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EXTERNAL PROPERNESS

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In this paper, we revisit the structural concept of properness. We distinguish between the properness of the whole system, here called internal properness, and the properness of the "observable part" of the system. We give geometric characterizations for this last properness concept, namely external properness.

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Notation, Geometric Algorithms, System, and Subspaces

Notation. Script capitals $\mathcal{V}$, $\underline{\mathcal{V}}$, ..., denote linear spaces with elements v, w, ...; $\{0\}$ is the zero subspace. The dimension of a space $\mathcal{V}$ is denoted $\dim(\mathcal{V})$. When $\mathcal{V} \subset \underline{\mathcal{V}}$, $\frac{\underline{\mathcal{V}}}{\mathcal{V}}$ or $\underline{\mathcal{V}}/\mathcal{V}$ stands for the quotient space $\underline{\mathcal{V}}$ modulo $\mathcal{V}$. The direct sum of independent spaces is written as $\oplus$. Given a linear map $X : \mathcal{V} \to \underline{\mathcal{V}}$, $\mathrm{Im}\, X = X\mathcal{V}$ denotes its image, and $\mathcal{K}_X$, or sometimes Ker X, denotes its kernel. We write $X^{-1}\mathcal{T}$ for the inverse image of the subspace $\mathcal{T}$ by the linear map X. We write $(Y^{-1}X)^{\eta}\mathcal{T}$ for $Y^{-1}X\big(Y^{-1}X(\cdots(Y^{-1}X\mathcal{T}))\big)$, η times. $\{x, y, z\}$ stands for the subspace spanned by the vectors x, y and z. e_i stands for the vector with a 1 in its ith component and 0 in its other components. $\mathbb{R}^+ = \{r \in \mathbb{R} :\ r \geq 0\}$. $\mathcal{W}^{\boldsymbol{T}}$ is the collection of all maps from $\boldsymbol{T}$ to $\mathcal{W}$. $\mathcal{C}^{\infty}(\mathbb{R}^+, \mathbb{R}^q)$ is the set of infinitely differentiable functions mapping from $\mathbb{R}^+$ to $\mathbb{R}^q$.

Geometric Algorithms. Given the maps $X :\ \mathcal{V} \to \underline{\mathcal{V}}$, $Y :\ \mathcal{V} \to \underline{\mathcal{V}}$ and $Z : \underline{\mathcal{V}} \to \mathcal{Z}$, and the subspaces $\mathcal{K} \subset \mathcal{V}$ and $\mathcal{L} \subset \underline{\mathcal{V}}$, we have the two following popular geometric algorithms (see mainly [22, 23, 28]):

ALG–V. – Algorithm for computing *the supremal* $(X, Y, \mathcal{L})$ *invariant subspace contained in* $\mathcal{K}$ ($[\mathcal{K} : X, Y, \mathcal{L}]$):

$$\mathcal{V}^0_{[\mathcal{K}:X,Y,\mathcal{L}]} = \mathcal{V}\ , \quad \mathcal{V}^{\mu+1}_{[\mathcal{K}:X,Y,\mathcal{L}]} = \mathcal{K} \cap X^{-1}\left(Y\mathcal{V}^{\mu}_{[\mathcal{K}:X,Y,\mathcal{L}]} + \mathcal{L}\right). \qquad \text{(ALG–V)}$$

which limit is $\mathcal{V}^*_{[\mathcal{K}:X,Y,\mathcal{L}]} = \sup\{\mathcal{T} \subset \mathcal{K} |\ X\mathcal{T} \subset Y\mathcal{T} + \mathcal{L}\}$. In the case that $\mathcal{L} = \mathrm{Im}\, Z$, we write $[\mathcal{K} : X, Y, Z]$ instead of $[\mathcal{K} : X, Y, \mathrm{Im}\, Z]$. In the case that $\mathcal{L} = \{0\}$, we write $[\mathcal{K} : X, Y]$ instead of $[\mathcal{K} : X, Y, \{0\}]$.

ALG–S. – Algorithm for computing *the infimal* $(\mathcal{L},Y,X)$ *invariant subspace contained in* $\mathcal{K}$, *related to* $\mathcal{L}$, *and initialized in* $\mathcal{R}$, where $\mathcal{R}\subset\mathcal{V}$ and $Y\mathcal{R}\subset X\mathcal{R}+\mathcal{L}$ ($[\mathcal{K},\mathcal{R}:\mathcal{L},Y,X]$):

$$\mathcal{S}^0_{[\mathcal{K},\mathcal{R}:\mathcal{L},Y,X]}=\mathcal{K}\cap\mathcal{R}\ ,\quad \mathcal{S}^{\mu+1}_{[\mathcal{K},\mathcal{R}:\mathcal{L},Y,X]}=\mathcal{K}\cap Y^{-1}\left(X\mathcal{S}^{\mu}_{[\mathcal{K},\mathcal{R}:\mathcal{L},Y,X]}+\mathcal{L}\right). \quad \text{(ALG–S)}$$

where the limit is $\mathcal{S}^*_{[\mathcal{K},\mathcal{R}:\mathcal{L},Y,X]}=\inf\{\mathcal{S}\subset\mathcal{K}|\ \mathcal{S}=Y^{-1}(X\mathcal{S}+\mathcal{L})\}$.
In the case that $\mathcal{L}=\{0\}$, we write $[\mathcal{K},\mathcal{R}:Y,X]$ instead of $[\mathcal{K},\mathcal{R}:\{0\},Y,X]$.
In the case that $\mathcal{R}=\{0\}$, we write $[\mathcal{K}:\mathcal{L},Y,X]$ instead of $[\mathcal{K},\{0\}:\mathcal{L},Y,X]$.

