KYBERNETIKA — VOLUME *44* (2008), NUMBER 1, PAGES 113–130

SEPARATION OF CONVEX POLYHEDRAL SETS WITH COLUMN PARAMETERS

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Separation is a famous principle and separation properties are important for optimization theory and various applications. In practice, input data are rarely known exactly and it is advisable to deal with parameters. In this article, we are concerned with the basic characteristics (existence, description, stability etc.) of separating hyperplanes of two convex polyhedral sets depending on parameters. We study the case, when parameters are situated in one column of the constraint matrix from the description of the given convex polyhedral set. We provide also a lot of examples carried out on PC.

Keywords: separating hyperplane, parameters, convex polyhedra, solution set, stability set

AMS Subject Classification: 90C31, 39B82, 52B99

1. INTRODUCTION

There are several kinds of separability of convex sets (cf. [8]). For the purpose of this paper it is convenient to introduce the following one.

Definition 1. Convex sets $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^n$ are called *strongly separable* if $\dim \mathcal{X} = \dim \mathcal{Y} = n$ and there exists a hyperplane $\mathcal{R} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{r}^T\boldsymbol{x} = s\}$ such that $\mathcal{X} \subseteq \overline{\mathcal{R}^-} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{r}^T\boldsymbol{x} \le s\}$, and $\mathcal{Y} \subseteq \overline{\mathcal{R}^+} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{r}^T\boldsymbol{x} \ge s\}$ hold. $\mathcal{R}$ is called *the separating hyperplane* of the sets $\mathcal{X}$, $\mathcal{Y}$.

We will use the following well known separation theorem (see e. g. [3, 7, 10]):

Theorem 1. Convex sets $\mathcal{X}$, $\mathcal{Y} \subset \mathbb{R}^n$ are strongly separable if and only if $\dim \mathcal{X} = \dim \mathcal{Y} = n$, and $\operatorname{int} \mathcal{X} \cap \operatorname{int} \mathcal{Y} = \emptyset$.

In this paper we study the strong separability of two convex polyhedral sets ($\widetilde{\boldsymbol{A}} \in \mathbb{R}^{m\times n}$, $\widetilde{\boldsymbol{C}} \in \mathbb{R}^{l\times n}$, $\widetilde{\boldsymbol{b}} \in \mathbb{R}^m$, $\widetilde{\boldsymbol{d}} \in \mathbb{R}^l$):

$$\widetilde{\mathcal{M}}_1 \equiv \{\boldsymbol{x} \in \mathbb{R}^n \mid \widetilde{\boldsymbol{A}}\boldsymbol{x} \le \widetilde{\boldsymbol{b}}\}, \tag{1}$$

$$\widetilde{\mathcal{M}}_2 \equiv \{\boldsymbol{x} \in \mathbb{R}^n \mid \widetilde{\boldsymbol{C}}\boldsymbol{x} \le \widetilde{\boldsymbol{d}}\}. \tag{2}$$

The first attempt to systematically study separation under uncertainty was done in [6], where we derived the basic separation properties of the sets (1), (2) with parameters on the right-hand side of inequalities. Some of there obtained results, which we need in this paper, will be presented in this section. In the following sections we study separation for the case when there are parameters in one column of the matrix $\widetilde{\boldsymbol{A}}$. Parameters in one line (column or row) is the most general case which still leads to quite strong results. In dealing with parameters, we are inspired by [1, 2, 9]. We will define so called solution set (Section 2) and in the sequel the so called stability sets (Section 4) and derive their description. The terms "solution" and "stability set" are taken over from [9], but the meaning is a bit different (we do not work with an objective function). Many examples of stability sets that were carried out on PC are presented in tables at the end of the paper. The Section 6 gives an application in the field of multiobjective programming.

Let us introduce some notation. Given a matrix $\boldsymbol{M}$, the expressions $\boldsymbol{M}_{i,\cdot}$, $\boldsymbol{M}_{\cdot,j}$ denote the ith row and the jth column of the matrix $\boldsymbol{M}$, respectively. Vector $\boldsymbol{e}_k$ denotes the kth unit vector. For given vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^k$, the expression $\boldsymbol{a} < \boldsymbol{b}$ means $a_i < b_i \;\forall i$. For any set $\mathcal{X}$ let us denote by $\overline{\mathcal{X}}$, $\operatorname{int}\mathcal{X}$, $\dim\mathcal{X}$, and $\operatorname{conv}\mathcal{X}$ the closure, the interior, the dimension, and the convex hull of $\mathcal{X}$, respectively. A sign of a real number $r \in \mathbb{R}$ is defined as

$$\operatorname{sgn}(r) = \begin{cases} 0 & r = 0, \\ 1 & r > 0, \\ -1 & r < 0. \end{cases}$$

Definition 2. A *basis* of a convex polyhedral set described by $\boldsymbol{Mx} = \boldsymbol{v}$, $\boldsymbol{x} \geq \boldsymbol{0}$ ($\boldsymbol{M} \in \mathbb{R}^{m\times n}$, $\boldsymbol{v} \in \mathbb{R}^m$, $m \leq n$) is any vector $B \in \{1,\ldots,n\}^m$ for which $\operatorname{rank}(\boldsymbol{M}_B) = m$ holds (where $\boldsymbol{M}_B$ means the restriction of the matrix $\boldsymbol{M}$ to the basic columns). A basis B is *feasible* if $\boldsymbol{M}_B^{-1}\boldsymbol{v} \geq \boldsymbol{0}$.

A sub-basis of the convex polyhedral set described by $\boldsymbol{Mx} \leq \boldsymbol{v}$ ($\boldsymbol{M} \in \mathbb{R}^{m\times n}$, $\boldsymbol{v} \in \mathbb{R}^m$) is any vector $S \in \{1,\ldots,m\}^k$, $1 \leq k \leq n$, for which $\operatorname{rank}(\boldsymbol{M}_S) = k$ holds (where $\boldsymbol{M}_S$ in this case means the restriction of the matrix $\boldsymbol{M}$ to the sub-basic rows). A sub-basis S is called *feasible* if $\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{M}_S\boldsymbol{x} = \boldsymbol{v}_S,\ \boldsymbol{M}_N\boldsymbol{x} \leq \boldsymbol{v}_N\} \neq \emptyset$ for $N = \{1,\ldots,m\} \setminus S$. A basis of $\boldsymbol{Mx} \leq \boldsymbol{v}$ is any n-elemental sub-basis.

Let us introduce

$$\mathcal{Q}^* \equiv \left\{ (\boldsymbol{u}, \boldsymbol{v}, v_{l+1}) \in \mathbb{R}^{m+l+1} \;\middle|\; \begin{pmatrix} \widetilde{\boldsymbol{A}}^T & \widetilde{\boldsymbol{C}}^T & \boldsymbol{0} \\ \widetilde{\boldsymbol{b}}^T & \widetilde{\boldsymbol{d}}^T & 1 \\ \boldsymbol{1}^T & \boldsymbol{1}^T & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ v_{l+1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ 0 \\ 1 \end{pmatrix},\ \begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ v_{l+1} \end{pmatrix} \geq \boldsymbol{0} \right\}.$$

With the help of the convex polytope $\mathcal{Q}^*$ we can describe all separating hyperplanes of $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$. Theorem 2 and Theorem 3 come from [6], but they have origin in [4, 5].

Theorem 2. Suppose that $\dim\widetilde{\mathcal{M}}_1 = \dim\widetilde{\mathcal{M}}_2 = n$, $\operatorname{int}\widetilde{\mathcal{M}}_1 \cap \operatorname{int}\widetilde{\mathcal{M}}_2 = \emptyset$. Let $(\boldsymbol{u}, \boldsymbol{v}, v_{l+1}) \in \mathcal{Q}^*$, $\boldsymbol{u}^T\widetilde{\boldsymbol{A}} \neq \boldsymbol{0}^T$, and $\eta \in \langle 0, v_{l+1}\rangle$ is arbitrary. Then

$$\mathcal{R} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{u}^T(\widetilde{\boldsymbol{A}}\boldsymbol{x} - \widetilde{\boldsymbol{b}}) = \eta\} \tag{3}$$

represents a separating hyperplane of the convex polyhedral sets $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$. Conversely, any separating hyperplane $\mathcal{R}$ of $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ can be expressed in the form of (3) for a certain $(\boldsymbol{u}, \boldsymbol{v}, v_{l+1}) \in \mathcal{Q}^*$, $\boldsymbol{u}^T \widetilde{\boldsymbol{A}} \neq \boldsymbol{0}^T$, and $\eta \in \langle 0, v_{l+1} \rangle$.

Theorem 3. Let $\dim \widetilde{\mathcal{M}}_1 = \dim \widetilde{\mathcal{M}}_2 = n$. Then the convex sets $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ are strongly separable if and only if $\mathcal{Q}^* \neq \emptyset$.

2. SOLUTION SET

From now on, we study the situation, when there are parameters in one column of the matrix $\widetilde{\boldsymbol{A}}$ from (1) instead of fixed values. We can assume without loss of generality that parameters are situated in the last column of $\widetilde{\boldsymbol{A}}$, i. e., $\widetilde{\boldsymbol{A}} = (\boldsymbol{A}\ \boldsymbol{\delta})$ for fixed matrix $\boldsymbol{A} \in \mathbb{R}^{m \times (n-1)}$ and vector of parameters $\boldsymbol{\delta} \in \mathbb{R}^m$. The problem will not be more complicated if there are parameters in the last column of the matrix $\widetilde{\boldsymbol{C}}$ from (2) as well, i. e., $\widetilde{\boldsymbol{C}} = (\boldsymbol{C}\ \boldsymbol{\mu})$ for fixed $\boldsymbol{C} \in \mathbb{R}^{l \times (n-1)}$ and vector of parameters $\boldsymbol{\mu} \in \mathbb{R}^l$. Let us introduce the family of convex polyhedral sets

$$\mathcal{M}_1(\boldsymbol{\delta}) \equiv \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\delta} x_n \leq \boldsymbol{b}\}, \tag{4}$$

$$\mathcal{M}_2(\boldsymbol{\mu}) \equiv \{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \boldsymbol{C}\boldsymbol{x} + \boldsymbol{\mu} x_n \leq \boldsymbol{d}\}, \tag{5}$$

where $\boldsymbol{b} \in \mathbb{R}^m$, $\boldsymbol{d} \in \mathbb{R}^l$. Assume that matrices $(\boldsymbol{A}\ \boldsymbol{b})$, $(\boldsymbol{C}\ \boldsymbol{d})$ do not contain the zero row.

Furthermore let us introduce

$$\mathcal{M}_1 \equiv \{\boldsymbol{x} \in \mathbb{R}^{n-1} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}, \tag{6}$$

$$\mathcal{M}_2 \equiv \{\boldsymbol{x} \in \mathbb{R}^{n-1} \mid \boldsymbol{C}\boldsymbol{x} \leq \boldsymbol{d}\}. \tag{7}$$

The following statements hold trivially.

