KYBERNETIKA — VOLUME *44* (2008), NUMBER 3, PAGES 373 – 384

PARETO OPTIMALITY IN THE KIDNEY EXCHANGE PROBLEM

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To overcome the shortage of cadaveric kidneys available for transplantation, several countries organize systematic kidney exchange programs. The kidney exchange problem can be modelled as a cooperative game between incompatible patient-donor pairs whose solutions are permutations of players representing cyclic donations. We show that the problems to decide whether a given permutation is not (weakly) Pareto optimal are NP-complete.

Keywords: barter exchange, kidney transplantation, Pareto optimality, NP-completeness

AMS Subject Classification: 91A12, 91A06, 68Q25

1. INTRODUCTION

The most effective currently known treatment for endstage renal failure is a kidney transplantation. The supply of cadaveric kidneys is insufficient for the fast growing demand. Fortunately, kidneys from living donors can be used, as a person can live with just one of his/her two kidneys. Improvements of operations techniques, which minimize the risk for a living donor and better survival rates of live-donor kidneys really lead to an increase of living donations. Usually, a donor is a genetic or an emotional relative of a patient. Yet not rarely, even when the donor's kidney is transplantable, it cannot be donated to the intended recipient because of ABO blood type incompatibility or a positive crossmatch (preformed antibodies) and the willing donor is lost. Therefore several countries or transplantcentres have started systematic kidney exchange programs, see their descriptions e. g. in [18, 19, 24, 25].

Although in literature *indirect* exchanges have also been studied [21, 29], in this paper we consider only direct exchanges, i. e. a donor donates only when his/her intended recipient receives a kidney from another living donor. In this case, the pool of patient-donor pairs is partitioned into disjoint cycles that represent the donations, e. g. cycle $A - B - C - A$ means that patient A receives the kidney of donor B, patient B the kidney of donor C and patient C of donor A.

In addition to various medical, ethical or legal problems, kidney exchange posed a lot of questions connected with the formulation of a suitable model, choice of optimality criteria etc. As the number of patients waiting for kidney transplantation is very high (the waiting list of the United Network for Organ Sharing in the U. S. A.

currently registers more than 70 000 candidates [27]), several authors studied the theoretical efficiency as well as practical performance of used algorithms on simulated or real data, see e. g. [3, 18, 21, 24, 25].

The kidney exchange problem is usually represented by a directed graph, whose vertices correspond to patient-donor pairs and there is a directed arc from vertex v to vertex u if the patient corresponding to vertex v can accept the kidney from the donor corresponding to vertex u. An exchange is a permutation π of vertices, on understanding that a patient in vertex v receives the kidney from vertex $\pi(v)$. If a vertex is on a cycle of length 1 in this permutation, i. e. if $\pi(v) = v$, patient v receives no kidney. Usually a *social welfare maximizing* exchange is sought, i. e. one that finds a kidney for a maximum number of patients, or one that maximizes the sum of utilities derived from obtaining kidneys according to this exchange [3]. Further, in the existing kidney exchange programs usually all the transplantations on an exchange cycle are performed simultaneously to avoid the danger that one of the donors might withdraw his commitment as soon as his/her intended recipient already received a transplantation. So the longer the cycle, the greater the logistical complications and danger of the exchange to fail. Therefore, the length of exchange cycles is usually a priori restricted (most often to 2) [22].

However, the social welfare criteria might be in conflict with the respect for patients' autonomy and their individual optimality criteria, so losses in efficiency might occur [29]. Therefore game-theoretical approaches have also been applied to kidney exchanges. In this paper we follow the model suggested in [8]. Here, the vertices of the exchange digraph are endowed with preferences over exchanges that combine the preferences over kidneys with the preferences over cycle lengths.

In the resulting 'kidney exchange game', computational complexity of the (weak) core has been intensively studied and several NP-completeness or even inapproximability results have been obtained [4, 9, 16]. In this paper we concentrate on (weak) Pareto optimal exchanges. Computational complexity of Pareto optimality in related problems, namely in the house allocation and roommates problem have been studied e. g. in [1] and [2]. For the kidney exchange game we show that even if it is quite easy to find a Pareto optimal exchange in the case with strict preferences of players over kidneys and a weakly Pareto optimal exchange in any case, it is an NP-complete problem to decide whether a given permutation is (weakly) Pareto optimal, and this holds in the case with strict preferences as well as in the case when all compatible kidneys are considered equally good.