System. In this paper we deal with dynamical systems $\Sigma=(\boldsymbol{T},\mathcal{W}=\mathcal{Y}\oplus\mathcal{U},\mathfrak{B})\in\mathfrak{L}^{p+m}$, where $\boldsymbol{T}=\mathbb{R}^+$ is the time set, $\mathcal{W}$ is the space of external variables, $\mathcal{Y}$ is the output space, $\mathcal{U}$ is the input space. Note that the splitting of $\mathcal{W}$ into two parts, $\mathcal{Y}$ and $\mathcal{U}$, is given *a priori*. This separation between inputs and outputs is often imposed by the requirements on the systems we are dealing with (see [16]). This setting is thus slightly different from the classic one introduced by Willems (see [25]) where the separation is made *a posteriori*: inputs are signals which are "causes" while outputs are "effects" and the separation is made in order to always get a proper transfer function matrix between those selected inputs and outputs. We work in the context where this separation is given *a priori*, which explains that properness has to be analyzed. Concerning the so-called behaviour $\mathfrak{B}$, our setting is also slightly different from that of Willems: the signals are supposed to be infinitely differentiable (as in [16]) and not locally integrable. This makes possible the *a priori* separation between u and y (see Remark 3.3.18 in [25]). Hence $\mathfrak{B}\subset\mathcal{C}^\infty(\mathbb{R}^+,\mathbb{R}^{p+m})\subset\mathcal{W}^{\boldsymbol{T}}$ is the behaviour of the system. $\mathfrak{L}^{p+m}$ is the set for which $\mathfrak{B}$ is the solution set of the following (E,A,B,C) representation [16]:

$$E\dot{x}(t)=Ax(t)+Bu(t)\quad;\quad y(t)=Cx(t) \tag{1}$$

where E and A: $\mathcal{X}\to\underline{\mathcal{X}}$, B: $\mathcal{U}\to\underline{\mathcal{X}}$, and C: $\mathcal{X}\to\mathcal{Y}$ are linear maps. The finite–dimensional spaces $\mathcal{X}$ and $\underline{\mathcal{X}}$ are the descriptor variable and equation spaces.

Representation (1) was introduced by Rosenbrock [26] who called it generalized; it is also usual to call it implicit, descriptor, or singular (see for example [19]). Let us note that:

1. We are working with usual time functions and not with generalized functions (distributions [10, 27]). In this framework there are no impulsions. As consequence, *minimality* is not considered in the context of generalized functions, but in the usual time framework, i. e. through the notion of external minimality [5, 17].
2. The system is defined in $\boldsymbol{T}=\mathbb{R}^+=[0,\ +\infty)$, which implies that the initial conditions are consistent and thus there are no internal switches for introducing initial conditions in the derivative actions, and thus the initial conditions are only present for the integrators.
3. The behaviour is contained in $\mathcal{C}^\infty(\mathbb{R}^+,\mathbb{R}^{p+m})$, which implies that all the involved time signal are smooth, i. e. all their time derivatives exist in $(0,\ +\infty)$. For $t=0$, it is sufficient to ask for the *maximally free variable* (i. e. the input part of the manifest variable) $\lim_{t\to 0_-}u(t)=u(0)$.

Also note that the dimensions of $\mathcal{X}$ and $\underline{\mathcal{X}}$ are not necessarily equal.

Subspaces. Related with the (E, A, B, C) representation (1) are the following subspaces:

The supremal (A, E, B) **invariant subspace contained in** $\mathcal{X}$ $\left(\mathcal{V}^{\mu}_{\mathcal{X}} := \mathcal{V}^{\mu}_{[\mathcal{X}:A,E,B]}\text{, with limit } \mathcal{V}^{*}_{\mathcal{X}}\right)$: $\mathcal{V}^{*}_{\mathcal{X}}$ characterizes (together with $E\mathcal{V}^{*}_{\mathcal{X}} + \mathrm{Im}\, B$) the set of all possible trajectories of (1) which are not identically zero for any input u [5]. Frankowska [13] called strict systems the (E, A, B, C) representations (1) satisfying $\mathcal{V}^{*}_{\mathcal{X}} = \mathcal{X}$.

The supremal (A, E) **invariant subspace contained in** $\mathcal{X}$ $\left(\mathcal{V}^{\mu}_{\mathcal{X}0} := \mathcal{V}^{\mu}_{[\mathcal{X}:A,E]}\text{,with limit } \mathcal{V}^{*}_{\mathcal{X}0}\right)$: Wong [31] and Armentano [3] characterized all the exponential trajectories of (1) by $\mathcal{V}^{*}_{\mathcal{X}0}$ (together with $E\mathcal{V}^{*}_{\mathcal{X}0}$). Loiseau [20] applied this subspace to general pencils in order to study structural properties with the Kronecker theory (see also [21]).

The supremal (A, E) **invariant subspace contained in** $\mathrm{Ker}\, C$ $\left(\mathcal{V}^{\mu}_{0} \coloneqq \mathcal{V}^{\mu}_{[\mathcal{K}_C:A,E]}\text{, with limit } \mathcal{V}^{*}_{0}\right)$: $\mathcal{V}^{*}_{0}$ characterizes (together with $E\mathcal{V}^{*}_{0}$) the set of all exponential trajectories of (1) which are unobservable at the output y [5].

The infimal (E, A) **invariant subspace** $\left(\mathcal{S}^{\mu}_{\mathcal{X}0} = \mathcal{S}^{\mu}_{[\mathcal{X},\mathcal{K}_E:E,A]}\text{, with limit } \mathcal{S}^{*}_{\mathcal{X}0}\right)$: Armentano [3] characterized the set of all trajectories of (1) due to pure differential actions by $\mathcal{S}^{*}_{\mathcal{X}0}$ (together with $A\mathcal{S}^{*}_{\mathcal{X}0}$). Loiseau [20] applied this subspace to general pencils in order to study structural properties with the Kronecker theory (see also [21]).

The supremal almost (A, E) **controllability subspace contained in** $\mathrm{Ker}\, C$ $\left(\mathcal{R}^{\mu}_{a0} = \mathcal{S}^{\mu}_{[\mathcal{K}_C,\mathcal{K}_E:E,A]}\text{, with limit } \mathcal{R}^{*}_{a0}\right)$: $\mathcal{R}^{*}_{a0}$ characterizes (together with $A\mathcal{R}^{*}_{a0}$) the set of all the trajectories of (1) due to pure differential actions with no influence on the input-output trajectories. Bonilla et al. [7] called $\mathcal{R}^{*}_{a0}$ the differential redundant subspace (see also [5]).