If $\mathcal{M}_1 \neq \emptyset$, then $\mathcal{M}_1(\boldsymbol{\delta}) \neq \emptyset\ \forall\, \boldsymbol{\delta} \in \mathbb{R}^m$ (since when $\boldsymbol{x} \in \mathcal{M}_1$, then $(\boldsymbol{x}, 0) \in \mathcal{M}_1(\boldsymbol{\delta})$). If $\dim \mathcal{M}_1 = n - 1$, then $\dim \mathcal{M}_1(\boldsymbol{\delta}) = n\ \forall\, \boldsymbol{\delta} \in \mathbb{R}^m$ (since when $\boldsymbol{x} \in \operatorname{int} \mathcal{M}_1$, then $(\boldsymbol{x}, 0) \in \operatorname{int} \mathcal{M}_1(\boldsymbol{\delta})$). Analogously for the set $\mathcal{M}_2$.

Definition 3. *The solution set* (for the strong separability of the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\delta})$ from (4) and $\mathcal{M}_2(\boldsymbol{\mu})$ from (5)) is the set of all $(\boldsymbol{\delta}, \boldsymbol{\mu}) \in \mathbb{R}^{m+l}$ such that the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\delta})$, $\mathcal{M}_2(\boldsymbol{\mu})$ are strongly separable.

Let us introduce

$$\mathcal{P}_1 \equiv \{\boldsymbol{\delta} \in \mathbb{R}^m \mid \dim \mathcal{M}_1(\boldsymbol{\delta}) = n\}, \tag{8}$$

$$\mathcal{P}_2 \equiv \{\boldsymbol{\mu} \in \mathbb{R}^l \mid \dim \mathcal{M}_2(\boldsymbol{\mu}) = n\}. \tag{9}$$

$$\mathcal{P} \equiv \mathcal{P}_1 \times \mathcal{P}_2 = \{(\boldsymbol{\delta}, \boldsymbol{\mu}) \in \mathbb{R}^{m+l} \mid \dim \mathcal{M}_1(\boldsymbol{\lambda}) = \dim \mathcal{M}_2(\boldsymbol{\mu}) = n\}, \tag{10}$$

$$\mathcal{U} \equiv \{(\boldsymbol{\delta}, \boldsymbol{\mu}) \in \mathbb{R}^{m+l} \mid \operatorname{int} \mathcal{M}_1(\boldsymbol{\delta}) \cap \operatorname{int} \mathcal{M}_2(\boldsymbol{\mu}) \neq \emptyset\}. \tag{11}$$

From Theorem 1, Theorem 3, and from definition of the sets $\mathcal{P}$ and $\mathcal{U}$ we get the following assertion.

Assertion 1.

(i) The solution set for $\mathcal{M}_1(\boldsymbol{\delta}),\ \mathcal{M}_2(\boldsymbol{\mu})$ is equal to $\mathcal{P} \setminus \mathcal{U}$.

(ii) We have $\mathcal{U} \subseteq \mathcal{P}$.

Assertion 2. If $\dim \mathcal{M}_1 = \dim \mathcal{M}_2 = n-1$ and the convex polyhedral sets $\mathcal{M}_1$ and $\mathcal{M}_2$ are not strongly separable, then the solution set for $\mathcal{M}_1(\boldsymbol{\delta}),\ \mathcal{M}_2(\boldsymbol{\mu})$ is empty.

Proof. From the assumptions of the assertion it follows that there exists a point $\boldsymbol{x}^0 \in \mathbb{R}^{n-1}$ such that $\boldsymbol{x}^0 \in \operatorname{int} \mathcal{M}_1 \cap \operatorname{int} \mathcal{M}_2$. Hence for all $\boldsymbol{\delta} \in \mathcal{P}_1,\ \boldsymbol{\mu} \in \mathcal{P}_2$ the inclusion $(\boldsymbol{x}^0, 0) \in \operatorname{int} \mathcal{M}_1(\boldsymbol{\delta}) \cap \operatorname{int} \mathcal{M}_2(\boldsymbol{\mu})$ holds and therefore the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\delta})$, $\mathcal{M}_2(\boldsymbol{\mu})$ are not strongly separable. □

Now we will be concerned with the description of the set $\mathcal{P}_1$. The description of $\mathcal{P}_2$ and $\mathcal{P}$ will be analogous.

Theorem 4. The set $\mathcal{P}_1$ has the description

$$\mathcal{P}_1 = \mathcal{V}_1 \cup -\mathcal{V}_1, \tag{12}$$

where

$$\mathcal{V}_1 = \{\boldsymbol{\delta} \in \mathbb{R}^m \mid \boldsymbol{h}_i^T \boldsymbol{\delta} > 0 \ \ \forall i \in I\} \tag{13}$$

and $\boldsymbol{h}_i,\ i \in I$, are extremal directions (vectors in directions of unbounded edges) of the convex polyhedral cone

$$\mathcal{N}_{\boldsymbol{A},\boldsymbol{b}} \equiv \{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T \boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{b}^T \boldsymbol{y} \le 0,\ \boldsymbol{y} \ge \boldsymbol{0}\}. \tag{14}$$

Proof. $\mathcal{P}_1$ is the set of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which $\operatorname{int} \mathcal{M}_1(\boldsymbol{\delta}) \neq \emptyset$ or equivalently

$$\{(\boldsymbol{x}, x_n) \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\delta} x_n \le \boldsymbol{b} - \boldsymbol{\varepsilon}\} \neq \emptyset \tag{15}$$

for an infinitesimal vector $\boldsymbol{\varepsilon} > \boldsymbol{0}$. The situation (15) holds for a vector $\boldsymbol{\delta}$ if and only if the problem

$$\max \{\boldsymbol{0}^T \boldsymbol{x} + 0 x_n \mid \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\delta} x_n \le \boldsymbol{b} - \boldsymbol{\varepsilon}\}$$

has an optimal solution. It follows from the theory of duality in linear programming that this is equivalent to the condition, that the problem

$$\min \{(\boldsymbol{b} - \boldsymbol{\varepsilon})^T \boldsymbol{y} \mid \boldsymbol{A}^T \boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{\delta}^T \boldsymbol{y} = 0,\ \boldsymbol{y} \ge \boldsymbol{0}\} \tag{16}$$

has an optimal solution. The set of feasible solutions to the problem (16) forms a convex polyhedral cone. Therefore the problem (16) has an optimal solution if and only if

$$\{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T \boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{\delta}^T \boldsymbol{y} = 0,\ \boldsymbol{y} \ge \boldsymbol{0},\ (\boldsymbol{b} - \boldsymbol{\varepsilon})^T \boldsymbol{y} < 0\} = \emptyset,$$

or, equivalently

$$\{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{\delta}^T\boldsymbol{y} = 0,\ \boldsymbol{y} \geq \boldsymbol{0},\ \boldsymbol{b}^T\boldsymbol{y} \leq 0,\ \boldsymbol{y} \neq \boldsymbol{0}\} = \emptyset.$$

Hence $\mathcal{P}_1$ is the set of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which

$$\{\boldsymbol{y} \in \mathcal{N}_{\boldsymbol{A},\boldsymbol{b}} \mid \boldsymbol{\delta}^T\boldsymbol{y} = 0\} = \{\boldsymbol{0}\} \tag{17}$$

holds. We claim that $\mathcal{P}_1 = \mathcal{V}_1 \cup -\mathcal{V}_1$.

Let $\boldsymbol{\delta}^0 \in \mathcal{V}_1$. Then $\boldsymbol{h}_i^T\boldsymbol{\delta}^0 > 0\ \forall i \in I$. Each nontrivial vector $\boldsymbol{y} \in \mathcal{N}_{\boldsymbol{A},\boldsymbol{b}}$ can be expressed as a linear combination $\boldsymbol{y} = \sum_{i\in I}\alpha_i\boldsymbol{h}_i$ for certain $\alpha_i \geq 0,\ \sum_{i\in I}\alpha_i > 0$. Therefore

$$\boldsymbol{y}^T\boldsymbol{\delta}^0 = \sum_{i\in I}\alpha_i\boldsymbol{h}_i^T\boldsymbol{\delta}^0 > 0$$

and the condition (17) holds. Analogously for $\boldsymbol{\delta}^0 \in -\mathcal{V}_1$.

Conversely, let $\boldsymbol{\delta}^0 \in \mathbb{R}^m$ and suppose that the condition (17) holds. Then either $\boldsymbol{y}^T\boldsymbol{\delta}^0 > 0$ for all nontrivial $\boldsymbol{y} \in \mathcal{N}_{\boldsymbol{A},\boldsymbol{b}}$ or $\boldsymbol{y}^T\boldsymbol{\delta}^0 < 0$ for all nontrivial $\boldsymbol{y} \in \mathcal{N}_{\boldsymbol{A},\boldsymbol{b}}$. In the first case we specially have $\boldsymbol{h}_i^T\boldsymbol{\delta}^0 > 0\ \forall i \in I$ and thus $\boldsymbol{\delta}^0 \in \mathcal{V}_1$. In the second case we analogously have $\boldsymbol{\delta}^0 \in -\mathcal{V}_1$. □

Theorem 5. The set $\mathcal{U}$ has the description

$$\mathcal{U} = \mathcal{U}_1 \cup -\mathcal{U}_1$$

where

$$\mathcal{U}_1 = \{(\boldsymbol{\delta},\boldsymbol{\mu}) \in \mathbb{R}^{m+l} \mid \boldsymbol{h}_i^T\boldsymbol{\delta} + \boldsymbol{g}_i^T\boldsymbol{\mu} > 0\ \ \forall i \in I\} \tag{18}$$

and $(\boldsymbol{h}_i^T, \boldsymbol{g}_i^T),\ i \in I$, are extremal directions of the convex polyhedral cone

$$\{(\boldsymbol{y},\boldsymbol{z}) \in \mathbb{R}^{m+l} \mid \boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{C}^T\boldsymbol{z} = \boldsymbol{0},\ \boldsymbol{b}^T\boldsymbol{y} + \boldsymbol{d}^T\boldsymbol{z} \leq 0,\ \boldsymbol{y},\boldsymbol{z} \geq \boldsymbol{0}\}. \tag{19}$$

P r o o f. The set $\mathcal{U}$ can be rewritten as $\mathcal{U} = \{(\boldsymbol{\delta},\boldsymbol{\mu}) \in \mathbb{R}^{m+l} \mid \dim(\mathcal{M}_1(\boldsymbol{\delta}) \cap \mathcal{M}_2(\boldsymbol{\mu})) = n\}$. When we apply Theorem 4 to the family of convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\delta}) \cap \mathcal{M}_2(\boldsymbol{\mu})$, we obtain the resulting description of the set $\mathcal{U}$. □

Let us introduce

$$\mathcal{P}_1' \equiv \{\boldsymbol{\delta} \in \mathbb{R}^m \mid \mathcal{M}_1(\boldsymbol{\delta}) \neq \emptyset\},$$
$$\mathcal{P}_2' \equiv \{\boldsymbol{\mu} \in \mathbb{R}^l \mid \mathcal{M}_2(\boldsymbol{\mu}) \neq \emptyset\}.$$

Now we will derive the description of the set $\mathcal{P}_1'$. The description of $\mathcal{P}_2'$ will be analogous.

