BIFURCATIONS FOR TURING INSTABILITY WITHOUT SO(2) SYMMETRY

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In this paper, we consider the Swift–Hohenberg equation with perturbed boundary conditions. We do not a priori know the eigenfunctions for the linearized problem since the SO(2) symmetry of the problem is broken by perturbation. We show that how the neutral stability curves change and, as a result, how the bifurcation diagrams change by the perturbation of the boundary conditions.

Keywords: perturbed boundary conditions, imperfect pitchfork bifurcation, Turing instability

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1. INTRODUCTION

It is well known that the Turing instability is the basic mechanism in the pattern formation problems. We usually consider reaction-diffusion equations with natural boundary conditions, such as Neumann or periodic boundary conditions. And the solutions to their problems automatically have SO(2) symmetry.

On the other hand, in [2], activator-inhibitor systems are considered with mixed boundary conditions which is not SO(2) symmetric. Namely, they analyze the system of two component reaction-diffusion equations which satisfy different boundary conditions, respectively.

One can also find a similar kind of study in the convection problem. In fact, Mizushima–Nakamura [5] studied linearized stability of the Rayleigh–Bénard problem with partially nonslip boundary conditions which are also not SO(2) symmetric. They observed the repulsion of the eigenvalues, which means the separation of the neutral stability curves for different modes, by changing the nonslip parameter. In addition, Kato–Fujimura [4] studied Rayleigh–Bénard convection with the boundary conditions which correspond to the one considered in [5]. Moreover, they obtained the global bifurcation diagram numerically, and they studied local bifurcation structure by the multiple scale method, as well.

In this paper, we consider the Swift–Hohenberg equation:

$$\frac{\partial w}{\partial t} = \left\{\nu - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2\right\} w - w^3, \quad t > 0,\ x \in (0, L/2). \tag{1}$$

with the following boundary conditions:

$$\begin{aligned} &w(t,0) = w(t,L/2) = 0, \\ &\delta w_x(t,0) - w_{xx}(t,0) = 0, \\ &\delta w_x(t,L/2) + w_{xx}(t,L/2) = 0, \end{aligned} \tag{2}$$

where $w = w(t,x)$ is real valued function, ν, $L > 0$ and $\delta \geq 0$ are parameters.

We analyze the linearized eigenvalue problem, and we shall study numerically the global bifurcation structures. Moreover, we study local bifurcation structures of stationary solutions to (1) with (2) by using the cubic normal forms.

2. LINEAR STABILITY AND SYMMETRY OF THE PROBLEM

In this section, we consider the linearized stability of (1) with (2). We will examine the case when $\delta = 0$, namely, we consider the linearized problem of (1) with the following boundary conditions:

$$w = w_{xx} = 0 \quad \text{at } x = 0, L/2. \tag{3}$$

If w is a smooth solution of (1) with (3), define

$$\hat{w}(t,x) := \begin{cases} w(t,x) & x \in (0,L/2) \\ -w(t,-x) & x \in (-L/2,0) \end{cases}$$

then, the solution is extended in $x \in (-L/2, L/2)$. Moreover, when $\delta = 0$, the solution to (1) with (2) can be extended to the solution on the whole line which is smooth and periodic function with period L. In particular, it can be extended to the solution witch is invariant under the mappings:

$$w(t,x) \to -w(t,-x), \quad w(t,x) \to w(t,x+L).$$

This implies that linearized eigenfunctions and eigenvalues around the trivial solution ($w \equiv 0$) are given by

$$w_m := \sin\Big(\frac{2\pi}{L}mx\Big), \quad \sigma_m := \nu - \Big(1 - \frac{4\pi^2}{L^2}m^2\Big)^2, \quad m \in \mathbb{Z}. \tag{4}$$

Thus, we can conclude that neutral stability curves are given by the following:

$$C_m = \left\{(L,\nu); \nu = \Big(1 - \frac{4\pi^2}{L^2}m^2\Big)^2\right\}, \quad m \in \mathbb{Z}.$$

It should be noted that when $\delta > 0$, we can not extend the solutions as an L-periodic function. However, the problem is invariant under the mappings: $w \to -w(t,x)$ and $w(t,x) \to w(t, L/2 - x)$ independent of δ.

