ is the MacNeille completion (completion by cuts) of the lattice E. It follows that E is supremum and infimum dense in $\hat{E}$ and hence E and $\hat{E}$ have the same set of all atoms (we identify E with $\varphi(E)$, where $\varphi : E \to \hat{E}$ is the embedding). Moreover, all suprema and infima existing in E are inherited in $\hat{E}$. This implies that for an atom a of E its $n_a a = p$ is an atom of $S(E)$ iff $n_a a$ is an atom of $S(\hat{E})$. Since we have $x \in S(\hat{E})$ iff there exists a set $\{a_\alpha \mid \alpha \in \mathcal{A}\}$ of atoms of $\hat{E}$ and hence atoms of E such that $x = \bigoplus\{n_{a_\alpha} a_\alpha \mid \alpha \in \mathcal{A}\} = \bigvee\{n_{a_\alpha} a_\alpha \mid \alpha \in \mathcal{A}\}$ [17, Theorem 3.3]) we see that $S(E) \subseteq S(\hat{E})$ and $S(\hat{E})$ is the MacNeille completion of $S(E)$. Further every block $\hat{B}$ of $S(\hat{E})$ is a MacNeille completion of a block B of $S(E)$ at which B and $\hat{B}$ correspond to the same maximal subset of pairwise compatible atoms of $S(E)$. Thus, by Theorem 6.1, $S(\hat{E})$ has at most five blocks. In [14] was proved that every complete lattice effect algebra is Archimedean. By Theorems 2.1, 3.2, 4.2, 5.3 and Corollary 2.3, there exists an (o)-continuous two-valued state ω on $S(\hat{E})$ and by the "State Smearing Theorem" ([17], Theorem 5.2) there exists a state $\hat{\omega}$ on $\hat{E}$ extending ω. Since E is a sub-effect algebra of $\hat{E}$, the restriction $\hat{\omega}|E$ of $\hat{\omega}$ is a state on E. □

In fact, we have proved more:

Theorem 6.3. Let $(E; \oplus, 0, 1)$ be a block-finite Archimedean atomic lattice effect algebra. Let the set $S(E)$ of all sharp elements of E have at most five blocks. Then there exists a state on E.

Remark 6.4. Let n be an arbitrary positive integer. It is easy to construct an Archimedean atomic lattice effect algebra E having n blocks such that $S(E)$ has a unique block (e. g., a horizontal sum of n finite chains or a complete atomic lattice effect algebra with $S(E) = C(E)$).

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