Theorem 6. Let us consider the convex polyhedral cone $\mathcal{N}_{\boldsymbol{A},\boldsymbol{b}}$ from (14). Let $\boldsymbol{g}_i$, $i \in I_1$, be extremal directions of $\mathcal{N}_{\boldsymbol{A},\boldsymbol{b}}$ with the property $\boldsymbol{g}_i^T\boldsymbol{b} < 0$ and let $\boldsymbol{h}_j$, $j \in I_2$, be extremal directions of $\mathcal{N}_{\boldsymbol{A},\boldsymbol{b}}$ with the property $\boldsymbol{h}_j^T\boldsymbol{b} = 0$. If $I_1 = \emptyset$, then $\mathcal{P}_1' = \mathbb{R}^m$. Otherwise the set $\mathcal{P}_1'$ has the description

$$\mathcal{P}_1' = \mathcal{V}_1' \cup -\mathcal{V}_1',$$

where

$$\mathcal{V}_1' = \{\boldsymbol{\delta} \in \mathbb{R}^m \mid \boldsymbol{g}_i^T\boldsymbol{\delta} > 0 \ \forall\, i \in I_1, \ \ \boldsymbol{h}_j^T\boldsymbol{\delta} \geq 0 \ \forall\, j \in I_2\}.$$

Proof. $\mathcal{P}_1'$ is the set of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which $\mathcal{M}_1(\boldsymbol{\delta}) \neq \emptyset$, i. e., the problem

$$\max\{\boldsymbol{0}^T\boldsymbol{x} + 0x_n \mid \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\delta}x_n \leq \boldsymbol{b}\}$$

has an optimal solution. It follows from the theory of duality in linear programming that this is equivalent to the condition, that the problem

$$\min\{\boldsymbol{b}^T\boldsymbol{y} \mid \boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}, \ \boldsymbol{\delta}^T\boldsymbol{y} = 0, \ \boldsymbol{y} \geq \boldsymbol{0}\} \tag{20}$$

has an optimal solution. The set of feasible solutions to the problem (20) forms a convex polyhedral cone. Therefore that the problem (20) has an optimal solution if and only if

$$\{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}, \ \boldsymbol{\delta}^T\boldsymbol{y} = 0, \ \boldsymbol{y} \geq \boldsymbol{0}, \ \boldsymbol{b}^T\boldsymbol{y} < 0\} = \emptyset. \tag{21}$$

Hence $\mathcal{P}_1'$ is the set of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which (21) holds. If $I_1 = \emptyset$, then $\boldsymbol{b}^T\boldsymbol{y} = 0$ for all $\boldsymbol{y} \in \mathcal{N}_{\boldsymbol{A},\boldsymbol{b}}$ and thus $\mathcal{P}_1' = \mathbb{R}^m$. Otherwise we assert that $\mathcal{P}_1' = \mathcal{V}_1' \cup -\mathcal{V}_1'$.

Let $\boldsymbol{\delta}^0 \in \mathcal{V}_1'$. Then $\boldsymbol{g}_i^T\boldsymbol{\delta}^0 > 0 \ \forall\, i \in I_1$ and $\boldsymbol{h}_j^T\boldsymbol{\delta}^0 \geq 0 \ \forall\, j \in I_2$. Each point $\boldsymbol{y} \in \{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}, \ \boldsymbol{b}^T\boldsymbol{y} < 0, \ \boldsymbol{y} \geq \boldsymbol{0}\}$ can be expressed as a linear combination $\boldsymbol{y} = \sum_{i\in I_1}\alpha_i\boldsymbol{g}_i + \sum_{j\in I_2}\beta_j\boldsymbol{h}_j$ for certain $\alpha_i, \beta_j \geq 0$, $\sum_{i\in I_1}\alpha_i > 0$. Therefore

$$\boldsymbol{y}^T\boldsymbol{\delta}^0 = \sum_{i\in I_1}\alpha_i\boldsymbol{g}_i^T\boldsymbol{\delta}^0 + \sum_{j\in I_2}\beta_j\boldsymbol{h}_j^T\boldsymbol{\delta}^0 > 0$$

and the condition (21) holds. Analogously for $\boldsymbol{\delta}^0 \in -\mathcal{V}_1'$.

Conversely assume, that $\boldsymbol{\delta}^0 \in \mathbb{R}^m$ and the condition (21) holds. Then either $\boldsymbol{y}^T\boldsymbol{\delta}^0 > 0$ for all $\{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}, \ \boldsymbol{b}^T\boldsymbol{y} < 0, \ \boldsymbol{y} \geq \boldsymbol{0}\}$ or $\boldsymbol{y}^T\boldsymbol{\delta}^0 < 0$ for all $\{\boldsymbol{y} \in \mathbb{R}^m \mid \boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}, \ \boldsymbol{b}^T\boldsymbol{y} < 0, \ \boldsymbol{y} \geq \boldsymbol{0}\}$. In the first case we specially have $\boldsymbol{g}_i^T\boldsymbol{\delta}^0 > 0$ $\forall\, i \in I_1$ and for infinitesimal $\varepsilon > 0$ also $(1-\varepsilon)\boldsymbol{h}_j^T\boldsymbol{\delta}^0 + \frac{\varepsilon}{|I_1|}\sum_{i\in I_1}\boldsymbol{g}_i^T\boldsymbol{\delta}^0 > 0 \ \forall\, j \in I_2$. Hence $(1-\varepsilon)\boldsymbol{h}_j^T\boldsymbol{\delta}^0 \geq 0$ for infinitesimal $\varepsilon > 0$, and thus $\boldsymbol{h}_j^T\boldsymbol{\delta}^0 \geq 0 \ \forall\, j \in I_2$. It follows that $\boldsymbol{\delta}^0 \in \mathcal{V}_1'$. In the second case we analogously have $\boldsymbol{\delta}^0 \in -\mathcal{V}_1'$. □

3. DESCRIPTION OF SEPARATING HYPERPLANES

Let us introduce

$$\mathcal{Q}^*(\boldsymbol{\delta},\boldsymbol{\mu}) \equiv \left\{(\boldsymbol{u},\boldsymbol{v},v_{l+1}) \in \mathbb{R}^{m+l+1} \mid \boldsymbol{Z}(\boldsymbol{\delta},\boldsymbol{\mu})\begin{pmatrix}\boldsymbol{u}\\ \boldsymbol{v}\\ v_{l+1}\end{pmatrix} = \boldsymbol{z}, \ (\boldsymbol{u},\boldsymbol{v},v_{l+1}) \geq \boldsymbol{0}\right\}, \tag{22}$$

where

$$\boldsymbol{Z}(\boldsymbol{\delta},\boldsymbol{\mu}) \equiv \begin{pmatrix} \boldsymbol{A}^T & \boldsymbol{C}^T & \boldsymbol{0} \\ \boldsymbol{\delta}^T & \boldsymbol{\mu}^T & 0 \\ \boldsymbol{b}^T & \boldsymbol{d}^T & 1 \\ \boldsymbol{1}^T & \boldsymbol{1}^T & 0 \end{pmatrix}, \quad \boldsymbol{z} \equiv \begin{pmatrix} \boldsymbol{0} \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For the explicit description of all separating hyperplanes of the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\delta}),\ \mathcal{M}_2(\boldsymbol{\mu})$ with $\boldsymbol{\delta} \in \mathcal{P}_1,\ \boldsymbol{\mu} \in \mathcal{P}_2$ and $\operatorname{int}\mathcal{M}_1(\boldsymbol{\delta}) \cap \operatorname{int}\mathcal{M}_2(\boldsymbol{\mu}) = \emptyset$ we can directly use Theorem 2.

Assertion 3. Let $\boldsymbol{\delta} \in \mathcal{P}_1,\ \boldsymbol{\mu} \in \mathcal{P}_2,\ (\boldsymbol{u},\boldsymbol{v},v_{l+1}) \in \mathcal{Q}^*(\boldsymbol{\delta},\boldsymbol{\mu})$. Suppose that $\big(\boldsymbol{u}^T\boldsymbol{A},\boldsymbol{u}^T\boldsymbol{\delta}\big) \neq (\boldsymbol{0}^T,0)$, and $\eta \in \langle 0, v_{l+1}\rangle$ is arbitrary. Then

$$\mathcal{R} = \{(\boldsymbol{x},x_n) \in \mathbb{R}^n \mid \boldsymbol{u}^T(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{\delta}x_n - \boldsymbol{b}) = \eta\} \tag{23}$$

represents a separating hyperplane of the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\delta}),\ \mathcal{M}_2(\boldsymbol{\mu})$. Conversely, any separating hyperplane $\mathcal{R}$ of convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\delta}),\ \mathcal{M}_2(\boldsymbol{\mu})$ can be expressed in the form of (23) for a certain $(\boldsymbol{u},\boldsymbol{v},v_{l+1}) \in \mathcal{Q}^*(\boldsymbol{\delta},\boldsymbol{\mu})$, $\big(\boldsymbol{u}^T\boldsymbol{A},\boldsymbol{u}^T\boldsymbol{\delta}\big) \neq (\boldsymbol{0}^T,0)$, and $\eta \in \langle 0, v_{l+1}\rangle$.

4. STABILITY SETS

In this section we deal with the so called stability sets. Stability sets are defined in a similar way as in [6, 9]. It is natural to define stability sets as sets of all $(\boldsymbol{\delta},\boldsymbol{\mu})$ such that all the sets $\mathcal{Q}^*(\boldsymbol{\delta},\boldsymbol{\mu})$ from (22) have the same system of feasible bases.

Definition 4. Let an arbitrary vector $(\boldsymbol{\delta}^0,\boldsymbol{\mu}^0)$ from the solution set $\mathcal{P} \setminus \mathcal{U}$ be given and suppose that the set $\mathcal{Q}^*(\boldsymbol{\delta}^0,\boldsymbol{\mu}^0)$ is nonempty. Denote by $\mathcal{S}$ the system of all feasible bases of the convex polyhedral set $\mathcal{Q}^*(\boldsymbol{\delta}^0,\boldsymbol{\mu}^0)$. *The stability set* corresponding to the system $\mathcal{S}$ is the closure of the set of all $(\boldsymbol{\delta},\boldsymbol{\mu}) \in \mathcal{P} \setminus \mathcal{U}$ under which all feasible bases from $\mathcal{S}$ remain feasible for $\mathcal{Q}^*(\boldsymbol{\delta},\boldsymbol{\mu})$.

Note that stability sets are defined as closed sets only for computational purposes. We will see later (Remark 1) that an additional point lies only on the border of the stability set.

Without loss of generality let us assume that

$$\operatorname{rank}\begin{pmatrix} \boldsymbol{A}^T & \boldsymbol{C}^T \\ \boldsymbol{1}^T & \boldsymbol{1}^T \end{pmatrix} = n.$$

Otherwise it would occur one of the following possibilities:

(i) If $\operatorname{rank}\begin{pmatrix} \boldsymbol{A}^T & \boldsymbol{C}^T \\ \boldsymbol{1}^T & \boldsymbol{1}^T \end{pmatrix} = \operatorname{rank}\begin{pmatrix} \boldsymbol{A}^T & \boldsymbol{C}^T \end{pmatrix}$, then $\mathcal{Q}^*(\boldsymbol{\delta},\boldsymbol{\mu}) = \emptyset\ \ \forall \boldsymbol{\delta},\boldsymbol{\mu}$ and the solution set is empty.