Let us notice here, that a similar situation has been discovered for other cooperative games too. As examples let us mention at least the linear production, flow and minimum-cost spanning tree games, where polynomial algorithms for finding a core element exist [13, 17, 20], but testing whether a given imputation in not in the core is NP-hard [10, 11].

Finally, it may be argued that our model is not suitable for kidney exchanges, as the risk involved in longer cycles is so high, that patients would rather give up a slightly better kidney and prefer to participate in an exchange cycle of length just two. However, let us notice that the proposed model can be used in the context of residence exchange fairs in Beijing [28] or for other barter exchange markets (for

DVDs, books, holiday houses or even shoes) referred to in [3].

The organisation of the paper is as follows. In Section 2 the kidney exchange game is formulated together with the studied solution concepts and results about the existence and finding of optimal permutations are reviewed. In Section 3 we concentrate on optimality testing and prove our NP-completeness results.

2. THE KIDNEY EXCHANGE GAME AND OPTIMAL PERMUTATIONS

The kidney exchange problem is represented by a finite simple digraph $G = (V, A)$ where each vertex represents a patient-donor pair (in general, a patient can have several donors, but this assumption can easily be dealt with). Loops are not allowed in G as they correspond to patients who have their own compatible donor and these usually do not take part in a kidney exchange program. An arc $(v, u) \in A$ if patient v can accept the kidney from donor u; we say that vertex u is *acceptable* for v. Moreover, we suppose that each vertex v has a linear ordering $\preceq_v$ of the acceptable vertices, meaning that patient v orders compatible kidneys from the medically most suitable one to the worst one.

If $u \preceq_v w$, we say that v prefers u to w. If $u \preceq_v w$ and $w \preceq_v u$, then v is *indifferent* between u and w, written $u \sim_v w$. If $u \preceq_v w$ but not $w \preceq_v u$, then v *strictly prefers* u to w, written $u \prec_v w$.

There are two extreme cases – the case with *strict preferences* where no indifferences in the preferences of vertices are allowed, and *dichotomous preferences* where each vertex is indifferent between all acceptable vertices.

Definition 1. An instance of *the kidney exchange game* (KE for short) is a triple $\Gamma = (V, G, \mathcal{O})$, where V is the set of patient-donor pairs (players), $G = (V, A)$ is a digraph and $\mathcal{O} = \{\preceq_v, v \in V\}$.

Definition 2. An *exchange* in a KE game $\Gamma = (V, G, \mathcal{O})$ is a permutation π of V such that $v \neq \pi(v)$ implies $(v, \pi(v)) \in A$ for each $v \in V$.

We say, that v is *uncovered* by permutation π iff $\pi(v) = v$. Otherwise, v is *covered* by π. In what follows, $C^\pi(v)$ denotes the cycle of π containing v and we represent a permutation by its cycles.

Further we will define an extension of preferences from vertices to preferences over permutations which incorporate cycle lengths as well.

Definition 3. Let $\Gamma = (V, G, \mathcal{O})$ be a KE game, $v \in V$ a player, π and σ permutations of V. We say that player v *prefers* permutation π to permutation σ, written $\pi \preceq_v \sigma$, if either

- v is uncovered by both π and σ, or
- v is covered by π and uncovered by σ, or
- v is covered by both π and σ and

(i) $\pi(v) \prec_v \sigma(v)$ or

(ii) $\pi(v) \sim_v \sigma(v)$ and $|C^\pi(v)| \leq |C^\sigma(v)|$.

Player v *strictly prefers* permutation π to permutation σ, written $\pi \prec_v \sigma$, if $\pi \preceq_v \sigma$ but not $\sigma \preceq_v \pi$.

Note that in the dichotomous case, preferences of players over permutations depend only on the lengths of cycles. More precisely, player v strictly prefers permutation π to permutation σ, if either v is covered in π and uncovered in σ, or if v is covered both in π and σ, but $C^\pi(v)$ is shorter than $C^\sigma(v)$.

With the players' preferences over permutations, we can define Pareto optimal and core permutations.