1. INTRODUCTION

The (E, A, B, C) representation (1) can describe proper systems, non-proper systems, systems with internal restrictions, and systems with internal structure variations (see for example [1, 8, 9, 11, 18, 29].

When people are interested in the physical implementation of control elements such as control laws, observers, filters, failure detectors, etc..., they are looking for square invertible systems without pure time derivative actions, namely they are looking for regular proper systems. So, let us then recall some definitions and characterizations:

Definition 1. (Gantmacher [14]) A pencil $[\lambda E - A]$ is regular if it is square and it has full generic rank, i. e. $\det[\lambda E - A]$ is not identically zero. Representation (1) is called regular if $[\lambda E - A]$ is regular.

Theorem 1. (Bernhard [4], Armentano [3], and Malabre [21]) A pencil $[\lambda E - A]$ is regular if and only if

$$\mathcal{X} = \mathcal{V}^*_{\mathcal{X}0} \oplus \mathcal{S}^*_{\mathcal{X}0}. \tag{2}$$

Definition 2. (Bernhard [4] and Armentano [3]) The (E, A, B, C) representation (1) is internally proper if the pencil $[\lambda E - A]$ is proper, namely $[\lambda E - A]$ is regular and has no infinite elementary divisor of order greater than 1, i. e. the dynamics of the system includes no derivator.

Kučera and Zagalak [15] characterized the dynamics of all proper systems that can be obtained from a regular representation (1) by applying descriptor variable feedback (see also [32]). Dai [12] called normal the (E, A, B, C) representations which are internally proper and characterized the descriptor variable feedbacks from which closed–loop systems have no infinite poles (no derivators); this property was called normalizability.

Let us point out that this notion of internal properness is related to the absence of pure time derivative actions in (1). In some situations, it is enough to get such a property on the input–output behaviour; for example Aplevich [2] defined the "properness" as the property of having no transmission poles at infinity, that is to say, having no pure time derivative actions in the input–output behaviour.

In this paper we are interested in finding geometric conditions which guarantee the properness concept stated by Aplevich. For this, we distinguish the properness concept of Definition 2, *internal properness*, and the properness concept of Aplevich [2], *external properness*. In Section 2 we formally define the external properness and we characterize it. In Section 3 we give two illustrative examples and, finally, in Section 4 we conclude.

2. EXTERNAL PROPERNESS

In order to formally define the external properness, we need to recall some basic concepts about external equivalence and external minimality:

Definition 3. (Willems [30]) Two representations are called externally equivalent if the corresponding sets of all possible trajectories for the external variables (external behaviours) are the same.

In representations like (1), the external variables are *a priori* split into two parts, $u(t)$ and $y(t)$.

Definition 4. (Kuijper [17] and Bonilla and Malabre [5]) The implicit representation (1), with $\mathcal{X}$ and $\underline{\mathcal{X}}$ not necessarily of the same dimension, is minimal among all externally equivalent representations of the same type if: 1) the corresponding descriptor equation has the least possible number of rows, and 2) the descriptor variable has the least possible number of components.

Bonilla and Malabre [5] showed that the (E, A, B, C) representation is externally equivalent to a minimal one (E_m, A_m, B_m, C_m), called the externally minimal part. Also Kuijper [17] gave necessary and sufficient conditions for external minimality:

Theorem 2. (Kuijper [17]) A given (E, A, B, C) representation is minimal, among all externally equivalent representations of the type (1), if and only if: (i) the matrix $[E\ B]$ is epic, (ii) the matrix $[E^T\ C^T]^T$ is monic, and (iii) the matrix $\left[\begin{smallmatrix} \lambda E - A \\ C \end{smallmatrix}\right]$ has full column rank for all complex number λ.

Theorem 3. (Bonilla and Malabre [5]) Any given (E, A, B, C) representation is externally equivalent to the minimal one (E_m, A_m, B_m, C_m), whose maps are uniquely defined as follows:

$$\begin{gathered} E_m \Pi_m = P_m E; \quad A_m \Pi_m = P_m A; \quad B_m = P_m B; \quad C_m \Pi_m = C \\ \Pi_m : \mathcal{X} \to \mathcal{V}^*_{\mathcal{X}} / (\mathcal{V}^*_0 + \mathcal{V}^*_{\mathcal{X}} \cap \mathcal{R}^*_{a0}) : \text{ canonical projection} \\ P_m : \underline{\mathcal{X}} \to (E\mathcal{V}^*_{\mathcal{X}} + \operatorname{Im} B)/(E\mathcal{V}^*_0 + A(\mathcal{V}^*_{\mathcal{X}} \cap \mathcal{R}^*_{a0})) : \text{ canonical projection.} \end{gathered} \tag{3}$$

In the light of these notions, we associate external properness with the properness of the external behaviour of the representation; more precisely:

Definition 5. The (E, A, B, C) representation (1) is externally proper if its externally minimal part is internally proper.

Let us state the first principal result:

Theorem 4. If $\mathcal{V}^*_0 = \{0\}$ [1] and $\mathcal{V}^*_{\mathcal{X}} = \mathcal{X}$ [2], then (1) is externally proper if and only if:

$$\mathcal{V}^*_{\mathcal{X}0} + \mathcal{S}^*_{\mathcal{X}0} = \mathcal{X}\ , \quad \mathcal{V}^*_{\mathcal{X}0} \cap \mathcal{S}^*_{\mathcal{X}0} \subset \mathcal{R}^*_{a0} \quad \text{and} \quad \dim \left(\frac{\mathcal{V}^*_{\mathcal{X}0} + (E^{-1}A)^2 \mathcal{R}^*_{a0}}{\mathcal{V}^*_{\mathcal{X}0} + E^{-1}A\mathcal{R}^*_{a0}} \right) = 0. \tag{4}$$

Before giving the proof, let us remark that there is no loss of generality when making these two assumptions; we do it, in order to avoid unnecessary algebraic complications. Indeed, if these two assumptions are not fulfilled we can take the quotient by $\mathcal{V}^*_0$ (to take out the maximal unobservable part) and take the restriction to $\mathcal{V}^*_{\mathcal{X}}$ (the set of all no null trajectories). Let us also note that in the case of externally minimal representations, these two assumptions are automatically fulfilled.