(ii) If $\operatorname{rank}\begin{pmatrix} \boldsymbol{A}^T & \boldsymbol{C}^T \\ \boldsymbol{1}^T & \boldsymbol{1}^T \end{pmatrix} > \operatorname{rank}\begin{pmatrix} \boldsymbol{A}^T & \boldsymbol{C}^T \end{pmatrix}$, then $\operatorname{rank}\begin{pmatrix} \boldsymbol{A}^T & \boldsymbol{C}^T \end{pmatrix} < n-1$ and in the description of $\mathcal{Q}^*(\boldsymbol{\delta},\boldsymbol{\mu})$ there are linear dependent equations, which we can remove.

Now we will derive the description of stability sets. Let $(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$ be from the solution set and B a feasible basis of the convex polytope $\mathcal{Q}^*(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$. Denote $\boldsymbol{D}(\boldsymbol{\delta}, \boldsymbol{\mu}) \equiv \boldsymbol{Z}_B(\boldsymbol{\delta}, \boldsymbol{\mu})$, $\boldsymbol{D} \equiv \boldsymbol{D}(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$. The basis B remains feasible for all values of parameters $\boldsymbol{\delta}$, $\boldsymbol{\mu}$ satisfying the relation

$$\boldsymbol{D}^{-1}(\boldsymbol{\delta}, \boldsymbol{\mu})\boldsymbol{z} \geq \boldsymbol{0}. \tag{24}$$

Define vectors $\boldsymbol{p} \in \mathbb{R}^{n+2}$, $\widetilde{\boldsymbol{q}}, \boldsymbol{q} \in \mathbb{R}^{m+l+1}$:

$$\boldsymbol{p} \equiv \boldsymbol{e}_n = \begin{pmatrix} \boldsymbol{0} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widetilde{\boldsymbol{q}} \equiv \begin{pmatrix} \boldsymbol{\delta} - \boldsymbol{\delta}^0 \\ \boldsymbol{\mu} - \boldsymbol{\mu}^0 \\ 0 \end{pmatrix}, \quad \boldsymbol{q} \equiv \begin{pmatrix} \boldsymbol{\delta} \\ \boldsymbol{\mu} \\ 0 \end{pmatrix}.$$

From the assumption $1 + \widetilde{\boldsymbol{q}}_B^T \boldsymbol{D}^{-1}\boldsymbol{p} \neq 0$ and the well known Sherman–Morrison formula it follows

$$\boldsymbol{D}^{-1}(\boldsymbol{\delta}, \boldsymbol{\mu}) = (\boldsymbol{D} + \boldsymbol{p}\widetilde{\boldsymbol{q}}_B^T)^{-1} = \boldsymbol{D}^{-1} - \frac{\boldsymbol{D}^{-1}\boldsymbol{p}\widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}}{1 + \widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}\boldsymbol{p}}.$$

Since for the choice $\boldsymbol{\delta} = \boldsymbol{\delta}^0$, $\boldsymbol{\mu} = \boldsymbol{\mu}^0$ the denominator $1+\widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}\boldsymbol{p} = 1$ (i. e. positive), assume moreover the following constraint

$$1 + \widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}\boldsymbol{p} > 0. \tag{25}$$

Let us rearrange the expression (24):

$$\begin{aligned} \boldsymbol{D}^{-1}(\boldsymbol{\delta}, \boldsymbol{\mu})\boldsymbol{z} &\geq \boldsymbol{0}, \\ \left(\boldsymbol{D}^{-1} - \frac{\boldsymbol{D}^{-1}\boldsymbol{e}_n\widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}}{1 + \widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}\boldsymbol{e}_n}\right)\boldsymbol{e}_{n+2} &\geq \boldsymbol{0}, \\ \boldsymbol{D}^{-1}_{\cdot,n+2} - \frac{\boldsymbol{D}^{-1}_{\cdot,n}\widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}_{\cdot,n+2}}{1 + \widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}_{\cdot,n}} &\geq \boldsymbol{0}. \end{aligned}$$

From the assumption (25) is this inequality equivalent to

$$\boldsymbol{D}^{-1}_{\cdot,n+2} + \boldsymbol{D}^{-1}_{\cdot,n+2}(\widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}_{\cdot,n}) - \boldsymbol{D}^{-1}_{\cdot,n}(\widetilde{\boldsymbol{q}}_B^T\boldsymbol{D}^{-1}_{\cdot,n+2}) \geq \boldsymbol{0}. \tag{26}$$

Since $\widetilde{\boldsymbol{q}}_B^T = \boldsymbol{q}_B^T - \boldsymbol{D}_{n,\cdot}$, the expression (26) is equivalent to

$$\boldsymbol{D}^{-1}_{\cdot,n+2} + \boldsymbol{D}^{-1}_{\cdot,n+2}\left((\boldsymbol{q}_B^T - \boldsymbol{D}_{n,\cdot})\boldsymbol{D}^{-1}_{\cdot,n}\right) - \boldsymbol{D}^{-1}_{\cdot,n}\left((\boldsymbol{q}_B^T - \boldsymbol{D}_{n,\cdot})\boldsymbol{D}^{-1}_{\cdot,n+2}\right) \geq \boldsymbol{0},$$

$$\boldsymbol{D}^{-1}_{\cdot,n+2}(\boldsymbol{q}_B^T\boldsymbol{D}^{-1}_{\cdot,n}) - \boldsymbol{D}^{-1}_{\cdot,n}(\boldsymbol{q}_B^T\boldsymbol{D}^{-1}_{\cdot,n+2}) \geq \boldsymbol{0}. \tag{27}$$

The expression (27) represents a system of linear inequalities with respect to the variables $\boldsymbol{\delta}$, $\boldsymbol{\mu}$.

Remark 1. Let us investigate the expression (25). It is equivalent to

$$1 + (\boldsymbol{q}_B^T - \boldsymbol{D}_{n,\cdot})\boldsymbol{D}^{-1}_{\cdot,n} > 0, \quad \text{or, to} \quad \boldsymbol{q}_B^T\boldsymbol{D}^{-1}_{\cdot,n} > 0. \tag{28}$$

When we multiply the system (27) by the vector $\boldsymbol{D}_{n+2,\cdot} \geq \boldsymbol{0}$, then we obtain

$$(\boldsymbol{D}_{n+2,\cdot}\boldsymbol{D}^{-1}_{\cdot,n+2})(\boldsymbol{q}_B^T\boldsymbol{D}^{-1}_{\cdot,n}) - (\boldsymbol{D}_{n+2,\cdot}\boldsymbol{D}^{-1}_{\cdot,n})(\boldsymbol{q}_B^T\boldsymbol{D}^{-1}_{\cdot,n+2}) \geq 0, \quad \boldsymbol{q}_B^T\boldsymbol{D}^{-1}_{\cdot,n} \geq 0.$$

Since the stability set is defined as a closed set, the constraint (28) is redundant.

Remark 2. (Description of stability sets) Given $\boldsymbol{\delta}^0$, $\boldsymbol{\mu}^0$ from the solution set $\mathcal{P} \setminus \mathcal{U}$. The stability set (corresponding to $\boldsymbol{\delta}^0$, $\boldsymbol{\mu}^0$) is the set of all $(\boldsymbol{\delta}, \boldsymbol{\mu}) \in \mathcal{P} \setminus \mathcal{U}$ satisfying the following systems of inequalities

$$\boldsymbol{D}^{-1}_{\cdot,n+2}(\boldsymbol{q}_B^T \boldsymbol{D}^{-1}_{\cdot,n}) - \boldsymbol{D}^{-1}_{\cdot,n}(\boldsymbol{q}_B^T \boldsymbol{D}^{-1}_{\cdot,n+2}) \geq \boldsymbol{0}$$

for all feasible bases B of the convex polytope $\mathcal{Q}^*(\boldsymbol{\delta}^0, \boldsymbol{\mu}^0)$ from (22). There is always a finite number of stability sets.

Example 1. Let

$$\boldsymbol{A} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \ \boldsymbol{C} = \begin{pmatrix} -1 \end{pmatrix}, \ \boldsymbol{d} = \begin{pmatrix} -2 \end{pmatrix}.$$

We will provide the description of the solution set and all stability sets.

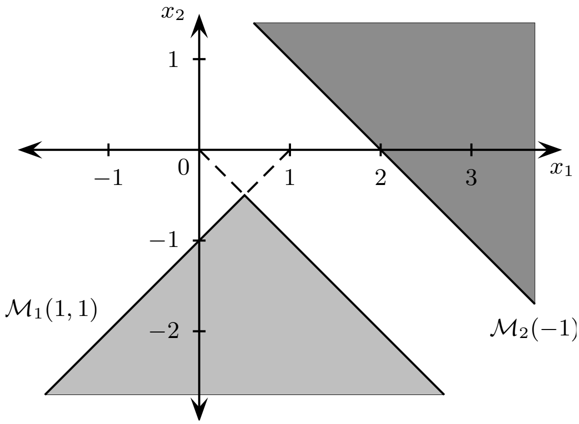

Fig. 1. Illustration to Example 1 for values $\boldsymbol{\delta} = (1,1)^T$, $\boldsymbol{\mu} = (-1)$.

Since the convex polyhedral cone $\mathcal{N}_{\boldsymbol{A},\boldsymbol{b}}$ from (14) contains only one extremal direction $\boldsymbol{h}_1 = (1,1)^T$, the set $\mathcal{P}_1$ (according to Theorem 4) is described as follows

$$\mathcal{P}_1 = \{\boldsymbol{\delta} \in \mathbb{R}^2 \mid \delta_1 + \delta_2 > 0\} \cup \{\boldsymbol{\delta} \in \mathbb{R}^2 \mid \delta_1 + \delta_2 < 0\}.$$

The set $\mathcal{P}_2$ is equal to $\mathbb{R}$, since the convex cone $\mathcal{N}_{\boldsymbol{C},\boldsymbol{d}} = \{\boldsymbol{y} \in \mathbb{R}^l \mid \boldsymbol{C}^T\boldsymbol{y} = \boldsymbol{0}, \ \boldsymbol{d}^T\boldsymbol{y} \leq 0, \ \boldsymbol{y} \geq \boldsymbol{0}\} = \{\boldsymbol{0}\}$ has no edge. The convex polyhedral set (19) has the description

$$\{(y_1, y_2, z_1) \in \mathbb{R}^3 \mid y_1 - y_2 - z_1 = 0, \ -y_2 - 2z_1 \leq 0, \ y_1, y_2, z_1 \geq 0\}$$

and has two edges in direction of $(h_1^1, h_2^1, g_1^1) = (1,1,0,1)^T$ and $(h_1^2, h_2^2, g_1^2) = (1,0,1,2)^T$. Hence the convex polyhedral set $\mathcal{U}_1$ from (18) is described as follows

$$\mathcal{U}_1 = \{(\delta_1, \delta_2, \mu_1) \in \mathbb{R}^3 \mid \delta_1 + \delta_2 > 0, \ \delta_1 + \mu_1 > 0\}.$$

The solution set is (according to Theorem 1) described by

$$\begin{aligned}\mathcal{P} \setminus \mathcal{U} =& \{(\delta_1, \delta_2, \mu_1) \in \mathbb{R}^3 \mid \delta_1 + \delta_2 > 0,\ \delta_1 + \mu_1 \le 0\} \cup \\ & \{(\delta_1, \delta_2, \mu_1) \in \mathbb{R}^3 \mid \delta_1 + \delta_2 < 0,\ \delta_1 + \mu_1 \ge 0\}.\end{aligned}$$

Now we will compute all stability sets according to Remark 2.