Definition 4. A coalition $S \subseteq V$ *weakly blocks* a permutation π if there exists a permutation σ of S such that each player in S prefers σ to π and at least one player in S strictly prefers σ to π. Coalition $S \subseteq V$ *strongly blocks* a permutation π if there exists a permutation σ of S such that each player in S strictly prefers σ to π.

Definition 5. A permutation π is *Pareto optimal* for game Γ, $\pi \in PO(\Gamma)$ for short, if the grand coalition V does not weakly block π. A permutation π is *weakly Pareto optimal* for game Γ, $\pi \in WPO(\Gamma)$ for short, if the grand coalition V does not strongly block π.

Definition 6. A permutation π is in the *core* $C(\Gamma)$ of game Γ if no coalition weakly blocks π, and it is in the *weak core* $WC(\Gamma)$ of game Γ if no coalition strongly blocks it.

As each strongly blocking coalition is also weakly blocking, we have

$$C(\Gamma) \subseteq WC(\Gamma) \subseteq WPO(\Gamma), \tag{1}$$
$$C(\Gamma) \subseteq PO(\Gamma) \subseteq WPO(\Gamma) \tag{2}$$

and the above inclusions can be proper [8].

In the case with strict preferences, the famous *Top Trading Cycles algorithm* (TTC for short) [26] can be used. The TTC algorithm was originally proposed for the house-swapping game where cycle lengths do not influence preferences of players. The permutation obtained by the TTC algorithm was shown to be in the core of the KE game in [7], however, only for strict preferences. In the case with indifferences, unlike in the original house-swapping game, the permutation given by the TTC algorithm may happen not to be in the core of the KE game, moreover, it is even NP-complete to decide whether $WC(\Gamma) \neq \emptyset$ and also whether $C(\Gamma) \neq \emptyset$ [6].

According to inclusions (1)–(2), TTC gives also a (weakly) Pareto optimal permutation for a KE game with strict preferences. On the other hand, in [8], it was argued that $PO(\Gamma)$ is always nonempty for each KE game Γ irrespective of indifferences. However, even in the case with dichotomous preferences it is NP-hard to find a permutation $\pi \in PO(\Gamma)$ [8] and this implies that finding a permutation in $PO(\Gamma)$ is NP-hard also in the general case with indifferences. Unlike this, it is always possible to find a permutation $\pi \in WPO(\Gamma)$ in polynomial time: given a KE game $\Gamma = (V, G, \mathcal{O})$, take the subgraph G_1 of G by including for each vertex $v \in V$ only arcs (v, u) such that $\{w; w \prec_v u\} = \emptyset$. As each vertex in G_1 has outdegree at least one, G_1 is not acyclic and a shortest cycle C can be found in polynomial time. Then $\pi = C \cup \{(v), v \notin C\} \in WPO(\Gamma)$. However, such a permutation may cover a very small number of vertices. Let us notice here that the problem of finding a weakly Pareto optimal permutation covering the maximum possible number of vertices remains open.

3. OPTIMALITY TESTING

In [5], Cechlárová and Hajduková presented a polynomial algorithm for testing whether a given permutation belongs to the (weak) core of a given KE game for general preferences. In this paper we show that testing (weak) Pareto optimality is computationally difficult. We consider the following decision problems: Ke–nonPO–test and Ke–nonWPO–test ask whether for a given KE game Γ permutation π is not Pareto optimal and not weakly Pareto optimal, respectively. Notice that both problems belong to the class NP, as when another permutation σ is given, it can be polynomially verified that each player of the grand coalition (strictly) improves compared to π. We prove that these problems are NP-complete for both extreme types of preferences – dichotomous and strict.

Theorem 1. Problem Ke-nonPO-test is NP-complete even in the special case of dichotomous preferences.

Proof. To prove the NP-hardness, we will use a polynomial transformation from the problem Exact 3-cover, shown to be NP-complete in [12]. In Exact 3-cover a finite set X, $|X| = 3q$ and a family $\mathcal{F}$ of three-element subsets of X are given. The question is whether a subfamily $\mathcal{F}'$ of $\mathcal{F}$ exists such that each element of X belongs to exactly one set from $\mathcal{F}'$.

For each instance $(X, \mathcal{F})$ of Exact 3-cover, we construct a KE game $\Gamma = (V, G, \mathcal{O})$ with dichotomous preferences and a permutation π.