Proof. The proof is given in 4 steps:

1. From (2), regularity is equivalent to: $\mathcal{X} = \mathcal{S}^*_{\mathcal{X}0} \oplus \mathcal{V}^*_{\mathcal{X}0}$. Also, the absence of infinite zeros of order greater than one is equivalent to (see [21]):

[1] All the exponential trajectories of (1) are observable.
[2] There are no trajectories identically equal to zero whatever be the input action.

$$\dim\left((\mathcal{V}^*_{\mathcal{X}0}+\mathcal{S}^2_{\mathcal{X}0})/(\mathcal{V}^*_{\mathcal{X}0}+\mathcal{S}^1_{\mathcal{X}0})\right)=0.$$

2. Let us next show that the (E_m, A_m, B_m, C_m) representation is internally proper if and only if:

$$\mathcal{X}=\mathcal{T}^*_1+\mathcal{T}^*_2,\ \ \mathcal{T}^*_1\cap\mathcal{T}^*_2=\text{Ker }\Pi_m \text{ and}$$
$$\dim\left(\tfrac{\mathcal{T}^*_2+\mathcal{T}^2_1}{\mathcal{T}^*_2+\mathcal{T}^1_1}\right)=\dim\left(\tfrac{(\mathcal{T}^*_2+\mathcal{T}^2_1)\cap\text{Ker }\Pi_m}{(\mathcal{T}^*_2+\mathcal{T}^1_1)\cap\text{Ker }\Pi_m}\right)$$

where (recall (ALG–V) and (ALG–S)):

$$\mathcal{T}^\mu_2=\mathcal{V}^\mu_{[\mathcal{X}:A,E,\mathcal{K}_{P_m}]}\ \text{ and }\mathcal{T}^\mu_1=\mathcal{S}^\mu_{[\mathcal{X},\mathcal{K}_{\Pi_m}:\mathcal{K}_{P_m},E,A]},\text{ for }\mu\geq 0.$$

Indeed, from the first item, the minimal (E_m, A_m, B_m, C_m) representation is regular if and only if: $\mathcal{X}_m=\mathcal{S}^*_{\mathcal{X}_m0}\oplus\mathcal{V}^*_{\mathcal{X}_m0}$, where $\mathcal{X}_m=\text{Im }\Pi_m$. Now from (3) and (ALG–S)–$[\mathcal{X}_m : E_m, A_m]$, we get:

$$\begin{aligned}\Pi^{-1}_m\mathcal{S}^{\mu+1}_{\mathcal{X}_m0} &= (E_m\Pi_m)^{-1}A_m\mathcal{S}^\mu_{\mathcal{X}_m0}=E^{-1}P^{-1}_mA_m\Pi_m\Pi^{-1}_m\mathcal{S}^\mu_{\mathcal{X}_m0}\\ &= E^{-1}P^{-1}_mP_mA\Pi^{-1}_m\mathcal{S}^\mu_{\mathcal{X}_m0}=E^{-1}(A\Pi^{-1}_m\mathcal{S}^\mu_{\mathcal{X}_m0}+\text{Ker }P_m)\end{aligned}$$

namely: $\mathcal{T}^\mu_1=\Pi^{-1}_m\mathcal{S}^\mu_{\mathcal{X}_m0}$. In a similar way: $\mathcal{T}^\mu_2=\Pi^{-1}_m\mathcal{V}^\mu_{\mathcal{X}_m0}$. And thus:

$$\mathcal{X}_m=\mathcal{S}^*_{\mathcal{X}_m0}\oplus\mathcal{V}^*_{\mathcal{X}_m0}\text{ if and only if }\mathcal{X}=\mathcal{T}^*_1+\mathcal{T}^*_2\text{ and }\mathcal{T}^*_1\cap\mathcal{T}^*_2=\text{Ker }\Pi_m$$

On the other hand:

$$\begin{aligned}&\dim\left(\frac{\mathcal{V}^*_{\mathcal{X}_m0}+\mathcal{S}^2_{\mathcal{X}_m0}}{\mathcal{V}^*_{\mathcal{X}_m0}+\mathcal{S}^1_{\mathcal{X}_m0}}\right)=\dim\left(\frac{\Pi_m\left(\mathcal{T}^*_2+\mathcal{T}^2_1\right)}{\Pi_m\left(\mathcal{T}^*_2+\mathcal{T}^1_1\right)}\right)\\ =\ &\dim\left(\frac{\mathcal{T}^*_2+\mathcal{T}^2_1}{\mathcal{T}^*_2+\mathcal{T}^1_1}\right)-\dim\left(\frac{(\mathcal{T}^*_2+\mathcal{T}^2_1)\cap\text{Ker }\Pi_m}{(\mathcal{T}^*_2+\mathcal{T}^1_1)\cap\text{Ker }\Pi_m}\right).\end{aligned}$$

And thus

$$\dim\left(\tfrac{\mathcal{V}^*_{\mathcal{X}_m0}+\mathcal{S}^2_{\mathcal{X}_m0}}{\mathcal{V}^*_{\mathcal{X}_m0}+\mathcal{S}^1_{\mathcal{X}_m0}}\right)=0$$
$$\text{if and only if }\dim\left(\tfrac{\mathcal{T}^*_2+\mathcal{T}^2_1}{\mathcal{T}^*_2+\mathcal{T}^1_1}\right)=\dim\left(\tfrac{(\mathcal{T}^*_2+\mathcal{T}^2_1)\cap\text{Ker }\Pi_m}{(\mathcal{T}^*_2+\mathcal{T}^1_1)\cap\text{Ker }\Pi_m}\right).$$

3. Let us now show that:

If $\mathcal{V}^*_0=\{0\}$ and $\mathcal{V}^*_\mathcal{X}=\mathcal{X}$ then $\mathcal{T}^\mu_1=\left(E^{-1}A\right)^\mu\mathcal{R}^*_{a0}$ and $\mathcal{T}^\mu_2=\mathcal{V}^\mu_{\mathcal{X}_0}+\mathcal{R}^*_{a0}$.