1. Choose $(\delta_1^1, \delta_2^1, \mu_1^1)$ from the solution set, e. g. in this way: $(\delta_1^1, \delta_2^1, \mu_1^1) = (1, 1, -1)$. The convex polytope $\mathcal{Q}^*(\delta_1^1, \delta_2^1, \mu_1^1)$ has only one feasible basis, $B = (1, 2, 3, 4)$, and the first stability set is described by the following system of inequalities
$$\delta_1 + \delta_2 > 0,\ \delta_1 + \mu_1 \le 0.$$
2. Choose $(\delta_1^2, \delta_2^2, \mu_1^2)$ from the solution set, but not from the first stability set, e. g. in this way: $(\delta_1^2, \delta_2^2, \mu_1^2) = (-1, -1, 1)$. The convex polytope $\mathcal{Q}^*(\delta_1^2, \delta_2^2, \mu_1^2)$ has only one feasible basis, $B = (1, 2, 3, 4)$, and the first stability set is described by the following system of inequalities
$$\delta_1 + \delta_2 < 0,\ \delta_1 + \mu_1 \ge 0.$$

We have obtained two stability sets (except degenerated stability sets, which have a dimension less than n) the solution set consists of.

Tables 1 – 2 contain further results obtained on `PC (x86), Pentium 4, 2.6 GHz, 512 MB RAM, Gentoo Linux`. Our source code was written in MATLAB 6.5. In each of the mentioned tables, the number of stability sets and the computing time (in minutes and seconds) is written down for given matrix $\boldsymbol{A}$, vector $\boldsymbol{b}$, matrix $\boldsymbol{C}$ and vector $\boldsymbol{d}$. The input data of $\boldsymbol{A}$, $\boldsymbol{C}$, $\boldsymbol{b}$, $\boldsymbol{d}$ were generated pseudorandomly. With the increase of m, l, n the number of stability sets and the computing time increases very rapidly.

5. A PERMANENT SEPARATING HYPERPLANE

Let us consider the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\delta})$, $\mathcal{M}_2(\boldsymbol{\mu})$ from (4), (5) with the property $\boldsymbol{\delta} \in \mathcal{Z}_1$, $\boldsymbol{\mu} \in \mathcal{Z}_2$, where $\mathcal{Z}_1 \subset \mathbb{R}^m$, $\mathcal{Z}_2 \subset \mathbb{R}^l$ are convex polytopes. Let us assume that $\mathcal{Z}_1 \subset \mathcal{P}_1'$ and $\mathcal{Z}_2 \subset \mathcal{P}_2'$. Moreover, we will assume for the sake of simplicity, that all the convex polyhedral sets $\mathcal{M}_1(\boldsymbol{\delta})$, $\boldsymbol{\delta} \in \mathcal{Z}_1$, $\mathcal{M}_2(\boldsymbol{\mu})$, $\boldsymbol{\mu} \in \mathcal{Z}_2$ contain at least one vertex. The question is, whether there exists a fixed hyperplane $\mathcal{R}$ such that:

$$\mathcal{M}_1(\boldsymbol{\delta}) \subseteq \overline{\mathcal{R}^-}\ \forall \boldsymbol{\delta} \in \mathcal{Z}_1,\quad \mathcal{M}_2(\boldsymbol{\mu}) \subseteq \overline{\mathcal{R}^+}\ \forall \boldsymbol{\mu} \in \mathcal{Z}_2.$$

Such a hyperplane $\mathcal{R}$ is called *a permanent separating hyperplane*. Note that a permanent separating hyperplane need not exist even if $\mathcal{M}_1(\boldsymbol{\delta})$, $\mathcal{M}_2(\boldsymbol{\mu})$ are separable for all $\boldsymbol{\delta} \in \mathcal{Z}_1$, $\boldsymbol{\mu} \in \mathcal{Z}_2$ (see Example 2). We will check the existence of a permanent separating hyperplane by the following process: Compute the convex hulls $\operatorname{conv}\left(\cup_{\boldsymbol{\delta} \in \mathcal{Z}_1} \mathcal{M}_1(\boldsymbol{\delta})\right)$ and $\operatorname{conv}\left(\cup_{\boldsymbol{\mu} \in \mathcal{Z}_2} \mathcal{M}_2(\boldsymbol{\mu})\right)$ and check separability of these convex hulls.

Table 1. Examples in $\mathbb{R}^2$, pseudorandom data.

matrix $\boldsymbol{A}$	vector $\boldsymbol{b}$	matrix $\boldsymbol{C}$	vector $\boldsymbol{d}$	number of stability sets	computing time
$\begin{pmatrix}1\\-1\end{pmatrix}$	$\begin{pmatrix}-1\\-1\end{pmatrix}$	$\begin{pmatrix}-3\\9\end{pmatrix}$	$\begin{pmatrix}-1\\-1\end{pmatrix}$	12	4 s
$\begin{pmatrix}-6\\-7\end{pmatrix}$	$\begin{pmatrix}-4\\12\end{pmatrix}$	$\begin{pmatrix}3\\2\\8\end{pmatrix}$	$\begin{pmatrix}-5\\9\\1\end{pmatrix}$	44	15 s
$\begin{pmatrix}-4\\-2\\-6\end{pmatrix}$	$\begin{pmatrix}-1\\9\\-4\end{pmatrix}$	$\begin{pmatrix}8\\1\\-5\end{pmatrix}$	$\begin{pmatrix}8\\-6\\2\end{pmatrix}$	90	1 min 19 s
$\begin{pmatrix}-2\\-2\\7\end{pmatrix}$	$\begin{pmatrix}1\\-1\\0\end{pmatrix}$	$\begin{pmatrix}-8\\-6\\-9\\-2\end{pmatrix}$	$\begin{pmatrix}11\\3\\-5\\12\end{pmatrix}$	206	4 min 8 s
$\begin{pmatrix}0\\8\\-6\\-9\end{pmatrix}$	$\begin{pmatrix}-4\\0\\-3\\9\end{pmatrix}$	$\begin{pmatrix}-6\\0\\-9\\-1\end{pmatrix}$	$\begin{pmatrix}3\\6\\10\\-5\end{pmatrix}$	968	29 min 51 s

Lemma 1. Let B_1 be a sub-basis of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta})$. Let us consider the following convex polyhedral cone

$$\mathcal{N}_{B_1} \equiv \{(\boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^m \mid \boldsymbol{A}_{B_1}^T \boldsymbol{y} + \boldsymbol{A}_{N_1}^T \boldsymbol{z} = \boldsymbol{0},\ \boldsymbol{b}_{B_1}^T \boldsymbol{y} + \boldsymbol{b}_{N_1}^T \boldsymbol{z} \leq 0,\ \boldsymbol{z} \geq \boldsymbol{0}\}, \tag{29}$$

where $N_1 \equiv \{1, \ldots, m\} \setminus B_1$. Let us denote by $(\boldsymbol{g}_i^{\boldsymbol{y}}, \boldsymbol{g}_i^{\boldsymbol{z}}),\ i \in I_1$, extremal directions of $\mathcal{N}_{B_1}$ with the property $(\boldsymbol{g}_i^{\boldsymbol{y}}, \boldsymbol{g}_i^{\boldsymbol{z}})^T(\boldsymbol{b}_{B_1}, \boldsymbol{b}_{N_1}) < 0$ and denote by $(\boldsymbol{h}_j^{\boldsymbol{y}}, \boldsymbol{h}_j^{\boldsymbol{z}}),\ j \in I_2$, extremal directions of $\mathcal{N}_{B_1}$ with the property $(\boldsymbol{h}_j^{\boldsymbol{y}}, \boldsymbol{h}_j^{\boldsymbol{z}})^T(\boldsymbol{b}_{B_1}, \boldsymbol{b}_{N_1}) = 0$. The set $\mathcal{S}_{B_1}$ of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which the sub-basis B_1 is feasible for $\mathcal{M}_1(\boldsymbol{\delta})$ has the following description:

If $I_1 = \emptyset$, then $\mathcal{S}_{B_1} = \mathbb{R}^m$. Otherwise

$$\mathcal{S}_{B_1} = \mathcal{S}_{B_1}^* \cup -\mathcal{S}_{B_1}^*,$$

where

$$\mathcal{S}_{B_1}^* = \{\boldsymbol{\delta} \in \mathbb{R}^m \mid (\boldsymbol{g}_i^{\boldsymbol{y}}, \boldsymbol{g}_i^{\boldsymbol{z}})^T(\boldsymbol{\delta}_{B_1}, \boldsymbol{\delta}_{N_1}) > 0 \ \forall i \in I_1,$$
$$(\boldsymbol{h}_j^{\boldsymbol{y}}, \boldsymbol{h}_j^{\boldsymbol{z}})^T(\boldsymbol{\delta}_{B_1}, \boldsymbol{\delta}_{N_1}) \geq 0 \ \forall j \in I_2\}.$$

Proof. Feasibility of the sub-basis B_1 preserves for the values $\boldsymbol{\delta} \in \mathbb{R}^m$ for which the set

$$\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}_{B_1}\boldsymbol{x} + \boldsymbol{\delta}_{B_1} x_n = \boldsymbol{b}_{B_1},\ \boldsymbol{A}_{N_1}\boldsymbol{x} + \boldsymbol{\delta}_{N_1} x_n \leq \boldsymbol{b}_{N_1}\}$$
$$= \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}_{B_1}\boldsymbol{x} + \boldsymbol{\delta}_{B_1} x_n \leq \boldsymbol{b}_{B_1},\ -\boldsymbol{A}_{B_1}\boldsymbol{x} - \boldsymbol{\delta}_{B_1} x_n \leq -\boldsymbol{b}_{B_1},\ \boldsymbol{A}_{N_1}\boldsymbol{x} + \boldsymbol{\delta}_{N_1} x_n \leq \boldsymbol{b}_{N_1}\}$$

Table 2. Examples in $\mathbb{R}^3$, $\mathbb{R}^4$, pseudorandom data.

