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A NEW NUMERICAL MODEL FOR PROPAGATION OF TSUNAMI WAVES

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A new model for propagation of long waves including the coastal area is introduced. This model considers only the motion of the surface of the sea under the condition of preservation of mass and the sea floor is inserted into the model as an obstacle to the motion. Thus we obtain a constrained hyperbolic free-boundary problem which is then solved numerically by a minimizing method called *the discrete Morse semi-flow*. The results of the computation in 1D show the adequacy of the proposed model.

Keywords: long waves, nonlinear hyperbolic equation, volume constraint, free boundary, variational method, discrete Morse semi-flow, FEM

AMS Subject Classification: 35L70, 47J30, 58E50, 76B15, 74J15, 35R35

1. INTRODUCTION

In this paper we present several basic ideas on a new model for the propagation of waves on the surface of a sea. The aim is to develop a model meeting the following requirements:

- it should be simple in the sense that it takes into account only the surface of the sea and not the movement of the whole body of water;
- it should be capable of treating the whole surface, i. e. not only in the deep water region but also near the shore including the climbing of waves onto land;
- it should accurately approximate the long wave equations in the deep water regions.

The first requirement is a practical one, since if we want to predict the strength of tsunami in certain regions based on the measurements of the initial stages of its propagation and known facts about its origin, we usually only have information about the surface displacement measured by satellites, buoys etc. Moreover, if we obtain a sufficiently precise model for the surface, one can expect that it will be computationally less demanding than any model for the entire body of water.

Existing models combine different approaches for the deep water region, for the region near the shore, and for inundation. It would be convenient if there was a model treating all these stages synthetically. Here, such a treatment is attempted by introducing the notion of an obstacle and a free boundary (i. e., the set of points where water touches the shore). Naturally, we also include the condition of volume preservation.

The last requirement of agreement with accurate models for long waves in deep water is obviously indispensable for a reliable model. In our case, this property is not taken into consideration when constructing the model but is merely checked after the construction.

Long waves (or shallow water waves) are characterized by a large ratio of wavelength to the depth of the sea and by a small ratio of amplitude compared to the depth with a certain asymptotical relation between these two ratios. Tsunami waves, usually generated by earthquakes or landslides, belong to this category. They have small amplitudes and are very fast in deep water (approximately $\sqrt{gh}$, where h is the depth and g is the gravitational constant) but when they enter the shoaling water of coastlines, their velocity diminishes and the wave height increases, striking the seashore with devastating force. There are many variants of models for such waves, see [1] for a concise summary and the references therein.

The new model is based on the minimization of a functional under the condition of preservation of mass and the obstacle constraint. The model is derived in the next section and the mathematical problem is formulated. In the third section the approximate problem, suitable for numerical computation, is introduced. The approximation uses the idea of discrete Morse semi-flow. The last section is devoted to the numerical scheme and to an experiment suggesting that the model might be useful.

2. DERIVATION OF THE MODEL

The standard technique for the derivation of equations for long waves starts from the incompressible irrotational Navier–Stokes equations. We also assume the incompressibility and irrotationality of the flow. The irrotationality particularly leads to the image of water in the form of layers of particles which are connected one with each of its neighbours in all directions like beads on strings. Therefore, we can consider that forming a wave is easier in shallow water than in the deep region, since there are fewer particles to be lifted than in deep sea. We apply this idea to the construction of a Lagrangian for the surface of the sea. For the sake of clarity, we carry out the derivation in the one-dimensional case.