Moreover:

$$\mathcal{T}^*_1=\mathcal{S}^*_{\mathcal{X}0}\quad\text{and}\quad\frac{\mathcal{T}^*_2+\mathcal{T}^2_1}{\mathcal{T}^*_2+\mathcal{T}^1_1}=\frac{\mathcal{V}^*_{\mathcal{X}0}+(E^{-1}A)^2\mathcal{R}^*_{a0}}{\mathcal{V}^*_{\mathcal{X}0}+E^{-1}A\mathcal{R}^*_{a0}}.$$

Indeed, from (3) we get for this case $\text{Ker }\Pi_m=\mathcal{R}^*_{a0}$ and $\text{Ker }P_m=A\mathcal{R}^*_{a0}$, and thus $\mathcal{T}^1_1=E^{-1}A\mathcal{R}^*_{a0}$, which implies (recall that $\mathcal{R}^*_{a0}\subset E^{-1}A\mathcal{R}^*_{a0}$): $\mathcal{T}^\mu_1=\left(E^{-1}A\right)^\mu\mathcal{R}^*_{a0}$.

Note that $E^{-1}A\mathcal{R}_{a0}^{*} \supset \operatorname{Ker} E = \mathcal{S}_{\mathcal{X}0}^{0} + \mathcal{R}_{a0}^{*}$. Let us then assume that $\left(E^{-1}A\right)^{\mu}$ $\mathcal{R}_{a0}^{*} \supset \mathcal{S}_{\mathcal{X}0}^{\mu-1} + \mathcal{R}_{a0}^{*}$. This implies $\left(E^{-1}A\right)^{\mu+1}\mathcal{R}_{a0}^{*} \supset \mathcal{S}_{\mathcal{X}0}^{\mu} + E^{-1}A\mathcal{R}_{a0}^{*} \supset \mathcal{S}_{\mathcal{X}0}^{\mu} + \mathcal{R}_{a0}^{*}$. On the other hand, $\mathcal{R}_{a0}^{*} \subset \mathcal{S}_{\mathcal{X}0}^{*}$, implies that: $\left(E^{-1}A\right)^{\mu}\mathcal{R}_{a0}^{*} \subset \mathcal{S}_{\mathcal{X}0}^{*}$. Therefore: $\mathcal{S}_{\mathcal{X}_0}^{*} = \mathcal{S}_{\mathcal{X}_0}^{*} + \mathcal{R}_{a0}^{*} \subset \left(E^{-1}A\right)^{\dim \mathcal{X}}\mathcal{R}_{a0}^{*} \subset \mathcal{S}_{\mathcal{X}_0}^{*}$, i.e. $\mathcal{T}_1^{*} = \left(E^{-1}A\right)^{\dim \mathcal{X}}\mathcal{R}_{a0}^{*} = \mathcal{S}_{\mathcal{X}0}^{*}$.
Now with the view that $\mathcal{T}_2^{0} = \mathcal{X} = \mathcal{V}_{\mathcal{X}0}^{0} = \mathcal{V}_{\mathcal{X}0}^{0} + \mathcal{R}_{a0}^{*}$, let us assume that $\mathcal{T}_2^{\mu} = \mathcal{V}_{\mathcal{X}_0}^{\mu} + \mathcal{R}_{a0}^{*}$. This assumption implies (remember that $E\mathcal{R}_{a0}^{*} \subset A\mathcal{R}_{a0}^{*}$):

$$\begin{aligned}
\mathcal{T}_2^{\mu+1} &= A^{-1}(E\mathcal{V}_{\mathcal{X}0}^{\mu} + E\mathcal{R}_{a0}^{*} + A\mathcal{R}_{a0}^{*}) = A^{-1}(E\mathcal{V}_{\mathcal{X}0}^{\mu} + A\mathcal{R}_{a0}^{*}) \\
&= A^{-1}E\mathcal{V}_{\mathcal{X}0}^{\mu} + \mathcal{R}_{a0}^{*} = \mathcal{V}_{\mathcal{X}0}^{\mu+1} + \mathcal{R}_{a0}^{*}
\end{aligned}$$

namely $\mathcal{T}_2^{\mu} = \mathcal{V}_{\mathcal{X}0}^{\mu} + \mathcal{R}_{a0}^{*}$, which implies $\mathcal{T}_2^{*} = \mathcal{V}_{\mathcal{X}0}^{*} + \mathcal{R}_{a0}^{*}$.
And since $\mathcal{R}_{a0}^{*} \subset (E^{-1}A)^{\eta}\mathcal{R}_{a0}^{*}$, for $\eta \geq 1$, we get: $\mathcal{T}_2^{*} + \mathcal{T}_1^{\mu} = \mathcal{V}_{\mathcal{X}0}^{*} + \mathcal{R}_{a0}^{*} + (E^{-1}A)^{\mu}\mathcal{R}_{a0}^{*} = \mathcal{V}_{\mathcal{X}0}^{*} + (E^{-1}A)^{\mu}\mathcal{R}_{a0}^{*}$.