Suppose that the elements of X are ordered $x_1, x_2, \dots, x_n$, $n = 3q > 3$, that $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ and $F_i = \{x_i^1, x_i^2, x_i^3\}$ (not necessarily obeying the order of elements of X). For each set $F_i \in \mathcal{F}$, there will be 9 vertices a_i^k, b_i^k, c_i^k, $k = 1, 2, 3$ in V and for each element $x_j \in X$ there will be one vertex x_j. The arcs of G are defined in Figure 1 in the form of incidence lists, where A_j is the set of those a_i^k that correspond to the occurrence of x_j as x_i^k in F_i.

In the obtained KE game, construct permutation

$$\pi = \left\{(x_1, \dots, x_n)(a_i^k, b_i^k, c_i^k),\ i = 1, \dots, m,\ k = 1, 2, 3\right\} \tag{3}$$

and for brevity, call $(x_1, \dots, x_n)$ the *long cycle*.

$$\begin{array}{lll} a_i^k: & b_i^k & i=1,\dots,m;\ k=1,2,3 \\ b_i^k: & x_i^k,\ c_i^k & i=1,\dots,m;\ k=1,2,3 \\ c_i^k: & c_i^{k+1},\ a_i^k & i=1,\dots,m;\ k=1,2,3 \text{ (cyclically)} \\ x_j: & A_j,\ x_{j+1} & j=1,\dots,n \text{ (cyclically)} \end{array}$$

Fig. 1. Arcs of G.

We will show, that $(X,\mathcal{F})$ admits an exact 3-cover if and only if permutation $\pi \notin PO(\Gamma)$.

Suppose that $(X,\mathcal{F})$ admits an exact 3-cover $\mathcal{F}' = \{F_i, i \in I\}$. Let us define permutation σ of V consisting of the following cycles:

$$(c_i^1, c_i^2, c_i^3), \quad (a_i^k, b_i^k, x_i^k) \quad k=1,2,3,\ i \in I \tag{4}$$

$$(a_i^k, b_i^k, c_i^k) \quad k=1,2,3,\ i \notin I, \tag{5}$$

As $\mathcal{F}'$ is an exact 3-cover, permutation σ is well defined and $|C^\sigma(v)| = 3$ for each player $v \in V$. Hence each player x_j strictly prefers σ to π and other players are indifferent between σ and π. So permutation π is weakly blocked via σ and therefore it is not Pareto optimal.

For the other direction, suppose that $\pi \notin PO(\Gamma)$ and σ weakly blocks it. We will show, that $\mathcal{F}$ admits an exact 3-cover.

As π covers each vertex, so does σ. We will moreover show, that σ consists only of 3-cycles. As G does not contain cycles of length 2, we must have $|C^\sigma(v)| = |C^\pi(v)| = 3$ for each player $v \in V' = \{a_i^k, b_i^k, c_i^k;\ i=1,\dots,m;\ k=1,2,3\}$. So vertices in V' cannot improve and therefore we must have $\sigma \prec_{x_j} \pi$ for at least one vertex x_j. Hence x_j cannot be on the long cycle. Then necessarily $\sigma(x_j) = a_i^k$ for some $a_i^k \in A_j$, which implies $C^\sigma(x_j) = C^\sigma(a_i^k) = (x_j, a_i^k, b_i^k)$.

If $\sigma(x_j) \neq x_{j+1}$ for some j, then also $C^\sigma(x_{j+1}) = (x_{j+1}, a_r^s, b_r^s)$, where $x_{j+1} \in F_r$ as its s^{th} element. By induction, we get that σ contains only cycles of length 3.

Further, if $(a_i^k, b_i^k, x_i^k) \in \sigma$ for some i and k, then necessarily $C^\sigma(c_i^k) = (c_i^1, c_i^2, c_i^3)$ and therefore also $(a_i^{k+1}, b_i^{k+1}, x_i^{k+1}) \in \sigma$ and $(a_i^{k+2}, b_i^{k+2}, x_i^{k+2}) \in \sigma$ (superscripts defined cyclically to be in {1,2,3}). So for each i, either $(a_i^k, b_i^k, x_i^k) \in \sigma$ for all k or for none. Hence if we set

$$I = \{i:\ (a_i^k, b_i^k, x_i^k) \in \sigma \text{ for some } k\},$$

then it is immediate that $\mathcal{F}' = \{F_i, i \in I\}$ is an exact 3-cover of $\mathcal{F}$.