A smooth function φ describes the sea bed. We consider a sufficiently large space interval $(0, l)$ so that there is no chance of waves coming up to its boundary. The surface of the water is expressed using the graph of a scalar function $\eta : [0, T] \times [0, l] \to \mathbb{R}$ (see Figure 1). There are two main conditions imposed on the function η:

$$\begin{gathered} \eta(t, x) \geq \varphi(x) \quad \forall\, x \in [0, l], \ \ \forall\, t \in [0, T], \\ \int_0^l (\eta(t, x) - \varphi(x))\, \mathrm{d}x = V \quad \forall\, t \in [0, T], \end{gathered} \tag{1}$$

where V denotes the volume of water in the sea. The second condition corresponds to the condition of preservation of mass. It is not completely natural but follows

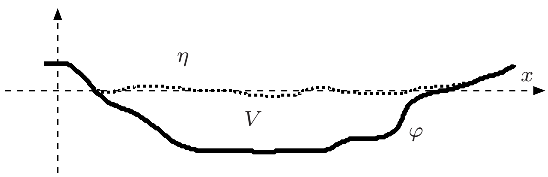

Fig. 1. Notation.

inevitably from the assumption of incompressibility, which is also put in the case of the Navier–Stokes model, and results in an infinite velocity of sound. Therefore, we have accepted this condition under the assumption that we have a connected mass of water bounded by lands (as in Figure 1) with the view of possibly modifying the requirement of overall mass-preservation to a local condition in numerical computations. Moreover, the initial perturbation of the bottom of the sea and consequently of the sea surface is usually volume-preserving which prevents unnatural spreading of impulses. We also neglect the effects of roundness and rotation of the Earth at this stage.

The Lagrangian of a function satisfying the above conditions is then defined as the difference between potential and kinetic energy:

$$L(\eta) = \frac{1}{2}\int_0^T \int_0^l \left(-\eta_t^2 + 2g\sqrt{1+\eta_x^2}(\eta-\varphi)\right) \mathrm{d}x\mathrm{d}t. \tag{2}$$

We set the problem in the following manner.

Problem 1. Find a stationary point of L in the convex set

$$\mathcal{K} = \{\eta \in H^1((0,T)\times(0,l));\;\; (\eta-\varphi)(0) = (\eta-\varphi)(l) = 0;\;\; \eta \text{ satisfies } (1)\}, \tag{3}$$

satisfying initial conditions $\eta(0,x) = \eta_0(x)$ and $\eta_t(0,x) = v_0(x)$.

The potential energy can also be interpreted as the energy needed to lift a string with constant mass density from the sea floor φ to the position η. The following lemma tells us that this form of potential energy is reasonable.

Lemma 1. Let ν be a nonnegative, smooth, even function increasing on $[0,\infty)$. Then the smooth minimizer of $\int_0^l \nu(\eta_x)(\eta-\varphi)\,\mathrm{d}x$ in $\mathcal{K}$ is a constant function in $\{\eta > \varphi\}$.

Proof. Since $0 \le \nu(0) < \nu(y)$ for every $y \in \mathbb{R}\setminus\{0\}$ and $\eta \ge \varphi$ on $[0,l]$, we have

$$\int_0^l \nu(\eta_x)(\eta-\varphi)\,\mathrm{d}x \ge \int_0^l \nu(0)(\eta-\varphi)\,\mathrm{d}x = \nu(0)V.$$

The existence of an x_0 in the region $\{\eta > \varphi\}$, for which $\eta_x(x_0) \neq 0$ and η_x is continuous at x_0, would lead to a sharp inequality in the above and thus to a contradiction with minimality. □

We approximate the square root in (2) in order to simplify the calculations and to be able to practically implement the model. We use the approximation $\sqrt{1+\eta_x^2} \approx 1 + \eta_x^2/2$ and omit the term $\int_0^l (\eta - \varphi)\,\mathrm{d}x$ which is constant due to (1):

$$\tilde{L}(\eta) = \frac{1}{2}\int_0^T \int_0^l \left(-\eta_t^2 + g\eta_x^2(\eta - \varphi)\right)\,\mathrm{d}x\mathrm{d}t. \tag{4}$$