Let us consider that how the neutral stability curves change when $\delta > 0$. Let $(L^{m,n}, \nu^{m,n})$ be the intersection point of two neutral stability curves C_m and C_n, $(m \neq n)$.

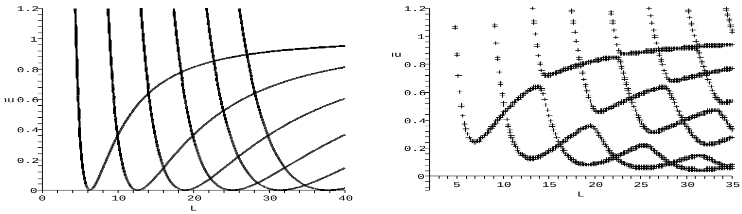

Fig. 1. Neutral stability curves drawn in (ν, L)-plane. [Left: They correspond to the critical curves for $C_1, C_2, \dots, C_6$ respectively from the left], [Right: The critical curves drown based on the numerical simulation when $\delta = 0.02$. The mth and nth curves avoid crossing when $m + n$ is even].

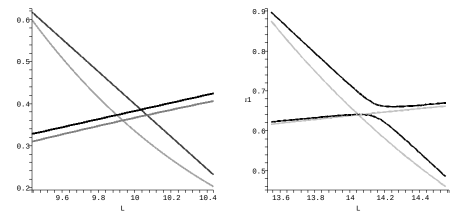

Fig. 2. The neutral stability curves drown in (L, ν) plane. The horizontal axis and vertical axis correspond to L and ν, respectively. Gray lines and black lines correspond to the neutral stability curves when $\delta = 0$ and $\delta = 0.03$ respectively. [Left: Neutral stability curves near $(L^{1,2}, \nu^{1,2})$], [Right: Neutral stability curves near $(L^{1,3}, \nu^{1,3})$].

Proposition 1. Let $m, n \in \mathbb{N}$, $(m \neq n)$. If $\delta > 0$ is sufficiently small, then the following holds. If $m + n$ is odd, then the neutral stability curves are given as two crossing curves in the neighborhood of $(L^{m,n}, \nu^{m,n})$. In addition, if $m + n$ is even, then the neutral stability curves are given as two hyperbolae in the neighborhood of $(L^{m,n}, \nu^{m,n})$.

The proof of Proposition 1 is given in [7]. Here, we give a sketch of the proof. The linearized eigenvalue problem of (1) with (2) is written as follows.

$$\begin{cases} \lambda w = \mathfrak{L}\, w, \\ w = \delta w_x \pm w_{xx} = 0 \text{ at } x = 0, L/2. \end{cases} \tag{5}$$

Here, $\mathfrak{L}$ denotes the linearized operator of the equation (1) around $w \equiv 0$, namely, it is defined as follows.

$$\mathfrak{L} := \left\{ \nu - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 \right\}.$$

We notice that problem (5) is self adjoint, that is, the following holds.

$$\langle \mathfrak{L} u, v \rangle_{L^2} = \langle u, \mathfrak{L} v \rangle_{L^2}.$$

Here, $\langle f, g\rangle_{L^2}$ denotes the standard L^2 inner product for real valued functions $f(x), g(x) \in L^2(0, L/2)$:

$$\langle f, g\rangle_{L^2} := \int_0^{L/2} f(x)g(x)\,\mathrm{d}x.$$

Therefore, all eigenvalues of the problem (5) are real.