As the above transformation is polynomial, we conclude that problem Ke–nonPO–test is NP-hard under dichotomous preferences. □

Theorem 2. Problem Ke–nonPO–test is NP-complete even in the special case of strict preferences.

P r o o f . To prove the NP-hardness, we use exactly the same polynomial transformation from the problem EXACT 3-COVER as in Theorem 1, but now the orders of entries in the incidence lists of vertices in Figure 1 define their strict preferences, with the entries in A_j ordered strictly, but arbitrarily.

The proof is also very similar, we just add some remarks.

When σ is defined by (4)−(5), $\sigma \prec_v \pi$ not only for players x_j, but also for all players $v \in \{c_i^k, b_i^k; i \in I\}$, as for them $\sigma(v) \prec_v \pi(v)$.

For the converse implication, realise that π is the most preferred permutation for players a_i^k, as $\pi(a_i^k)$ is the only acceptable vertex for a_i^k and $C^\pi(a_i^k)$ is shortest possible. Hence if the grand coalition weakly blocks π via a permutation σ, then necessarily $\sigma(a_i^k) = \pi(a_i^k) = b_i^k$ and players a_i^k cannot improve. Further, as $C^\sigma(a_i^k) = C^\sigma(b_i^k)$, we have $|C^\sigma(b_i^k)| = 3$ for each b_i^k. To have at least one player who strictly improves, σ must contain at least one cycle of the form (a_i^k, b_i^k, x_i^k) and the rest of the proof follows. □

Theorem 3. KE–NONWPO–TEST is NP-complete even in the special case of dichotomous preferences.

P r o o f . We will use a polynomial transformation from the problem RESTRICTED SAT shown to be NP-complete in [15]. In this problem one asks whether a Boolean formula B in CNF containing n Boolean variables $x_1, x_2, \ldots, x_n$ and m clauses $K_1, K_2, \ldots, K_m$, such that each variable appears exactly twice nonnegated and exactly twice negated in B is satisfiable.

For each instance B of RESTRICTED SAT we construct a KE game $\Gamma = (V, G, \mathcal{O})$ with dichotomous preferences and a permutation π.

For each variable x_j, $j = 1, 2, \ldots, n$ there will be a *variable cell* $\Gamma(x_j)$ of 8 *variable players* $x_j^1, x_j^2, y_j^1, y_j^2, z_j^1, z_j^2, w_j^1, w_j^2$ where x_j^1 (y_j^1) corresponds to the first and x_j^2 (y_j^2) to the second occurrence of literal x_j $(\overline{x}_j)$. Players $x_j^1, x_j^2, y_j^1, y_j^2$ will be called *proper* variable players.

For each clause $K_k = \{p_k^1, p_k^2, \ldots, p_k^{i_k}\}$, $k = 1, 2, \ldots, m$ (without loss of generality we suppose $i_k > 1$ for each k) there is a *clause cell* $\Gamma(K_k)$ consisting of $4i_k$ *clause players* $p_k^i, t_k^i, q_k^i, r_k^i,\ i = 1, \ldots, i_k$. Player p_k^i corresponds to the ith entry of K_k and will be called a *proper clause* player, $i = 1, \ldots, i_k$.

Hence game Γ consists of $24n$ players; there are 8 variable players for each variable and 4 clause players for each of $4n$ literals.

We will use the following notation: for each proper variable player v, $c(v)$ denotes the proper clause player corresponding to the position of the corresponding literal in B; and for each proper clause player c, the corresponding proper variable player will be denoted by $v(c)$.

The arc set of G is defined in Figure 2 and the construction is illustrated in Figures 3 and 4. In Figure 4 the notation $f(a)$ and $s(a)$ denotes the first and the second vertex in the preference list of player a.