Lemma 1 is applicable to this functional, too. Since the stationary point η is continuous, taking the first variation of this functional, one can reduce the formulation by means of a differential equation:

$$\begin{aligned} &\eta_{tt}(t,x) - g[\eta_x(t,x)(\eta(t,x) - \varphi(x))]_x + \frac{g}{2}\eta_x(t,x)^2 = \lambda(t) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \{(t,x) : \eta(t,x) > \varphi(x)\} \\ &\eta(t,x) = \varphi(x) \qquad \text{in } \{(t,x) : \eta(t,x) \leq \varphi(x)\}, \end{aligned} \tag{5}$$

where

$$\lambda(t) = \frac{1}{V}\int_0^l \Big(\eta_{tt}(\eta - \varphi) + g\eta_x(\eta-\varphi)_x(\eta - \varphi) + \frac{g}{2}\eta_x^2(\eta-\varphi)\Big)\,\mathrm{d}x \tag{6}$$

is a Lagrange multiplier depending on time.

We have obtained a hyperbolic free-boundary problem with volume constraint. Now, we would like to show that these equations accurately describe the propagation of long waves in deep water. To this end, we introduce scaling parameters ε and σ which relate the typical amplitude of the wave a to the typical depth of the sea H, and the typical depth H to the typical wavelength d, respectively:

$$\varepsilon = \frac{a}{H}, \qquad \sigma = \frac{H}{d}.$$

Tsunami waves in deep water have generally very small values of ε and σ with $\varepsilon \approx \sigma^2$.

Further, we scale the variables t, x, φ and η, so as to obtain variables $\hat{t}, \hat{x}, \hat{\varphi}$ and $\hat{\eta}$, which are of order $\mathcal{O}(1)$:

$$\hat{t} = \frac{\sqrt{gH}}{d}t, \qquad \hat{x} = \frac{x}{d}, \qquad \hat{\varphi} = \frac{\varphi}{H}, \qquad \hat{\eta} = \frac{\eta}{a}.$$

Equation (5) rewritten in the new variables becomes

$$\hat{\eta}_{\hat{t}\hat{t}} - \left(\hat{\eta}_{\hat{x}}\left(\varepsilon\hat{\eta} - \hat{\varphi}\right)\right)_{\hat{x}} + \frac{1}{2}\varepsilon\hat{\eta}_{\hat{x}}^2 = \hat{\lambda}.$$

One can immediately see that the leading part of the equation for long waves in deep water is

$$\eta_{tt} + g(\eta_x\varphi)_x = \lambda'$$

with

$$\lambda' = \frac{1}{V} \int_0^l (-\eta_{tt}\varphi + g\eta_x\varphi\varphi_x)\, \mathrm{d}x.$$

Let us, moreover, suppose that the sea has constant depth H. Then, inserting a wave of the form $a\sin(kx - \omega t)$ in the last equation, we get the relation for the phase velocity of the wave in the form

$$C = \frac{\omega}{k} = \sqrt{gH}.$$

As $kH = H/d = \sigma$ and $\tanh\sigma = \sigma + \mathcal{O}(\sigma^3)$, this corresponds quite well to the result from linear wave theory which states that

$$C^2 = \frac{g}{k}\tanh(kH).$$

Remark. In the potential energy term of (2), we have used the weight corresponding to the depth $\eta - \varphi$. Since the considerations in the beginning of this section do not determine a specific form for this weight, we could use some other weight in the form of a function of the depth (i. e., $F(\eta - \varphi)$). However, by calculations similar to those above, we would obtain the deep-water phase velocity $\sqrt{gF(H)}$, which is expected to be near $\sqrt{gH}$. This justifies the adopted form of the potential energy and suggests the possibility of modifying the weight function in the shallow parts in order to obtain better results both physically and mathematically (the difficulty in proving the existence of approximate solutions constructed in the next section is caused mainly by the vanishing of $\eta - \varphi$ on the free boundary – this could be avoided by a suitable modification of the weight function).