We rewrite the linearized eigenvalue problem (5) as follows.

$$\frac{\mathrm{d}}{\mathrm{d}x} W = M(\lambda)\, W, \tag{6}$$

where

$$W := \begin{pmatrix} w \\ w_1 \\ w_2 \\ w_3 \end{pmatrix}, \quad w_j = \frac{\partial^j w}{\partial x^j}, \quad (j = 1, 2, 3), \tag{7}$$

and

$$M(\lambda) = \begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ \nu - 1 & 0 & -2 & -\lambda \end{pmatrix}. \tag{8}$$

Since we are interested in the parameter region which gives 0-eigenvalue, we set $\lambda = 0$. Let $\vec{\zeta}_j$ $(j = 1, \ldots, 4)$ be the eigenvectors of $M(0)$. Then, we obtain the general solution of (6) as follows.

$$W = c_1\vec{\zeta}_1 e^{\Lambda_+ x} + c_2\vec{\zeta}_2 e^{-\Lambda_+ x} + c_3\vec{\zeta}_3 e^{\Lambda_- x} + c_4\vec{\zeta}_4 e^{-\Lambda_- x}. \tag{9}$$

Here, $\Lambda_\pm$ denotes eigenvalues of $M(0)$, and c_j $(j = 1, 2, 3, 4)$ are arbitrary constants. From boundary conditions (2), we obtain the system of linear equations as follows.

$$P(L, \nu, \delta)\ C \ = \ 0, \tag{10}$$

where $P(L, \nu, \delta)$ is a 4×4 matrix and C denotes a vector $(c_1, c_2, c_3, c_4)^t$. Thus, the neutral stability curves are given as the set of parameters at which (10) has nontrivial solutions as follows.

$$\{(L, \nu, \delta) \in \mathbb{R}^3; g(L, \nu, \delta) = 0\}. \tag{11}$$

Here, $g(L, \nu, \delta) := \det P(L, \nu, \delta)$. Let $1 \gg \delta > 0$ and $m, n \in \mathbb{Z}$. Without loss of generality, we assume $m > n$. Then, we obtain the Taylor expansion of $g(L, \nu, \delta)$ near (L, ν, δ)=$(L^{m,n}, \nu^{m,n}, 0)$ as follows.

$$\begin{aligned} g(L, \nu, \delta) \ &= \ (\hat{L}^{m,n}, \hat{\nu}^{m,n}, \delta) H_{m,n} (\hat{L}^{m,n}, \hat{\nu}^{m,n}, \delta)^t \\ &\quad + O(|(\hat{L}^{m,n}) + (\hat{\nu}^{m,n}) + \delta|^3). \end{aligned} \tag{12}$$

Here, $(\hat{L}^{m,n}, \hat{\nu}^{m,n}) := (L, \nu) - (L^{m,n}, \nu^{m,n})$, and $H_{m,n}$ is Hesse matrix of $g(L, \nu, \delta)$ at $(L^{m,n}, \nu^{m,n}, 0)$. And as a result, if the sum $m + n$ is even, $H_{m,n}$ has two positive eigenvalues and third one is negative. On the other hand, if the sum $m + n$ is odd, $H_{m,n}$ has a 0-eigenvalue and other two eigenvalues are opposite sign. Finally, according to the classification theorem on the quadratic curves, conclusions of the proposition holds.

3. NUMERICAL STUDY TO GLOBAL BIFURCATION STRUCTURE

In this section, we show the global bifurcation structure of stationary solutions to (1) with (2) based on numerical simulations(Figure 3). We can see that several solution branches are folded with loops when $\delta = 0.05$. That is, when $\delta = 0$ the mix mode branch bifurcates from pure mode branch as pitchfork bifurcation. When $\delta > 0$, the pitchfork bifurcations are broken since the problem loses the SO(2) symmetry. Moreover, we can see that the imperfections of the pitchfork bifurcation close to the intersection points between mth and nth branches when $m + n$ is even.