matrix $\boldsymbol{A}$	vector $\boldsymbol{b}$	matrix $\boldsymbol{C}$	vector $\boldsymbol{d}$	number of stability sets	computing time
$\begin{pmatrix} 5 & 5 \\ 7 & 1 \\ -6 & -6 \end{pmatrix}$	$\begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix}$	$\begin{pmatrix} -8 & 9 \\ 4 & 8 \\ 2 & 2 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 6 \\ 11 \end{pmatrix}$	41	25 s
$\begin{pmatrix} -7 & 8 \\ 3 & 7 \\ 8 & -9 \end{pmatrix}$	$\begin{pmatrix} -6 \\ 6 \\ 6 \end{pmatrix}$	$\begin{pmatrix} -5 & -3 \\ -3 & 1 \\ 9 & 8 \\ 1 & 7 \end{pmatrix}$	$\begin{pmatrix} 7 \\ -4 \\ -6 \\ -3 \end{pmatrix}$	429	10 min 45 s
$\begin{pmatrix} 0 & -3 \\ 5 & -3 \\ 0 & 8 \\ 9 & 9 \\ -7 & -7 \end{pmatrix}$	$\begin{pmatrix} 7 \\ 6 \\ 11 \\ 0 \\ -2 \end{pmatrix}$	$\begin{pmatrix} 7 & -1 \\ -4 & 5 \\ -5 & 6 \end{pmatrix}$	$\begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix}$	608	29 min 56 s
$\begin{pmatrix} 8 & 5 & 3 \\ -7 & 1 & 1 \\ 2 & 8 & 4 \end{pmatrix}$	$\begin{pmatrix} 7 \\ -4 \\ 7 \end{pmatrix}$	$\begin{pmatrix} 4 & 5 & -2 \\ 6 & 8 & -2 \\ -4 & -7 & -2 \\ 2 & 6 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 \\ -3 \\ -6 \\ -2 \end{pmatrix}$	44	27 s
$\begin{pmatrix} 7 & 6 & -2 \\ -6 & 3 & 3 \\ -9 & 0 & -2 \\ 8 & -4 & 7 \end{pmatrix}$	$\begin{pmatrix} -3 \\ 2 \\ -3 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 4 & 5 & -7 \\ -4 & -1 & -4 \\ -2 & -5 & -4 \\ -7 & -3 & 7 \end{pmatrix}$	$\begin{pmatrix} 3 \\ 1 \\ 12 \\ -1 \end{pmatrix}$	131	3 min 56 s

is non-empty. Consider the convex polyhedral cone

$$\begin{aligned}\{(\boldsymbol{y}^1, \boldsymbol{y}^2, \boldsymbol{z}) \in \mathbb{R}^{m+|B_1|} \mid &\boldsymbol{A}_{B_1}^T \boldsymbol{y}^1 - \boldsymbol{A}_{B_1}^T \boldsymbol{y}^2 + \boldsymbol{A}_{N_1}^T \boldsymbol{z} = \boldsymbol{0}, \\ &\boldsymbol{b}_{B_1}^T \boldsymbol{y}^1 - \boldsymbol{b}_{B_1}^T \boldsymbol{y}^2 + \boldsymbol{b}_{N_1}^T \boldsymbol{z} \le 0,\ \boldsymbol{y}^1, \boldsymbol{y}^2, \boldsymbol{z} \ge \boldsymbol{0}\}.\end{aligned} \tag{30}$$

Denote by $(\boldsymbol{g}_i^1, \boldsymbol{g}_i^2, \boldsymbol{g}_i^{\boldsymbol{z}})$, $i \in I_1'$, the extremal directions of the convex polyhedral cone (30) with the property $(\boldsymbol{g}_i^1, \boldsymbol{g}_i^2, \boldsymbol{g}_i^{\boldsymbol{z}})^T(\boldsymbol{b}_{B_1}, -\boldsymbol{b}_{B_1}, \boldsymbol{b}_{N_1}) < 0$ and by $(\boldsymbol{h}_j^1, \boldsymbol{h}_j^2, \boldsymbol{h}_j^{\boldsymbol{z}})$, $j \in I_2'$, the extremal directions of (30) with the property $(\boldsymbol{h}_j^1, \boldsymbol{h}_j^2, \boldsymbol{h}_j^{\boldsymbol{z}})^T(\boldsymbol{b}_{B_1}, -\boldsymbol{b}_{B_1}, \boldsymbol{b}_{N_1}) = 0$. After substitution $\boldsymbol{y} \equiv \boldsymbol{y}^1 - \boldsymbol{y}^2$, $\boldsymbol{g}_i^{\boldsymbol{y}} \equiv \boldsymbol{g}_i^1 - \boldsymbol{g}_i^2$, $\boldsymbol{h}_j^{\boldsymbol{y}} \equiv \boldsymbol{h}_j^1 - \boldsymbol{h}_j^2$ we obtain the statement of Lemma 1 according to Theorem 6: If a vector $(\boldsymbol{g}_i^1, \boldsymbol{g}_i^2, \boldsymbol{g}_i^{\boldsymbol{z}})$ represents an extremal direction of (30), then the vector $(\boldsymbol{g}_i^{\boldsymbol{y}}, \boldsymbol{g}_i^{\boldsymbol{z}})$ is zero or represents an extremal direction of (29), and vice versa. Likewise for vectors $(\boldsymbol{h}_j^1, \boldsymbol{h}_j^2, \boldsymbol{h}_j^{\boldsymbol{z}})$ and $(\boldsymbol{h}_j^{\boldsymbol{y}}, \boldsymbol{h}_j^{\boldsymbol{z}})$. □

Let $\boldsymbol{\delta}^0 \in \mathbb{R}^m$ and B_1 be any feasible sub-basis of $\mathcal{M}_1(\boldsymbol{\delta}^0)$. Let us introduce

$$\mathcal{S}_{B_1}^{\boldsymbol{\delta}^0} \equiv \begin{cases} \mathbb{R}^m & \text{if } I_1 = \emptyset \text{ (from Lemma 1)}, \\ \mathcal{S}_{B_1}^* & \text{if } \boldsymbol{\delta}^0 \in \mathcal{S}_{B_1}^*, \\ -\mathcal{S}_{B_1}^* & \text{if } \boldsymbol{\delta}^0 \in -\mathcal{S}_{B_1}^*. \end{cases}$$

Lemma 2. Let $\boldsymbol{\delta}^0 \in \mathbb{R}^m$ and S be an arbitrary $(n-1)$-elemental sub-basis of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta}^0)$. Let us assume that the basis S determines an edge of $\mathcal{M}_1(\boldsymbol{\delta}^0)$ unbounded in the direction of $(\boldsymbol{h}^0, h_n^0) \neq (\mathbf{0}, 0)$ and this edge originates from the vertex corresponding to the basis $S \cup \{i\}$ for a certain $i \in \{1, \ldots, m\} \setminus S$. Then the set $\mathcal{H}_S^i$ of all $\boldsymbol{\delta} \in \mathbb{R}^m$ for which the edge, corresponding to the sub-basis S, represents an unbounded edge of $\mathcal{M}_1(\boldsymbol{\delta})$ originating from the vertex determined by the basis $S \cup \{i\}$, has the following description:

If $h_n^0 = 0$, then $\mathcal{H}_S^i = \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$. Otherwise

$$\mathcal{H}_S^i = \left\{\boldsymbol{\delta} \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0} \mid (\delta_j - \boldsymbol{A}_{j,\cdot}^T \boldsymbol{A}_S^{-1} \boldsymbol{\delta}_S) \operatorname{sgn}(h_n^0) \leq 0 \ \forall j \in \{1, \ldots, m\} \setminus (S \cup \{i\})\right\}. \quad (31)$$

P r o o f. Any unbounded edge of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta})$ corresponding to the sub-basis S and originating from the vertex determined by the basis $S \cup \{i\}$ is described by the system

$$\boldsymbol{A}_S \boldsymbol{x} + \boldsymbol{\delta}_S x_n = \boldsymbol{b}_S, \ \ \boldsymbol{A}_{i,\cdot} \boldsymbol{x} + \delta_i x_n \leq b_i$$

and is unbounded in direction which represents (according to [11]) a nontrivial solution to

$$\boldsymbol{A}_S \boldsymbol{x} + \boldsymbol{\delta}_S x_n = \mathbf{0}, \ \ \boldsymbol{A}_{i,\cdot} \boldsymbol{x} + \delta_i x_n \leq 0, \quad (32)$$

whereas the inequalities $\boldsymbol{A}_{j,\cdot} \boldsymbol{x} + \delta_j x_n \leq 0$ with $j \in \{1, \ldots, m\} \setminus (S \cup \{i\})$ if added to (32) are redundant. For the special case when $(\boldsymbol{x}, x_n) = (\boldsymbol{h}^0, h_n^0)$, $\boldsymbol{\delta} = \boldsymbol{\delta}^0$ we have

$$\boldsymbol{A}_S \boldsymbol{h}^0 + \boldsymbol{\delta}_S^0 h_n^0 = \mathbf{0}, \ \ \boldsymbol{A}_{i,\cdot} \boldsymbol{h}^0 + \delta_i^0 h_n^0 \leq 0.$$

If $h_n^0 = 0$, then the vector $(\boldsymbol{h}^0, h_n^0)$ is obviously an extremal direction of $\mathcal{M}_1(\boldsymbol{\delta})$ for all $\boldsymbol{\delta} \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$. Otherwise, the matrix $\boldsymbol{A}_S$ must be nonsingular (since rank $\left(\boldsymbol{A}_S \ \boldsymbol{\delta}_S^0\right) = n-1$). From (32) we have $\boldsymbol{x} = -\boldsymbol{A}_S^{-1} \boldsymbol{\delta}_S x_n$ and consequently

$$\left(\delta_i - \boldsymbol{A}_{i,\cdot} \boldsymbol{A}_S^{-1} \boldsymbol{\delta}_S\right) x_n \leq 0.$$