3. THE DISCRETE MORSE SEMI–FLOW

In this section, we present a method that can be used to solve the problem derived in the previous section and potentially also to obtain theoretical results. This method discretizes time and constructs the approximate solution by minimizing the discretized functional on each time level. The method is called the discrete Morse semi-flow and was first introduced in [2] and analyzed and applied to various problems, e. g., in [3, 4, 6, 7].

In the subsequent text we also consider an outer force f, which allows us to treat the initial perturbance in the sea surface caused mostly by sudden changes in the shape of the bottom. We assume that $f \in L^2((0,T) \times (0,l))$ is supported far from the coast (i. e., it does not interfere with the free boundary, precisely the support of $f \subset \{(t,x) : \eta(t,x) > \varphi(x)\}$).

We create an equidistant partition of the interval $[0,T]$ into N subintervals and denote $h = T/N$. We define a new unknown function u by the formula $u = \eta - \varphi$ and determine initial functions $u_0 = \eta_0 - \varphi$ and $u_1 = u_0 + hv_0$. Our method is formulated inductively in the following way.

Problem 2. For $n = 2, 3, \dots$, find minimizer u_n of the functional

$$L_n(u) := \int_0^l \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2} \chi_{u>0} \,\mathrm{d}x + \frac{g}{2} \int_0^l (u+\varphi)_x^2 |u| \,\mathrm{d}x + \int_0^l fu \,\mathrm{d}x, \quad (7)$$

in the function set

$$\mathcal{K}' := \left\{ u \in H_0^1(0,l);\ \int_0^l u\chi_{u>0} \,\mathrm{d}x = V \right\}. \quad (8)$$

Here $\chi_{u>0}$ is the characteristic function of the set $\{x \in [0,l] : u(x) > 0\}$.

The existence of minimizers is an important issue. However, we have not yet succeeded in proving the existence for this cubic constrained functional and we leave this problem for the next report. We have to check that the minimizers u_n satisfy the obstacle constraint.

Lemma 2. Minimizers u_n of the functional L_n in $\mathcal{K}'$ are nonnegative.

P r o o f . Suppose there is $x_0 \in [0,l]$ such that $u_n(x_0) < 0$. The Sobolev imbedding theorem ensures the continuity of u_n and therefore, there is an interval I, where $u_n < 0$. Let us take the function $\tilde{u}_n = u_n \chi_{u_n>0} \in \mathcal{K}'$ and estimate $L_n(\tilde{u}_n) - L_n(u_n)$:

$$\begin{aligned} L_n(\tilde{u}_n) - L_n(u_n) &= \frac{g}{2} \int_0^l \left((\tilde{u}_n + \varphi)_x^2 |\tilde{u}_n| - (u_n + \varphi)_x^2 |u_n| \right) \mathrm{d}x \\ &\le -\frac{g}{2} \int_I (u_n + \varphi)_x^2 |u_n| \,\mathrm{d}x < 0. \end{aligned}$$

This is in contradiction with the minimality of $L_n(u_n)$. □

In the remaining part of this section, we shall study the connection between the sequence of minimizers $\{u_n\}_n$ obtained in Problem 2 and the weak solution of the original problem defined naturally as a function $w \in H^2(0,T,H^1(0,l))$ satisfying

$$\begin{aligned} &\int_0^T \int_0^l \left(-w_t \phi_t + g(w+\varphi)_x w \phi_x + \frac{g}{2}(w+\varphi)_x^2 \phi + f\phi \right) \mathrm{d}x\mathrm{d}t - \int_0^l v_0 \phi(0,x) \,\mathrm{d}x \\ &= \frac{1}{V} \int_0^T \int_0^l \left(w_{tt} w + g(w+\varphi)_x w_x w + \frac{g}{2}(w+\varphi)_x^2 w + fw \right) \mathrm{d}x \left(\int_0^l \phi \,\mathrm{d}x \right) \mathrm{d}t \end{aligned}$$

for each $\phi \in C_0^\infty((0,T) \times (0,l) \cap \{w > 0\})$ and $w = 0$ in the complement of $\{w > 0\}$ and the initial condition $w(0,x) = u_0(x)$ in the sense of traces.