4. Finally, let us note that (remember that $\mathcal{S}_{\mathcal{X}0}^{*} \supset \mathcal{R}_{a0}^{*}$):

$$\mathcal{T}_1^{*} + \mathcal{T}_2^{*} = \mathcal{S}_{\mathcal{X}0}^{*} + \mathcal{V}_{\mathcal{X}0}^{*} + \mathcal{R}_{a0}^{*} = \mathcal{S}_{\mathcal{X}0}^{*} + \mathcal{V}_{\mathcal{X}0}^{*} \quad \text{and}$$
$$\mathcal{T}_1^{*} \cap \mathcal{T}_2^{*} = \mathcal{S}_{\mathcal{X}_0}^{*} \cap (\mathcal{V}_{\mathcal{X}0}^{*} + \mathcal{R}_{a0}^{*}) = \mathcal{S}_{\mathcal{X}0}^{*} \cap \mathcal{V}_{\mathcal{X}0}^{*} + \mathcal{R}_{a0}^{*}$$

which prove the first two conditions in (4).
Also:

$$\begin{aligned}
\dim\left(\frac{(\mathcal{T}_2^{*} + \mathcal{T}_1^{2}) \cap \operatorname{Ker} \Pi_m}{(\mathcal{T}_2^{*} + \mathcal{T}_1^{1}) \cap \operatorname{Ker} \Pi_m}\right) &= \dim\left(\frac{(\mathcal{V}_{\mathcal{X}0}^{*} + \mathcal{R}_{a0}^{*} + \mathcal{T}_1^{2}) \cap \mathcal{R}_{a0}^{*}}{(\mathcal{V}_{\mathcal{X}0}^{*} + \mathcal{R}_{a0}^{*} + \mathcal{T}_1^{1}) \cap \mathcal{R}_{a0}^{*}}\right) \\
&= \dim\left(\frac{\mathcal{R}_{a0}^{*}}{\mathcal{R}_{a0}^{*}}\right) = 0
\end{aligned}$$

which proves the third condition in (4) and completes the proof. □

Let us note that the three conditions (i)–(iii) of Theorem 2 are related with the subspaces of Theorem 3 as follows (see [6]): (i) iff $E\mathcal{V}_{\mathcal{X}}^{*} + \operatorname{Im} B = \underline{\mathcal{X}}$, (ii) iff $\mathcal{R}_{a0}^{*} = \{0\}$, and (iii) iff $\mathcal{V}_0^{*} = \{0\}$[3].
In addition, if the representation is regular[4], we get:

[3] When there are no algebraic restrictions on the input space $\mathcal{U}$ (equivalently $E^{-1}\operatorname{Im} B \subset \mathcal{V}_{\mathcal{X}}^{*}$), (i) is equivalent to $\mathcal{V}_{\mathcal{X}}^{*} = \mathcal{X}$ (the system is strict in the sense of Frankowska [13]).
[4] In this case, the behaviour is specified in the following more explicit way: $\Sigma = (\mathbb{R}^{+}, \mathbb{R}^{m+p}, \mathfrak{B})$, with $\mathfrak{B} = \mathfrak{B}_{[A,B,C,D]}^{\exp} \oplus \mathfrak{B}_{[N,\Gamma,\Theta]}^{\mathrm{pol}}$, where the exponential and polynomial behaviours, $\mathfrak{B}_{[A,B,C,D]}^{\exp}$ and $\mathfrak{B}_{[N,\Gamma,\Theta]}^{\mathrm{pol}}$, are defined as:

$$\begin{aligned}
\mathfrak{B}_{[A,B,C,D]}^{\exp} = \Big\{&(w_{\mathrm{in}}, w_{\mathrm{out}}^{\exp}) \in \mathcal{C}^{\infty}(\mathbb{R}^{+}, \mathbb{R}^{m+p}) \big| \ \exists\, x_{e,0} \in \mathbb{R}^{n_e}, \\
&w_{\mathrm{out}}^{\exp}(t) = C\Big(\mathrm{e}^{At}x_{e,0} + \textstyle\int_0^t \mathrm{e}^{A(t-\tau)}Bw_{\mathrm{in}}(\tau)\mathrm{d}\tau\Big) + Dw_{\mathrm{in}}(t)\Big\}
\end{aligned}$$

$$\mathfrak{B}_{[N,\Gamma,\Theta]}^{\mathrm{pol}} = \Big\{(w_{\mathrm{in}}, w_{\mathrm{out}}^{\mathrm{pol}}) \in \mathcal{C}^{\infty}(\mathbb{R}^{+}, \mathbb{R}^{m+p}) \big| \ w_{\mathrm{out}}^{\mathrm{pol}}(t) = \Theta \textstyle\sum_{j=1}^{n_p-1} N^j \Gamma \frac{\mathrm{d}^j}{\mathrm{d}t^j} w_{\mathrm{in}}(t)\Big\}$$

where w_{in} is the input variable and $w_{\mathrm{out}}^{\exp}$ and $w_{\mathrm{out}}^{\mathrm{pol}}$ are the output variables.

Corollary 1. If $\mathcal{V}_0^* = \{0\}$ [1], $\mathcal{V}_{\mathcal{X}}^* = \mathcal{X}$ [2], and $\mathcal{X} = \mathcal{V}_{\mathcal{X}0}^* \oplus \mathcal{S}_{\mathcal{X}0}^*$ [5], then (1) is externally proper if and only if

$$E^{-1}A\mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}0}^*. \tag{5}$$

Proof. If the implicit representation (1) is proper, we get from (2): $\mathcal{X} = \mathcal{V}_{\mathcal{X}0}^* \oplus \mathcal{S}_{\mathcal{X}0}^*$. In this case, the first two conditions in (4) are automatically satisfied. Since $(E^{-1}A)^\mu \mathcal{R}_{a0}^* \subset (E^{-1}A)^{\dim \mathcal{X}} \mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}_0}^*$, we get: $\mathcal{V}_{\mathcal{X}0}^* \cap (E^{-1}A)^\mu \mathcal{R}_{a0}^* = \mathcal{V}_{\mathcal{X}_0}^* \cap \mathcal{S}_{\mathcal{X}0}^* \cap (E^{-1}A)^\mu \mathcal{R}_{a0}^* = \{0\}$. Then:

$$\dim\left(\frac{\mathcal{V}_{\mathcal{X}0}^* + (E^{-1}A)^2\mathcal{R}_{a0}^*}{\mathcal{V}_{\mathcal{X}_0}^* + E^{-1}A\mathcal{R}_{a0}^*}\right) = \dim\left(\frac{\mathcal{V}_{\mathcal{X}0}^* \oplus (E^{-1}A)^2\mathcal{R}_{a0}^*}{\mathcal{V}_{\mathcal{X}0}^* \oplus E^{-1}A\mathcal{R}_{a0}^*}\right) = \dim\left(\frac{(E^{-1}A)^2\mathcal{R}_{a0}^*}{E^{-1}A\mathcal{R}_{a0}^*}\right).$$