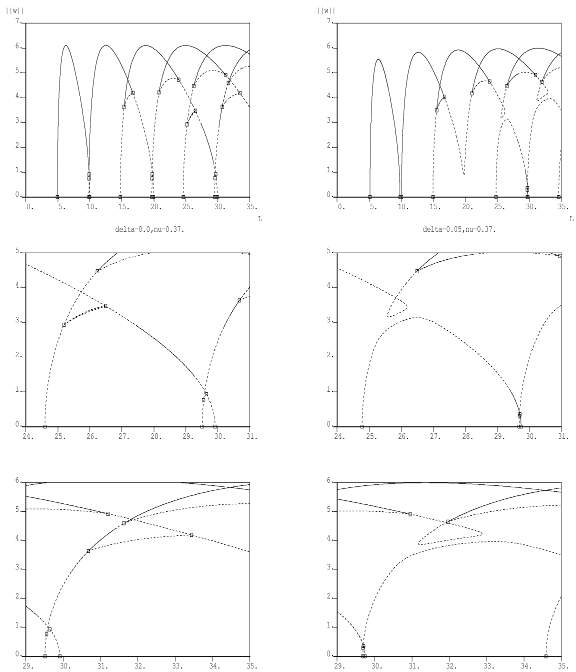

Fig. 3. Bifurcation diagrams of (1) with (2) for $\nu = 0.37$. The horizontal axis and vertical axis are L and $||w||$, respectively. [Top left: $\delta = 0$], [Top right: $\delta = 0.05$], [Middle: Close up around interaction point of third and fifth branches (Left: $\delta = 0$, Right: $\delta = 0.05$)], [Bottom: Close up to interaction point of fourth and sixth branches (Left: $\delta = 0$, Right : $\delta = 0.05$)].

4. BIFURCATION ANALYSIS TO SHE

We shall study the local bifurcation around the degenerate points to understand the qualitative change of the bifurcation diagram which we saw in the previous section. Let $\sigma_{l,\delta}$, $\phi_{l,\delta}(x)$, $l \in \mathbb{N}$ be the linearized eigenvalues and eigenfunctions of (1) with

(2). More precisely, $\phi = \phi_{l,\delta}(x)$ solves

$$\begin{aligned} &\sigma\phi = -\phi_{xxxx} - 2\phi_{xx} - (1-\nu)\phi, \\ &\phi = \delta\phi_x \pm \phi_{xx} = 0 \text{ at } x = 0, L/2 \end{aligned} \tag{13}$$

with eigenvalues $\sigma = \sigma_{l,\delta}$ for $l \in \mathbb{N}$. We note that $\sigma_{l,0} = \sigma_l = \nu - (1 - 4l^2\pi^2/L^2)^2$ and $\phi_{l,0} = w_l = \sin(2\pi lx/L)$.

Substituting the eigenfunction expansion:

$$w(t,x) = \sum_{l\in N} a_l(t)\phi_{l,\delta}(x)$$

into (1), we have

$$\dot{a}_j = \sigma_{j,\delta}a_j - \Big\langle \Big(\sum_{l\in N} a_l\phi_{l,\delta} \Big)^3,\ \phi_{j,\delta} \Big\rangle_{L^2} \Big/ ||\phi_{j,\delta}||^2_{L^2}, \quad j \in \mathbb{N}. \tag{14}$$

We define

$$B(r) := \{(L,\nu); (\hat{L}^{m,n})^2 + (\hat{\nu}^{m,n})^2 < r^2\}.$$

Theorem. Let $m, n \in \mathbb{N}$ $(n \neq m)$. There exist a positive constant ε such that for $\delta < O(\varepsilon^3)$, the cubic normal form of (1) with (2) on the center manifold are given as follows if $(L,\nu) \in B(\varepsilon) \setminus B(\varepsilon^2)$

$$\begin{cases} \dot{a}_m = (\sigma_{m,\delta} + Aa_m^2 + Ba_n^2)a_m \\ \dot{a}_n = (\sigma_{n,\delta} + Ca_m^2 + Da_n^2)a_n \qquad \text{(if } m+n \text{ is odd).} \end{cases} \tag{15}$$

$$\begin{cases} \dot{a}_m = (\sigma_{m,\delta} + Aa_m^2 + Ba_n^2)a_m + Ea_n^3 + Fa_na_m^2 \\ \dot{a}_n = (\sigma_{n,\delta} + Ca_m^2 + Da_n^2)a_n + Ga_m^3 + Ha_n^2a_m \qquad \text{(if } m+n \text{ is even).} \end{cases} \tag{16}$$

Here, A, B, C, D, E, F, G, H are depend on δ, m, n, and it holds that $A, B, C, D < 0$ and $AD - BC < 0$. In addition, for $\delta = 0$, $m \neq 3n$ (or $n \neq 3m$), it holds that $E = F = G = H = 0$. Moreover, (15) is robust against perturbations in higher order terms.