The construction will be completed by the definition of permutation π. For each variable x_j, $j = 1, \ldots, n$ we put

$$(x_j^1, z_j^1, y_j^1, w_j^1, x_j^2, z_j^2, y_j^2, w_j^2) \in \pi,$$

$x_j^i:$	z_j^i	$i=1,2;$	$j=1,\dots,n$	
$z_j^i:$	$c(y_j^{3-i}),y_j^i$	$i=1,2;$	$j=1,\dots,n$	
$y_j^i:$	w_j^i	$i=1,2;$	$j=1,\dots,n$	
$w_j^i:$	$c(x_j^i),x_j^{3-i}$	$i=1,2;$	$j=1,\dots,n$	
$p_k^i:$	$v(p_k^i),r_k^i$	$i=1,\dots,i_k;$	$k=1,\dots,m$	
$r_k^i:$	q_k^i	$i=1,\dots,i_k;$	$k=1,\dots,m$	
$q_k^i:$	t_k^i	$i=1,\dots,i_k;$	$k=1,\dots,m$	
$t_k^i:$	r_k^{i+1},p_k^{i+1}	$i=1,\dots,i_k;$	$k=1,\dots,m;$	(subscripts defined cyclically)

Fig. 2. Arcs of G.

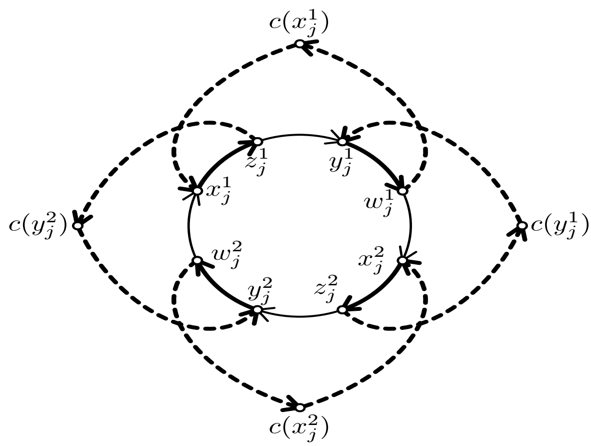

Fig. 3. Variable cell with adjacent inter-cell cycles.

and for each clause K_k, , $k=1,\dots,m$ we put

$$(p_k^1,r_k^1,q_k^1,t_k^1,p_k^2,r_k^2,q_k^2,t_k^2,\dots,p_k^{i_k},r_k^{i_k},q_k^{i_k},t_k^{i_k})\in\pi.$$

These cycles will be called *perimeter* cycles. Notice that perimeter cycles in variable cells have length 8, while perimeter cycles in clause cells are of length $4i_k$, $k=1,\dots,m$.

As π covers all players, let us first analyze how all players can be covered by cycles not longer than in π. It is easy to see that for players of a variable cell $\Gamma(x_j)$ there are only three possibilities, all players are:

(i) either on the perimeter cycle, or

(ii) in two cycles $(x_j^i,z_j^i,y_j^i,w_j^i,c(x_j^i))$, $i=1,2$ (let us call them *T-cycles*), or

(iii) in two cycles $(y_j^i,w_j^i,x_j^{3-i},z_j^{3-i},c(y_j^i))$, $i=1,2$ called *F-cycles.*

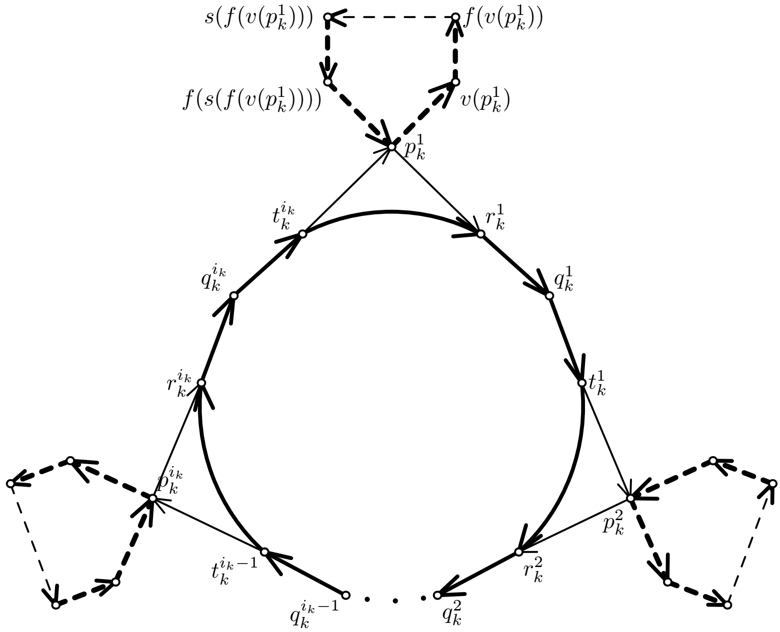

Fig. 4. Clause cell with adjacent inter-cell cycles.