Thus

$$\dim\left(\frac{\mathcal{V}_{\mathcal{X}0}^* + (E^{-1}A)^2\mathcal{R}_{a0}^*}{\mathcal{V}_{\mathcal{X}0}^* + E^{-1}A\mathcal{R}_{a0}^*}\right) = 0$$

$$\text{if and only if}\quad E^{-1}A\mathcal{R}_{a0}^* = (E^{-1}A)^2\mathcal{R}_{a0}^* = (E^{-1}A)^{\dim \mathcal{X}}\mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}0}^*. \qquad \square$$

Let us note that in [7] the sufficient condition $\mathcal{R}_{a0}^* = \mathcal{S}_{\mathcal{X}0}^*$ was used.

3. ILLUSTRATIVE EXAMPLES

Example 1. Let us consider the system of Figure 1, which (E, A, B, C) representation is ($\bar{x} = \left[\begin{array}{cc|c} x & z & \xi^T \end{array}\right]^T$):

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \dot{\bar{x}} = \left[\begin{array}{cc|ccc} -\beta & -\varepsilon^2 & 0 & 0 & 0 \\ 1/\varepsilon & -1/\varepsilon & 0 & 0 & 1/\varepsilon \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \bar{x} + \left[\begin{array}{c} 0 \\ 0 \\ \hline -1 \\ 0 \\ 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \end{array}\right] \bar{x}. \tag{6}$$

In order to enhance the involved subspaces, let us pre-multiply (6) by L and let us apply the change of variable $\overline{x} = R\widetilde{x}$, with (see [24] for computation details): $R = \left[\begin{array}{c|ccc} \mathrm{I}_2 & \begin{array}{c} -\varepsilon \\ -1/\varepsilon^2 \end{array} & \begin{array}{c} 0 \\ 1/\varepsilon \end{array} & \begin{array}{c} 0 \\ 0 \end{array} \\ \hline 0 & & \mathrm{I}_3 & \end{array}\right]$ and $L = \left[\begin{array}{c|ccc} \mathrm{I}_2 & \begin{array}{c} -\varepsilon\beta - 1 \\ 1 - 1/\varepsilon^3 \end{array} & \begin{array}{c} \varepsilon \\ 1/\varepsilon^2 \end{array} & \begin{array}{c} 0 \\ -1/\varepsilon \end{array} \\ \hline 0 & & \mathrm{I}_3 & \end{array}\right]$. We obtain in this way:

$$\left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \dot{\widetilde{x}} = \left[\begin{array}{cc|ccc} -\beta & -\varepsilon^2 & 0 & 0 & 0 \\ 1/\varepsilon & -1/\varepsilon & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right] \widetilde{x} + \left[\begin{array}{c} (\varepsilon\beta + 1) \\ (1/\varepsilon^3 - 1) \\ \hline -1 \\ 0 \\ 0 \end{array}\right] u$$

$$y = \left[\begin{array}{cc|ccc} 0 & 1 & -1/\varepsilon^2 & 1/\varepsilon & 0 \end{array}\right] \widetilde{x}. \tag{7}$$

[5] The pencil $[\lambda E - A]$ is regular.

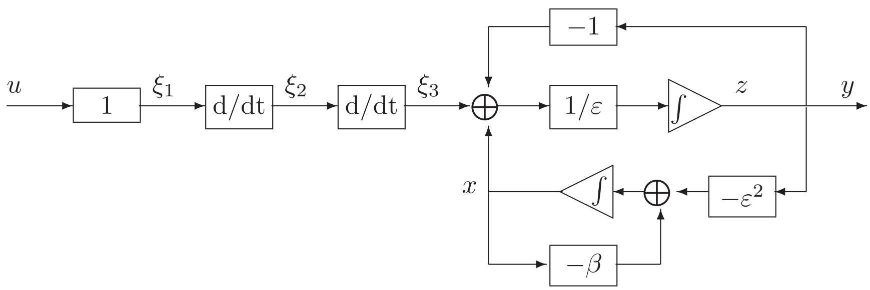

Fig. 1. Example 1.

Applying algorithms (ALG–S)–$[\mathcal{K}_C, \mathcal{K}_E : E, A]$ and (ALG–S)–$[\mathcal{X}, \mathcal{K}_E : E, A]$ to (7), we get:

$$\mathcal{R}^*_{a0} = \{e_5\}\,; \quad E^{-1}A\mathcal{R}^*_{a0} = \{e_4, e_5\}\,; \quad \mathcal{S}^*_{\mathcal{X}0} = \{e_3, e_4, e_5\}$$

We can see that: $\mathcal{S}^*_{\mathcal{X}0} \neq E^{-1}A\mathcal{R}^*_{a0}$. Then from Corollary 1, (6) is not externally proper. Indeed, its transfer function is non proper:

$$G(\mathrm{s}) = \mathrm{s}^2 \frac{\mathrm{s} + \beta}{\varepsilon \mathrm{s}^2 + (1 + \beta\varepsilon)\mathrm{s} + \beta + \varepsilon^2}$$

Example 2. Let us consider the system of Figure 2, which (E, A, B, C) representation is ($\bar{x} = \left[\begin{array}{cc|c} x & z^T & \xi^T \end{array}\right]^T$):

$$\begin{aligned}
\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \dot{\overline{x}} &= \left[\begin{array}{ccc|ccc} -\beta & -\varepsilon^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon & 0 & 0 & 1/\varepsilon^2 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \overline{x} \\
&\quad + \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right]^T u \\
y &= \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right] \overline{x}.
\end{aligned} \tag{8}$$