The equation

$$\delta_i - \boldsymbol{A}_{i,\cdot} \boldsymbol{A}_S^{-1} \boldsymbol{\delta}_S = \det\left(\boldsymbol{A}_S^{-1}\right) \cdot \det \begin{pmatrix} \boldsymbol{A}_S & \boldsymbol{\delta}_S \\ \boldsymbol{A}_{i,\cdot} & \delta_i \end{pmatrix}$$

holds. The determinant $\det \begin{pmatrix} \boldsymbol{A}_S & \boldsymbol{\delta}_S \\ \boldsymbol{A}_{i,\cdot} & \delta_i \end{pmatrix}$ has a constant sign for all $\boldsymbol{\delta} \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$, since if for certain $\boldsymbol{\delta}^1, \boldsymbol{\delta}^2 \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$

$$d_1 = \det \begin{pmatrix} \boldsymbol{A}_S & \boldsymbol{\delta}_S^1 \\ \boldsymbol{A}_{i,\cdot} & \delta_i^1 \end{pmatrix} > 0 \ \text{ and } \ d_2 = \begin{pmatrix} \boldsymbol{A}_S & \boldsymbol{\delta}_S^2 \\ \boldsymbol{A}_{i,\cdot} & \delta_i^2 \end{pmatrix} < 0,$$

hold, then for the convex combination $\boldsymbol{\delta}^3 \equiv \frac{1}{|d_1|+|d_2|}(|d_2|\boldsymbol{\delta}^1 + |d_1|\boldsymbol{\delta}^2) \in \mathcal{S}_{S\cup\{i\}}^{\boldsymbol{\delta}^0}$ we have

$$\det \begin{pmatrix} \boldsymbol{A}_S & \boldsymbol{\delta}_S^3 \\ \boldsymbol{A}_{i,\cdot} & \delta_i^3 \end{pmatrix} = 0,$$

which contradicts the feasibility of the basis $S \cup \{i\}$. Hence an element x_n from (32) has a constant sign for all $\boldsymbol{\delta} \in \mathcal{S}^{\boldsymbol{\delta}^0}_{S\cup\{i\}}$, namely $\operatorname{sgn}(h^0_n)$. A vector $(\boldsymbol{x}, x_n)$ belongs to the unbounded edge of $\mathcal{M}_1(\boldsymbol{\delta})$ provided $\boldsymbol{A}_{j,\cdot}\boldsymbol{x} + \delta_j x_n \leq 0$ for all $j \in \{1,\dots,m\} \setminus (S \cup \{i\})$. Hence

$$-\boldsymbol{A}^T_{j,\cdot}\boldsymbol{A}_S^{-1}\boldsymbol{\delta}_S x_n + \delta_j x_n \leq 0 \ \ \forall j \in \{1,\dots,m\} \setminus (S \cup \{i\}),$$

or, equivalently

$$\left(\delta_j - \boldsymbol{A}^T_{j,\cdot}\boldsymbol{A}_S^{-1}\boldsymbol{\delta}_S\right) \operatorname{sgn}(h^0_n) \leq 0 \ \ \forall j \in \{1,\dots,m\} \setminus (S \cup \{i\}). \qquad \square$$

Let $\boldsymbol{\delta}^0 \in \mathcal{Z}_1$. Denote by $\mathfrak{S}$ a family of all feasible bases of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta}^0)$ and by $\mathfrak{H}$ a family of pairs (S, i), where S is a feasible sub-basis of $\mathcal{M}_1(\boldsymbol{\delta}^0)$ to which it corresponds an unbounded edge originating from a vertex determined by the basis $S \cup \{i\}$. Let us introduce

$$\mathcal{Z}_1(\mathfrak{S},\mathfrak{H}) \equiv \mathcal{Z}_1 \cap \left(\cap_{B\in\mathfrak{S}} \mathcal{S}^{\boldsymbol{\delta}^0}_B\right) \cap \left(\cap_{(S,i)\in\mathfrak{H}} \mathcal{H}^i_S\right).$$

The set $\mathcal{Z}_1(\mathfrak{S},\mathfrak{H})$ represents a set of all $\boldsymbol{\delta} \in \mathcal{Z}_1$ such that all bases from $\mathfrak{S}$ are (according to Lemma 1) feasible for $\mathcal{M}_1(\boldsymbol{\delta})$ and the family of unbounded edges is preserved (Lemma 2). In this way we can divide the set $\mathcal{Z}_1$ into the sets $\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k),\ k \in K$, where K is a finite index set. Each set $\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k),\ k \in K$, represents a convex set, a closure of which is a convex polytope.

Assertion 4. Let $k \in K$ and $\mathfrak{S}_k \neq \emptyset$. Let us assume that the set $\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)$ is closed and denote by $\boldsymbol{\delta}_i,\ i \in V_k$, all vertices of the convex polytope $\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)$. Then the set $\operatorname{conv}\left(\cup_{\boldsymbol{\delta}\in\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)}\mathcal{M}_1(\boldsymbol{\delta})\right)$ represents a convex polyhedral set and the equation

$$\operatorname{conv}\left(\bigcup_{\boldsymbol{\delta}\in\mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)} \mathcal{M}_1(\boldsymbol{\delta})\right) = \operatorname{conv}\left(\bigcup_{i\in V_k} \mathcal{M}_1(\boldsymbol{\delta}_i)\right)$$

holds.

P r o o f. We will prove that for an arbitrary $\boldsymbol{\delta}^1, \boldsymbol{\delta}^2 \in \mathcal{Z}_1(\mathfrak{S}_k,\mathfrak{H}_k)$ and an arbitrary convex combination $\boldsymbol{\delta}^c \equiv (1-c)\boldsymbol{\delta}^1 + c\boldsymbol{\delta}^2$, $c \in (0,1)$ we have

$$\mathcal{M}_1(\boldsymbol{\delta}^c) \subseteq \operatorname{conv}\left(\mathcal{M}_1(\boldsymbol{\delta}^1) \cup \mathcal{M}_1(\boldsymbol{\delta}^2)\right).$$

To prove this it is sufficient to show that all vertices of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta}^c)$ can be expressed as a convex combination of vertices of $\mathcal{M}_1(\boldsymbol{\delta}^1)$, $\mathcal{M}_1(\boldsymbol{\delta}^2)$ and all extremal directions of $\mathcal{M}_1(\boldsymbol{\delta}^c)$ can be expressed as non-negative linear combinations of extremal directions of $\mathcal{M}_1(\boldsymbol{\delta}^1)$, $\mathcal{M}_1(\boldsymbol{\delta}^2)$. Let $B \in \mathfrak{S}_i$ be a basis of $\mathcal{M}_1(\boldsymbol{\delta})$ and denote by $\boldsymbol{v}^1$, $\boldsymbol{v}^2$, and $\boldsymbol{v}^c$ the vertex corresponding to the basis B of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta}^1)$, $\mathcal{M}_1(\boldsymbol{\delta}^2)$, and $\mathcal{M}_1(\boldsymbol{\delta}^c)$, respectively. Next denote $\boldsymbol{M} \equiv \left(\boldsymbol{A}_B \ \boldsymbol{\delta}^1_B\right)$. According to the well-known Sherman–Morrison formula, we get

$$\boldsymbol{v}^c = \left(\boldsymbol{M} + c(\boldsymbol{\delta}^2_B - \boldsymbol{\delta}^1_B)\boldsymbol{e}^T_n\right)^{-1}\boldsymbol{b}_B = \left(\boldsymbol{M}^{-1} - \frac{c\boldsymbol{M}^{-1}(\boldsymbol{\delta}^2_B - \boldsymbol{\delta}^1_B)\boldsymbol{e}^T_n\boldsymbol{M}^{-1}}{1 + c\boldsymbol{e}^T_n\boldsymbol{M}^{-1}(\boldsymbol{\delta}^2_B - \boldsymbol{\delta}^1_B)}\right)\boldsymbol{b}_B. \quad (33)$$

Since B is a feasible basis of $\mathcal{M}_1(\boldsymbol{\delta}^c)$ for all $c \in \langle 0,1\rangle$, the denominator of expression (33) has a constant sign for all $c \in \langle 0,1\rangle$. Hence the expression (33) represents a monotone function of a variable c with $c \in \langle 0,1\rangle$ and, therefore, the vertex $\boldsymbol{v}^c$ is a convex combination of $\boldsymbol{v}^1$, $\boldsymbol{v}^2$.

Let $(S,i) \in \mathfrak{H}_k$, where S is an $(n-1)$-elemental sub-basis determining unbounded edge of $\mathcal{M}_1(\boldsymbol{\delta}^c)$. Let us denote a vector in direction of this unbounded edge for $\mathcal{M}_1(\boldsymbol{\delta}^1)$, $\mathcal{M}_1(\boldsymbol{\delta}^2)$, and $\mathcal{M}_1(\boldsymbol{\delta}^c)$ as $(\boldsymbol{h}^1, h_n^1)$, $(\boldsymbol{h}^2, h_n^2)$, and $(\boldsymbol{h}^c, h_n^c)$, respectively. From the proof of Lemma 2 we have $\operatorname{sgn}(h_n^1) = \operatorname{sgn}(h_n^2) = \operatorname{sgn}(h_n^c)$. Consider the following three cases.

1. If $h_n^1 = h_n^2 = h_n^c = -1$, then
$$\boldsymbol{\delta}^1 = \boldsymbol{A}_S\boldsymbol{h}^1,\ \boldsymbol{\delta}^2 = \boldsymbol{A}_S\boldsymbol{h}^2,\ \boldsymbol{\delta}^c = \boldsymbol{A}_S\boldsymbol{h}^c.$$
Hence we obtain $\mathbf{0} = \boldsymbol{A}_S\big(\boldsymbol{h}^c - (1-c)\boldsymbol{h}^1 - c\boldsymbol{h}^2\big)$. From the nonsingularity of the matrix $\boldsymbol{A}_S$ it follows that $(\boldsymbol{h}^c, h_n^c) = (1-c)(\boldsymbol{h}^1, h_n^1) + c(\boldsymbol{h}^2, h_n^2)$.
2. The case $h_n^1 = h_n^2 = h_n^c = 1$ is analogous to the previous one.
3. In the case $h_n^1 = h_n^2 = h_n^c = 0$ all the vectors $\boldsymbol{h}^1$, $\boldsymbol{h}^2$, $\boldsymbol{h}^c$ determine the same direction. □

According to the Assertion 4 we can, under certain assumption, reduce the computation of the convex hull of an infinite number of convex polyhedral sets to finite number (for an explicit description of convex hulls see [5]). In this way we can reduce the whole computation of $\operatorname{conv}\big(\cup_{\boldsymbol{\delta}\in\mathcal{Z}_1}\mathcal{M}_1(\boldsymbol{\delta})\big)$ to a computation of a convex hull of finitely many convex polyhedral sets, since

$$\operatorname{conv}\left(\bigcup_{\boldsymbol{\delta}\in\mathcal{Z}_1}\mathcal{M}_1(\boldsymbol{\delta})\right) = \operatorname{conv}\left(\bigcup_{k\in K}\ \bigcup_{i\in V_k}\mathcal{M}_1(\boldsymbol{\delta}_i)\right).$$

Example 2. Given

$$\boldsymbol{A} = \begin{pmatrix}-3\\3\\-3\end{pmatrix},\ \boldsymbol{b} = \begin{pmatrix}0\\12\\15\end{pmatrix},\ \mathcal{Z}_1 = \left\{\boldsymbol{\delta}\in\mathbb{R}^3 \mid \boldsymbol{\delta} = (1,1,2)^T + t(1,-1,2)^T,\ t\in\langle 0,6\rangle\right\}$$

and $\mathcal{M}_2(\boldsymbol{\mu}) = \left\{\boldsymbol{x}\in\mathbb{R}^2 \mid \begin{pmatrix}1 & 3\\-1 & -2\end{pmatrix}\boldsymbol{x} \le \begin{pmatrix}14\\-12\end{pmatrix}\right\}$ is fixed. We will compute the convex hull $\operatorname{conv}\big(\cup_{\boldsymbol{\delta}\in\mathcal{Z}_1}\mathcal{M}_1(\boldsymbol{\delta})\big)$ and check the existence of a permanent separating hyperplane.