As the lengths of the T-cycles and F-cycles are 5, in cases (*ii*) and (*iii*) all players from $\Gamma(x_j)$ are better off than under π. Moreover, as these cycles contain one player from a clause cell, we will call them *proper inter-cell* cycles. Notice that any other cycle involving players from several different cells has length greater than 8, so only proper inter-cell cycles could be used when we do not want to make anybody worse off.

Let us now look at a clause cell $\Gamma(K_k)$. If its non-proper clause players are to be covered without making anybody worse off, they must be on cycles that use only arcs within $\Gamma(K_k)$. Otherwise at least two variable cells have to be crossed, getting the cycle length greater than 8, which will necessarily make some proper variable player worse off. Hence the only possibility to cover all players from $\Gamma(K_k)$ is either by the perimeter cycle, or by having some proper clause players p_k^i, $i \in I$ in their corresponding proper inter-cell cycles and the remaining players of $\Gamma(K_k)$ on the common cycle which uses the perimeter arcs and the corresponding *shortcuts* (t_k^{i-1}, r_k^i), $i \in I$. In the latter case all players of $\Gamma(K_k)$ simultaneously strictly improve compared to π.

Now suppose that B is satisfied by some Boolean valuation. Create a permutation σ as follows: cover each variable cell $\Gamma(x_j)$ for which x_j is true by two T-cycles and by two F-cycles if x_j is false. As B is satisfied, each clause cell is crossed by at least one proper inter-cell cycle. Let the remaining players of each clause cell be on the common cycle with corresponding shortcuts. It is easy to see that $2 \leq |C^{\sigma}(v)| < |C^{\pi}(v)|$ for each $v \in V$, so $\sigma \prec_v \pi$ for each $v \in V$.

For the other direction, suppose that B is not satisfiable, but all players have strictly improved. Then all variable players are on T-cycles or F-cycles and each clause cell is crossed by at least one proper inter-cell cycle.

Let us now define a Boolean valuation as follows: for each used T-cycle assign the underlying variable true and for each used F-cycle make the corresponding variable false. Clearly such a valuation is not contradictory and it is easy to see that it satisfies B, a contradiction.

Hence B is satisfiable if and only if $\pi \notin WPO$. As the construction is polynomial, we conclude that KE–NONWPO–TEST is NP-complete. □

Theorem 4. KE–NONWPO–TEST problem is NP-complete even in the special case of strict preferences.

Proof. We will use the same transformation as in Theorem 3 but now we interpret the order of entries in Figure 2 as preferences of players.

The argument is also identical, we just notice that in addition to getting shorter cycles, each player v, for whom $\sigma(v) \neq \pi(v)$, strictly prefers $\sigma(v)$ to $\pi(v)$. □

4. CONCLUSION

The exchange game studied in this paper turns out to be a computationally and structurally interesting cooperative game. It is applicable not only in the context of kidney exchanges, but also in other barter markets, where it can be assumed that participants care about the lengths of exchange cycles. Let us stress here, that in the case of strict preferences a core (and hence also Pareto-optimal) permutation can always be found. Our results indicate some limits when trying to find alternative permutations that could perhaps be more favourable by some criteria. Moreover, the studied model puts a great emphasis on avoiding any blocking, even by coalitions containing a great number of players. It would be interesting to look for permutations which could only be improved upon by cooperation of many players, as in this case their coordination may be critical and so the real danger of disrupting the current exchange will not be so big.

Note added in the proof: On April 9, 2008, the news was published about the world's first simultaneous cyclic exchange of length 6, performed at the John Hopkins Hospital in Maryland, U.S.A., see
http://news.bbc.co.uk/go/pr/fr/-/hi/health/7338437.stm

ACKNOWLEDGEMENT

This work was supported by the VEGA grants 1/3001/06, 1/3128/06, VVGS grant 36/2006 and Science and Technology Assistance Agency contract No. APVT-20-004104.

(Received June 21, 2007.)

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