Let u be the minimizer of L_n, select an arbitrary function $\xi \in C_0^\infty(0,l)$ with support inside $\{u > 0\}$ and for $\varepsilon > 0$ set

$$u_\varepsilon = \frac{u + \varepsilon\xi}{1 + \frac{\varepsilon}{V} \int \xi \,\mathrm{d}x}.$$

If ε is small enough, we have $u_\varepsilon \in \mathcal{K}'$, $u_\varepsilon \geq 0$ and $\chi_{u>0} = \chi_{u_\varepsilon>0}$ and we can easily calculate the limit

$$A = \lim_{\varepsilon \to 0+} \frac{L_n(u_\varepsilon) - L_n(u)}{\varepsilon}.$$

Since $A = 0$, we obtain for each $\xi \in C_0^\infty((0,l) \cap \{u > 0\})$ the identity

$$\int_0^l \left(\frac{u - 2u_{n-1} + u_{n-2}}{h^2} \xi + g(u+\varphi)_x u \xi_x + \frac{g}{2}(u+\varphi)_x^2 \xi + f\xi \right) \mathrm{d}x = \int_0^l \lambda_n \xi \, \mathrm{d}x,$$

where λ_n is defined by

$$\lambda_n = \frac{1}{V} \int_0^l \left(\frac{u - 2u_{n-1} + u_{n-2}}{h^2} u\chi_{u>0} + g(u+\varphi)_x u_x u + \frac{g}{2}(u+\varphi)_x^2 u + fu \right) \mathrm{d}x.$$

The value λ_n can be called a Lagrange multiplier because it comes in naturally through considering the variation of the functional $L_n(u) - \lambda_n \int_0^l u\chi_{u>0} \, \mathrm{d}x$ in $H_0^1(0,l)$.

In order to get a time-dependent function, we interpolate the minimizers in time and define functions u^h, $\bar{u}^h$ and $\bar{\lambda}^h$ as

$$\begin{aligned} \bar{u}^h(x,t) &= u_n(x), \\ u^h(x,t) &= \frac{t-(n-1)h}{h} u_n(x) + \frac{nh-t}{h} u_{n-1}(x), \\ \bar{\lambda}^h(t) &= \lambda_n, \end{aligned}$$

for $(t,x) \in ((n-1)h, nh] \times (0,l)$, $n = 1,2,\ldots,N$. For $t = 0$ we define $u^h(0,x) = \bar{u}^h(0,x) = u_0(x)$. Functions u^h, $\bar{u}^h$ satisfy

$$\begin{aligned} &\int_h^T \int_\Omega \left(\frac{u_t^h(t) - u_t^h(t-h)}{h} \phi + g(\bar{u}^h+\varphi)_x \bar{u}^h \phi_x + \frac{g}{2}(\bar{u}^h+\varphi)_x^2 \phi + f\phi \right) \mathrm{d}x\mathrm{d}t \\ &\qquad = \int_h^T \int_\Omega \bar{\lambda}^h \phi \, \mathrm{d}x\mathrm{d}t, \qquad \forall\, \phi \in C_0^\infty([0,T) \times (0,l) \cap \{u^h > 0\}), \\ &u^h \equiv 0 \quad \text{in} \quad (h,T) \times (0,l) \setminus \{u^h > 0\}, \end{aligned}$$

and, therefore, we shall call them *approximate weak solutions*.

One can see the similarity between the equations for a weak solution and approximate weak solution. Nevertheless, to show the convergence (as $h \to 0$) is rather nontrivial. In [6], we faced the same problem for a similar quadratic functional and were able to prove the existence and regularity of minimizers in higher dimensions but not the convergence. The convergence for a quadratic functional with volume constraint but without free boundary was shown in [5].