1. Choose $\boldsymbol{\delta}^1 \in \mathcal{Z}_1$, e.g. as $\boldsymbol{\delta}^1 = (1,1,2)^T$. The family of all feasible bases of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta}^1)$ is $\mathfrak{S}_1 = \{(1,2)\}$. The convex polyhedral cone $\mathcal{N}_{(1,2)}$ from (29) has two extremal directions $\boldsymbol{g}_1 = (-1,-1,0)^T$ and $\boldsymbol{h}_1 = (-9,-5,4)^T$. Hence
$$\mathcal{S}^{\boldsymbol{\delta}^1}_{(1,2)} = \{\boldsymbol{\delta}\in\mathbb{R}^3 \mid -\delta_1-\delta_2 < 0,\ -9\delta_1 - 5\delta_2 + 4\delta_3 \le 0\}.$$

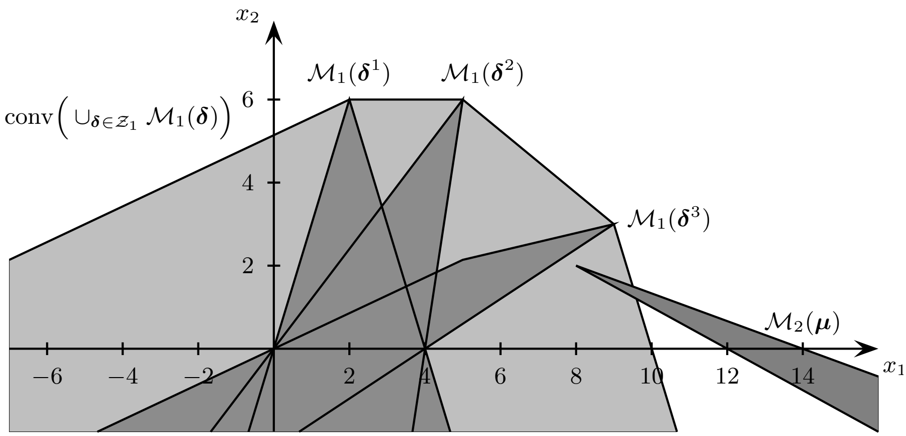

Fig. 2. Illustration to Example 2.

The convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta}^1)$ has two extremal directions $(-1,-3)$ (it corresponds to the sub-basis (1) originating from a vertex determined by the basis $(1,2)$) and $(1,-3)$ (it corresponds to the sub-basis (2) originating from a vertex determined by the basis $(1,2)$). Hence $\mathfrak{H}_1 = \{((1),2),((2),1)\}$. Sets $\mathcal{H}^2_{(1)}$, $\mathcal{H}^1_{(2)}$ have according to (31) the description

$$\mathcal{H}^2_{(1)} = \{\boldsymbol{\delta} \in \mathcal{S}^{\boldsymbol{\delta}^1}_{(1,2)} \mid -\delta_1+\delta_3 \geq 0\},$$
$$\mathcal{H}^1_{(2)} = \{\boldsymbol{\delta} \in \mathcal{S}^{\boldsymbol{\delta}^1}_{(1,2)} \mid \delta_2+\delta_3 \geq 0\}.$$

The convex polytope $\mathcal{Z}_1(\mathfrak{S}_1,\mathfrak{H}_1)$ is equal to

$$\begin{aligned}\mathcal{Z}_1(\mathfrak{S}_1,\mathfrak{H}_1) &= \mathcal{Z}_1 \cap \mathcal{S}^{\boldsymbol{\delta}^1}_{(1,2)} \cap \mathcal{H}^2_{(1)} \cap \mathcal{H}^2_{(1)} \\ &= \big\{\boldsymbol{\delta} \in \mathbb{R}^3 \mid \boldsymbol{\delta} = (1,1,2)^T + t(1,-1,2)^T,\ t \in \langle 0, \tfrac{3}{2}\rangle\big\}\end{aligned}$$

and consists of two vertices $\boldsymbol{\delta}^1$ and $\boldsymbol{\delta}^2 = (\frac{5}{2},-\frac{1}{2},5)^T$.

2. Choose $\boldsymbol{\delta}^3 \in \mathcal{Z}_1 \setminus \mathcal{Z}_1(\mathfrak{S}_1,\mathfrak{H}_1)$, e.g. as $\boldsymbol{\delta}^3 = (7,-5,14)^T$. The family of all feasible bases of the convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta}^3)$ is $\mathfrak{S}_2 = \{(1,3),(2,3)\}$. The convex polyhedral cone $\mathcal{N}_{(1,3)}$ from (29) has two extremal directions $\boldsymbol{g}_2 = (1,-1,0)^T$, $\boldsymbol{h}_2 = (9,-4,5)^T$. Hence

$$\mathcal{S}^{\boldsymbol{\delta}^3}_{(1,3)} = \{\boldsymbol{\delta} \in \mathbb{R}^3 \mid \delta_1-\delta_3 > 0,\ 9\delta_1+5\delta_2-4\delta_3 \geq 0\}.$$

The convex polyhedral cone $\mathcal{N}_{(2,3)}$ has two extremal directions $\boldsymbol{g}_3 = (-1,-1,0)^T$, $\boldsymbol{h}_3 = (5,-4,9)^T$ and hence

$$\mathcal{S}^{\boldsymbol{\delta}^3}_{(2,3)} = \{\boldsymbol{\delta} \in \mathbb{R}^3 \mid -\delta_2-\delta_3 > 0,\ 9\delta_1+5\delta_2-4\delta_3 \geq 0\}.$$

The convex polyhedral set $\mathcal{M}_1(\boldsymbol{\delta}^3)$ has two extremal directions: $(-7,-3)$ (it corresponds to the sub-basis (1) originating from a vertex determined by the basis $(1,3)$) and $(-5,-3)$ (it corresponds to the sub-basis (2) originating from a vertex determined by the basis $(2,3)$). Hence $\mathfrak{H}_2 = \{((1),3),((2),3)\}$. Sets $\mathcal{H}^3_{(1)}$, $\mathcal{H}^3_{(2)}$ have, according to (31), the description

$$\mathcal{H}^3_{(1)} = \{\boldsymbol{\delta} \in \mathcal{S}^{\boldsymbol{\delta}^3}_{(1,3)} \mid \delta_1 + \delta_2 \geq 0\},$$
$$\mathcal{H}^3_{(2)} = \{\boldsymbol{\delta} \in \mathcal{S}^{\boldsymbol{\delta}^3}_{(2,3)} \mid \delta_1 + \delta_2 \geq 0\}.$$

The convex polytope $\mathcal{Z}_1(\mathfrak{S}_2,\mathfrak{H}_2)$ is equal to

$$\begin{aligned}\mathcal{Z}_1(\mathfrak{S}_2,\mathfrak{H}_2) &= \mathcal{Z}_1 \cap \mathcal{S}^{\boldsymbol{\delta}^3}_{(1,3)} \cap \mathcal{S}^{\boldsymbol{\delta}^3}_{(2,3)} \cap \mathcal{H}^3_{(1)} \cap \mathcal{H}^3_{(2)} = \\ &= \left\{\boldsymbol{\delta} \in \mathbb{R}^3 \mid \boldsymbol{\delta} = (1,1,2)^T + t(1,-1,2)^T,\ t \in \langle \tfrac{3}{2}, 6\rangle\right\}\end{aligned}$$

and consists of two vertices $\boldsymbol{\delta}^2$ a $\boldsymbol{\delta}^3$.

Altogether we obtain

$$\begin{aligned}\operatorname{conv}\left(\bigcup_{\boldsymbol{\delta}\in\mathcal{Z}_1} \mathcal{M}_1(\boldsymbol{\delta})\right) &= \operatorname{conv}\big(\mathcal{M}_1(\boldsymbol{\delta}^1) \cup \mathcal{M}_1(\boldsymbol{\delta}^2) \cup \mathcal{M}_1(\boldsymbol{\delta}^3)\big) \\ &= \left\{\boldsymbol{x} \in \mathbb{R}^n \middle| \begin{pmatrix} -3 & 7 \\ 0 & 1 \\ 3 & 4 \\ 3 & 1 \end{pmatrix} \boldsymbol{x} \leq \begin{pmatrix} 36 \\ 6 \\ 39 \\ 30 \end{pmatrix}\right\}.\end{aligned}$$

There is no permanent separating hyperplane (since the sets $\operatorname{conv}\big(\cup_{\boldsymbol{\delta}\in\mathcal{Z}_1} \mathcal{M}_1(\boldsymbol{\delta})\big)$ and $\mathcal{M}_2(\boldsymbol{\mu})$ are not separable), even though $\mathcal{M}_1(\boldsymbol{\delta})$, $\mathcal{M}_2(\boldsymbol{\mu})$ are strongly separable for all $\boldsymbol{\delta} \in \mathcal{Z}_1$.

6. APPLICATION IN MULTIOBJECTIVE PROGRAMMING

In this section we show an application of the proposed theory in multiobjective programming. Let us consider a multiobjective program

$$\max\{\boldsymbol{Cx} \mid \boldsymbol{x} \in \mathcal{M}\},$$

where $\mathcal{M} \equiv \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{Ax} \leq \boldsymbol{b}\}$, $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{C} \in \mathbb{R}^{l\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$. Let $\boldsymbol{x}^0 \in \mathcal{M}$ be a weakly efficient solution, i. e., there is no $\boldsymbol{x} \in \mathcal{M}$ with $\boldsymbol{Cx} > \boldsymbol{Cx}^0$. Alternatively, weak efficiency of $\boldsymbol{x}^0$ can be characterized as separability by a hyperplane of two convex polyhedral sets, $\mathcal{M}$ and $\{\boldsymbol{x} \mid \boldsymbol{Cx} \geq \boldsymbol{Cx}^0\}$, or after transation,

$$\{\boldsymbol{x} \mid \boldsymbol{Ax} \leq \boldsymbol{b} - \boldsymbol{Ax}^0\} \text{ and } \{\boldsymbol{x} \mid \boldsymbol{Cx} \geq \boldsymbol{0}\}.$$

As long as there are uncertainties or measurement errors in one column coefficients of the cost matrix $\boldsymbol{C}$, they can be modelled by column parameters and the theory derived in previous sections is applicable.

7. CONCLUSION

In this article, we were concerned with separation properties of two convex polyhedral sets $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ depending on parameters. Parameters arise in one column of the constraint matrix from the description of these convex polyhedral sets. The situation, when there are parameters on the right-hand side of inequalities was dealt with in [6]. The situation, when parameters arise in one row of the constraint matrix, is the subject of further research. We defined the so called solution set (a set of parameters for which $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ are strongly separable) and stability sets (sets of parameters for which separability of $\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_2$ has the same characteristics). To stability sets, one could apply various kinds of postoptimality analysis (parametric analysis, sensitivity analysis or tolerance analysis – see e. g. [2]), but it goes outside the scope of this paper. We provided also several examples which were carried out on a computer. One section was devoted to determining the so called permanent separating hyperplane which separates $\mathcal{M}_1(\boldsymbol{\delta})$ and $\mathcal{M}_2(\boldsymbol{\mu})$ for all values of parameters $\boldsymbol{\delta}, \boldsymbol{\mu}$ from a given convex polytope. Eventually, we showed how this theory is applicable in multiobjective programming.

(Received September 20, 2006.)

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