In order to enhance the involved subspaces, let us pre-multiply (6) by L' and let us apply the change of variable $\overline{x} = R'\widetilde{x}$, with (see [24] for computation details): $R' = \left[\begin{array}{c|ccc} \mathrm{I}_3 & \begin{matrix} 0 \\ 1/\varepsilon^2 \\ -2/\varepsilon^3 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1/\varepsilon^2 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline 0 & & \mathrm{I}_3 & \end{array}\right]$ and $L' = \left[\begin{array}{c|ccc} \mathrm{I}_3 & \begin{matrix} 1 \\ 2/\varepsilon^3 \\ -3/\varepsilon^4 \end{matrix} & \begin{matrix} 0 \\ -1/\varepsilon^2 \\ 2/\varepsilon^3 \end{matrix} & \begin{matrix} 0 \\ 0 \\ -1/\varepsilon^2 \end{matrix} \\ \hline 0 & & \mathrm{I}_3 & \end{array}\right]$. We obtain

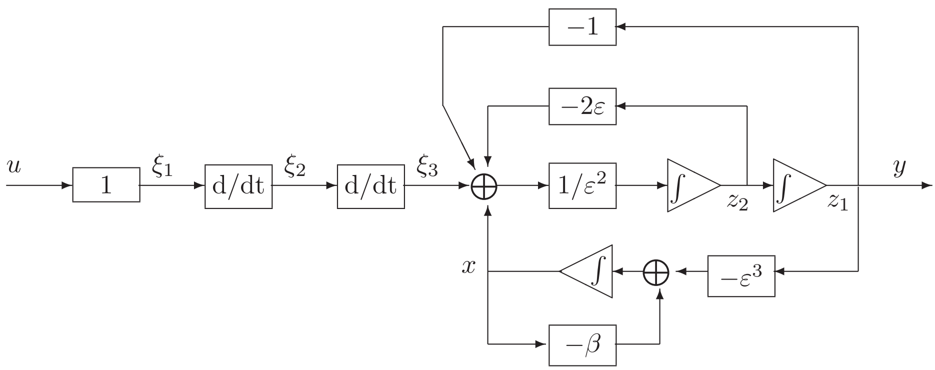

Fig. 2. Example 2.

in this way:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \dot{\widetilde{x}} = \left[\begin{array}{ccc|ccc} -\beta & -\varepsilon^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \widetilde{x}$$
$$+ \left[\begin{array}{ccc|ccc} -1 & -2/\varepsilon^3 & 3/\varepsilon^4 & -1 & 0 & 0 \end{array}\right] u$$
$$y = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1/\varepsilon^2 & 0 & 0 \end{array}\right] \widetilde{x}. \tag{9}$$

Applying algorithms (ALG–S)–$[\mathcal{K}_C, \mathcal{K}_E : E, A]$ and (ALG–S)–$[\mathcal{X}, \mathcal{K}_E : E, A]$ to (9), we get:

$$\mathcal{R}^*_{a0} = \{e_5, e_6\}\,; \quad E^{-1}A\mathcal{R}^*_{a0} = \{e_4, e_5, e_6\}\,; \quad \mathcal{S}^*_{\mathcal{X}0} = \{e_4, e_5, e_6\}$$

We can see that $\mathcal{S}^*_{\mathcal{X}0} = E^{-1}A\mathcal{R}^*_{a0}$. Then from Corollary 1, (8) is externally proper. Indeed, it is externally equivalent to the following state space representation (see Figure 3):

$$\dot{\widehat{x}} = \begin{bmatrix} -\beta & -\varepsilon^2 & 0 \\ 0 & 0 & 1 \\ 1/\varepsilon^2 & -1/\varepsilon^2 & -2/\varepsilon \end{bmatrix} \widehat{x} + \begin{bmatrix} -1 \\ -2/\varepsilon^3 \\ 3/\varepsilon^4 \end{bmatrix} u \tag{10}$$
$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \widehat{x} + (1/\varepsilon^2)u.$$

Its transfer function is proper:

$$G(\mathrm{s}) = \mathrm{s}^2 \frac{\mathrm{s} + \beta}{\varepsilon^2\mathrm{s}^3 + (\varepsilon^2\beta + 2\varepsilon)\mathrm{s}^2 + (2\beta\varepsilon + 1)\mathrm{s} + \beta + \varepsilon^3}$$

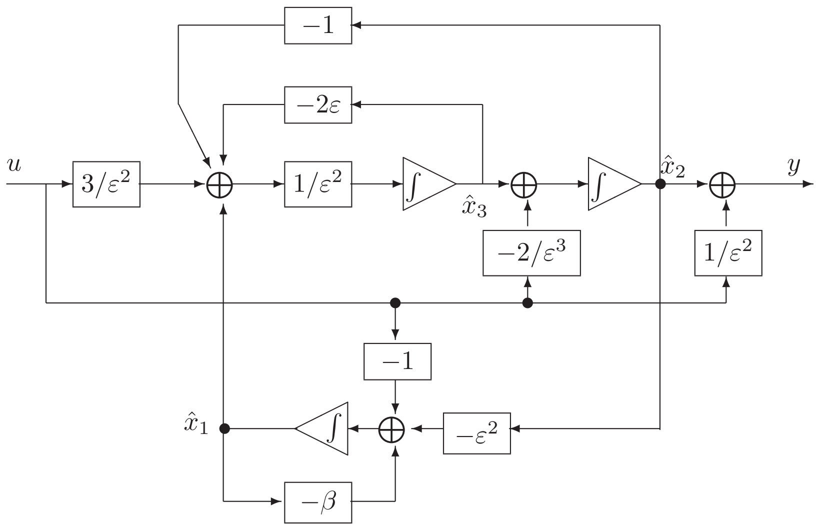

Fig. 3. Equivalent system of (8).

4. CONCLUSION

In this paper we have revisited the structural concept of properness. We have distinguished between the properness of the whole representation (here named *internal properness*) and the properness of its externally minimal part (here named *external properness*). We have given geometric characterizations for *external properness*.

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