4. NUMERICAL EXPERIMENT

Here we show the results of an experiment based on the presented numerical model. The settings are as follows: The domain under consideration is chosen so that its left boundary is far into the sea and the right boundary is on land, where there is no danger of waves coming. On the left boundary, a homogeneous Neumann boundary condition is prescribed and the right boundary satisfies a homogeneous Dirichlet boundary condition. The depth of the sea on the left part is set to -2.2 and the sea floor then rises to the right in three slopes to the shore (see Figure 2). The initial values u_0 and u_1 are both set to zero. An outer force simulating an upheaval of the sea bed is applied near the left boundary, generating a long wave.

The program uses a standard finite element method to express the unknown function u_n and the domain is split into 150 elements. However, the coefficients corresponding to the basis functions are not arbitrary in this case, due to the volume preservation condition which constrains them to a hyperplane. We search for the minimizer of the functional

$$L_n(u) := \int_0^l \frac{|u - 2u_{n-1} + u_{n-2}|^2}{2h^2}\,\mathrm{d}x + \frac{g}{2}\int_0^l (u+\varphi)_x^2 u_{n-1}\,\mathrm{d}x + \int_0^l fu\,\mathrm{d}x$$

in this constrained space and subsequently cut off the parts of the solution which are not above the obstacle. The volume thus changes and is adjusted by multiplying by a suitable constant. In practical computation, there is almost no truncation since the changes of volume which is of order 10^2 do not exceed 10^{-3}. Note that the minimized functional is quadratic due to the simplification in the second term.

The results are shown in Figure 2. The motion naturally depends on the form of the initial outer force and on the shape of the seafloor. We have chosen two types of outer forces and two types of shapes for the floor in the numerical experiment in order to have a glimpse of the dependence. Outer force f_1 is a volume-preserving force having the value of 0.6 at the 4th node and the value of -0.6 at the 33rd node of the triangulation at the first time level only. The outer force f_2 is equal to zero except at the 4th node, where it is set to -0.6 on the first time level. In the case of f_2 we have also tried to insert an underwater mountain in the way of the wave.

Although the results differ slightly in each case, it is generally observed that the long wave slows down and becomes steeper and higher when it arrives at the shallow region. At the same time, a well-known phenomenon occurs, namely, that the water withdraws from the coast for a moment. A steep wave is formed which slowly penetrates high into the land and subsequently withdraws with a greater speed. The inundation is then repeated on a smaller and smaller scale.

The above observations perfectly fit the properties of tsunami waves observed and measured in reality. However, apart from this qualitative agreement, it is necessary to compare the quantitative results with real experiments, which remains a future task. The comparison with solutions of other models such as the KdV equation, the RLW equation, or Boussinesq systems would also be of interest, but these models are not able to treat the area near the shore. In any case, the qualitative agreement of this model with real phenomena, and the simplicity of the model suggesting the possibility of fast computation, seem quite promising and I believe that parameter tuning and introduction of new parameters (such as a weighted volume constraint) will yield a serviceable model.