The proof of Theorem is given in [7]. Here, we give a sketch of the proof. Let $j, l \in \mathbb{N}$ and $j + l$ be even. Then, the following equality holds.

$$\sigma_{j,\delta} < \phi_{j,\delta}, \phi_{l,0} >_{L^2} = \sigma_{l,0} < \phi_{j,\delta}, \phi_{l,0} >_{L^2} + k_0 l\{(-1)^l \phi''_{j,\delta}(L/2) - \phi_{j,\delta}(0)''\}. \tag{17}$$

Here, $k_0 = 2\pi/L$. Moreover, eigenfunctions $\phi_{j,\delta}$, $(j = 1, 2, 3, \dots)$ satisfy the following properties

$$\phi_{2l-1,\delta}(x) = \phi_{2l-1,\delta}(L/2 - x), \quad \phi_{2l,\delta}(x) = -\phi_{2l,\delta}(L/2 - x), \quad (l \in \mathbb{N}).$$

We construct the center manifolds for (14) which are expressed as $a_l = h_l(a_m, a_n)$, $(l \neq m, n)$ for $|a_m|, |a_n| < O(\varepsilon)$, $|\sigma_{m,\delta}|, |\sigma_{n,\delta}| < O(\varepsilon^2)$, $\delta < O(\varepsilon^3)$. Let $l \in \mathbb{N}$, $l \neq m, n$. As a result, for $|a_m|, |a_n| < O(\varepsilon)$, $|\sigma_{m,\delta}|, |\sigma_{n,\delta}| < O(\varepsilon^2)$, $\delta < O(\varepsilon^3)$, we have $|h_l(a_n, a_m)| < O(\varepsilon^4)$. It follows that the equation (14) is reduced to

$$\begin{cases} \dot{a}_m = \sigma_{m,\delta}a_m - \big\langle (a_m\phi_{m,\delta} + a_n\phi_{n,\delta})^3, \phi_{m,\delta} \big\rangle_{L^2} / ||\phi_{m,\delta}||^2_{L^2} + O(\varepsilon^4) \\ \dot{a}_n = \sigma_{n,\delta}a_n - \big\langle (a_m\phi_{m,\delta} + a_n\phi_{n,\delta})^3, \phi_{n,\delta} \big\rangle_{L^2} / ||\phi_{n,\delta}||^2_{L^2} + O(\varepsilon^4) \end{cases}$$

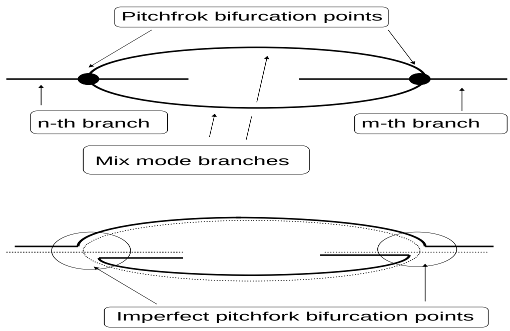

Fig. 4. Schematic pictures of the bifurcation structures.
[Above: The case $m+n$ is odd], [Below: The case $m+n$ is even].