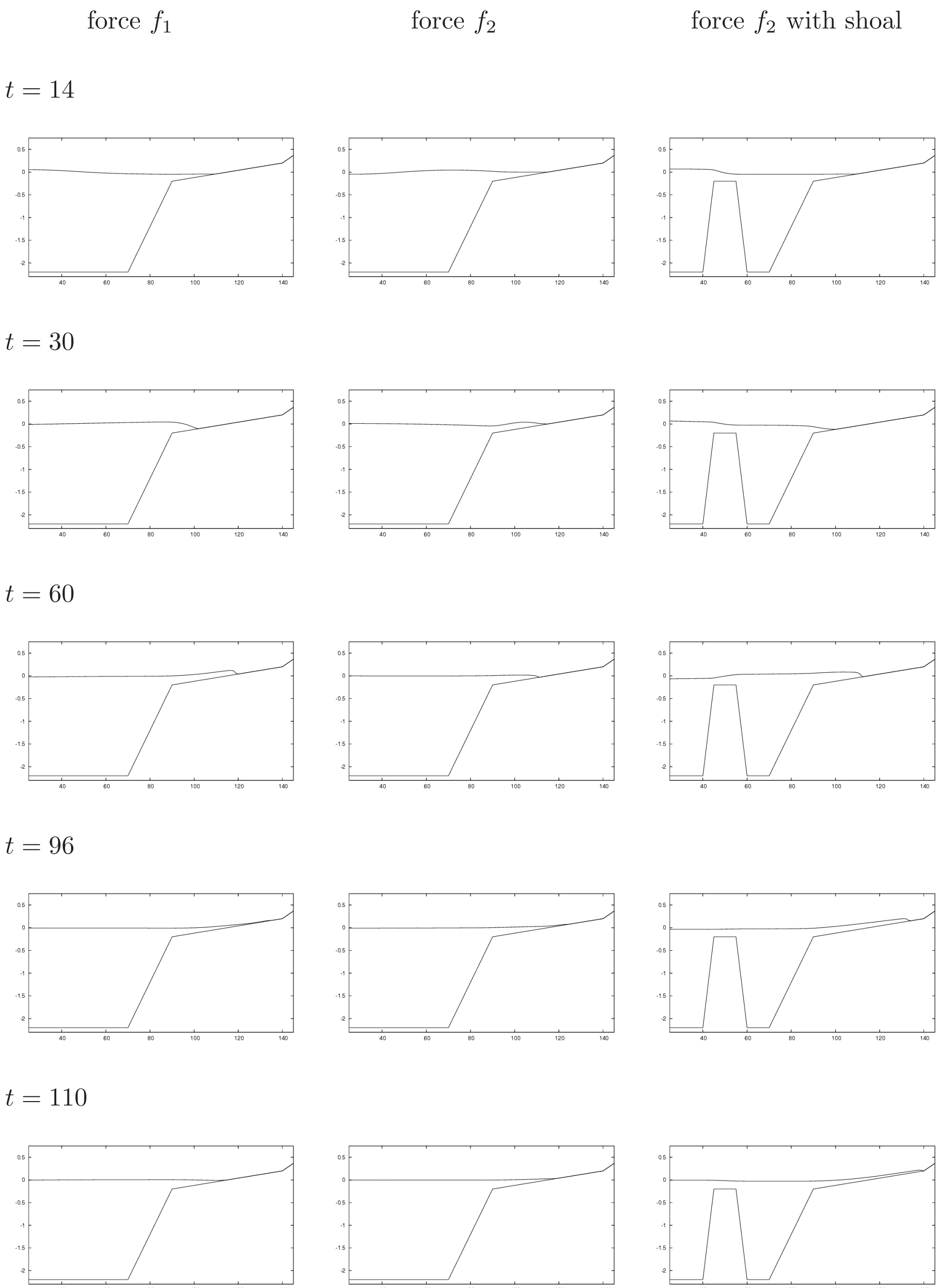

Fig. 2. Numerical results at different times.

5. CONCLUSION

A completely new model for the propagation of tsunami waves has been proposed. Unlike existing models, this model is able to handle all phases of the wave evolution, i. e., generation, propagation, shoaling and inundation. The model is based on minimization of a hyperbolic functional under the restrictive conditions of the obstacle in the form of a seabed and volume preservation. The resulting mathematical problem seems difficult but it is possible to realize it numerically by the use of the discrete Morse semi-flow. The simulation produced satisfying results.

Nevertheless, there are still many problems to be solved before the model can be pronounced applicable. Namely, the mathematical theory must be developed (existence and regularity of minimizers in higher dimensions, convergence of approximate solutions) and computational results should be compared to real experiments with simultaneous parameter tuning.

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