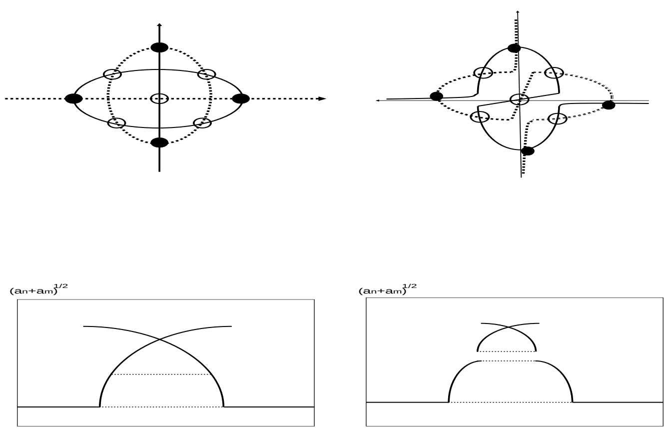

Fig. 5. [Above: Phase portraits of normal forms for $\sigma_{\delta,m} = \sigma_{\delta,n} > 0$. The horizontal axis and vertical axis are a_m and a_n, respectively. Left: The case when $m+n$ is odd. Right: The case when $1 \gg \delta > 0$ and $m+n$ is even], [Below: The bifurcation diagram of equilibriums of the normal forms. The vertical axis is $\sqrt{a_n^2 + a_m^2}$. Left: The case when $m+n$ is odd. Right: The case when $m+n$ is even and $1 \gg \delta > 0$].

Moreover, it holds that the normal form of (14) are invariant under the mapping $(a_m, a_n) \to (-a_m, -a_n)$ since (1) with (2) is invariant under the mapping $w(t,x) \to$

$-w(t,x)$. More precisely, the nonlinear terms of the normal form are expressed as

$$\sum_{\substack{p,q,j\in\mathbb{N}\\ p+q=2j+1}} C^{(p,q)}\ a_n^p\ a_m^q \tag{18}$$

Now we divide the proof into two parts.

Part 1. We prove Theorem in the case when $m+n$ is odd. Without loss of generality, we assume that m is odd and n is even. We represent the eigenfunction expansion as follows:

$$\sum_{j\in\mathbb{N}} a_j\phi_{j,\delta} = \sum_{j_i\in\mathbb{N}} a_{2j_1-1}\phi_{2j_1-1,\delta} + \sum_{j_2\in\mathbb{N}} a_{2j_2}\phi_{2j_2,\delta}.$$

The equation (1) with (2) is invariant under the change of variable: $x \to L/2 - x$. And using symmetry property of the eigenfunctions, it follows that the normal form is invariant under the mappings: $(a_m, a_n) \to (a_m, -a_n)$ and $(a_m, a_n) \to (-a_m, a_n)$. Thus, we obtain the normal form to (14) as follows.

$$\begin{aligned} \dot{a}_m &= \sigma_{m,\delta}a_m + A\ a_m^3 + B\ a_n^2 a_m + h.o.t, \\ \dot{a}_n &= \sigma_{n,\delta}a_n + C\ a_m^2 a_n + D\ a_n^3 + h.o.t. \end{aligned} \tag{19}$$

And we can find that A, B, C, $D< 0$ and $AD - BC < 0$ for $0 < \delta \ll 1$. Moreover, we can verify that (15) is robust against perturbations in higher order terms.

Part 2. In this part, we prove Theorem in the case when $m+n$ is even. Without loss of generality, we assume $m > n$. Then, the cubic normal form is given as follows:

$$\begin{cases} \dot{a}_m = \sigma_{m,\delta}a_m + A\ a_m^3 + B\ a_n^2 a_m + E\ a_n^3 + F\ a_m^2 a_n, \\ \dot{a}_n = \sigma_{n,\delta}a_n + C\ a_m^2 a_n + D\ a_n^3 + G\ a_m^3 + H\ a_n^2 a_m. \end{cases} \tag{20}$$

And we can verify that $A, B, C, D < 0$, $AD - BC < 0$, $H = 3E$ and $F = 3G$. Moreover, in the case when $m \neq 3n$ and $\delta = 0$, it holds that $E = G = F = H = 0$.

By the normal form analysis, we can understand the local bifurcation structure of stationary solutions to (1) with (2). More precisely, when $m+n$ is odd, the bifurcation structure of the equilibrium to (15) is robust for small perturbation $\delta > 0$. On the other hand, if $m+n$ is even, the bifurcation structure of the equilibrium to (16) is changed under the perturbation with small $\delta > 0$ (see Figure 5).

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