KYBERNETIKA — VOLUME *43* (2007), NUMBER 6, PAGES 797 – 806

RIGOROUS NUMERICS FOR SYMMETRIC HOMOCLINIC ORBITS IN REVERSIBLE DYNAMICAL SYSTEMS

Yasuaki Hiraoka

We propose a new rigorous numerical technique to prove the existence of symmetric homoclinic orbits in reversible dynamical systems. The essential idea is to calculate Melnikov functions by the exponential dichotomy and the rigorous numerics. The algorithm of our method is explained in detail by dividing into four steps. An application to a two dimensional reversible system is also treated and the existence of a symmetric homoclinic orbit is rigorously verified as an example.

Keywords: rigorous numerics, exponential dichotomy, homoclinic orbits

AMS Subject Classification: 65G20, 65P30

1. INTRODUCTION

In the theory of dynamical systems, a solution which connects two fixed points (say p_1 and p_2) is called a heteroclinic orbit and, in the case of $p_1 = p_2$, we call it a homoclinic orbit. It is well known that homoclinic and heteroclinic orbits play a crucial role in the study of dynamical systems, since the existence of these solutions are closely related to global bifurcation structures, appearances of chaos and so on (e. g., see [5]). Due to its importance, many works to study the existence of homoclinic and heteroclinic orbits have been done so far, and we have now an accomplished tool to analytically treat these orbits, which is called Melnikov functions [5, 12].

Suppose we have a heteroclinic orbit $w(t)$ connecting two hyperbolic fixed points in an unperturbed problem. Then, in the theory of Melnikov functions, we can investigate the persistence of the heteroclinic orbit under small perturbations by the original heteroclinic orbit $w(t)$ and an exponential dichotomy property [3]. From this strategy, bifurcation problems of homoclinic and heteroclinic orbits have been studied well [1, 9, 10]. However, it is obvious from the assumption that this theory can not be used to study homoclinic and heteroclinic orbits unless we obtain the original solution $w(t)$. Namely, we can not apply the theory to directly study the existence of these orbits, although it is a very powerful tool for bifurcation problems. In the practical problems, it often occurs that we can not obtain $w(t)$ and, consequently, can not regard the problems as perturbation settings. This is the motivation of our study.

In this paper, we propose a numerical verification method to show the existence of symmetric homoclinic orbits in reversible dynamical systems. This is an extension of the Melnikov theory in the following sense that we do not need a known homoclinic orbit $w(t)$, but we replace it by numerical approximate solutions.

The problem we consider is supposed that a reversible dynamical system (definition of the reversibility is described in Section 2)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \quad x \in \mathbb{R}^N \tag{1}$$

has a hyperbolic fixed point at the origin. Here we assume that the vector field f is smooth. Let us suppose that we have an approximate numerical homoclinic solution

$$\left\{(\xi^i, t_i) \mid \xi^i \in \mathbb{R}^N, t_i \in \mathbb{R},\ i = 0, 1, \dots, K\right\}, \tag{2}$$

which is usually obtained by numerical simulations. In this setting, the purpose of this paper is to present a rigorous numerical method to prove the existence of symmetric homoclinic orbits in a neighborhood of the numerical solution (2).

Let us here remark that the reversible assumption can be essentially removed and it may be possible to extend to general heteroclinic bifurcation problems. Moreover, since our method is based on the rigorous numerics of Melnikov functions, the method may be also applied to the stability analysis of traveling pulses in reaction diffusion equations with one space dimension [8]. This potential to the stability analysis seems remarkable comparing to some known rigorous numerical methods dealing with homoclinic and heteroclinic orbits [13, 15]. Namely, it may be possible that we verify not only the existence of a traveling pulse, which corresponds to a homoclinic or heteroclinic orbit in the moving coordinate, but also its stability simultaneously. These extensions will be discussed in future work [6].

2. ALGORITHM

We propose an algorithm consisting of the following four steps for the numerical verifications of homoclinic orbits:

Step 1. Construction of an approximate solution

Step 2. Enclosure of a fundamental matrix solution

Step 3. Construction of the stable manifold

Step 4. Analysis for an intersection of the stable and unstable manifolds.

The basic strategy is to rigorously calculate a Melnikov function by using an approximate numerical homoclinic solution (2) and an exponential dichotomy property. We refer to the paper [9] for a comparison to the original Melnikov type argument.

In this paper, we impose the following hypotheses on the dynamical system (1):

(H1): We assume $N=2n$ and S-reversibility. That is to say, the vector field satisfies $f(Sx)=-Sf(x)$ for a linear map $S:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ with $S^2 = I_{2n}$. Here I_{2n} is the identity map on $\mathbb{R}^{2n}$.

(H2): Eigenvalues of the linearized matrix $f_x(0)$ at the origin are given by

$$\{\pm\lambda_i \mid i = 1, \ldots, n,\ \mathrm{Re}\lambda_i > 0,\ \lambda_1 < \mathrm{Re}\lambda_j (j \geq 2)\}.$$

From the reversibility, we can show that if λ is an eigenvalue then so is $-\lambda$. Therefore hypothesis (H2) actually imposes $0 < \lambda_1 \in \mathbb{R}$ and $\lambda_1 < \mathrm{Re}\lambda_j,\ j = 2, \ldots, n$.

A homoclinic orbit $h(t)$ in the reversible system (1) is called symmetric if $h(-t) = Sh(t)$ for all t. Since we only deal with symmetric homoclinic orbits in this paper, we prepare a numerical homoclinic solution (2) as the following form

$$\{(\xi^i, t_i) \mid i = 0, \pm 1, \ldots, \pm K, \xi^{-i} = S\xi^i, t_{-i} = -t_i, \xi^{\pm K} \approx 0\}. \tag{3}$$

It should be noted that ξ^0 is selected at a point on the S-invariant subspace $\mathrm{Fix}(S) := \{x \in \mathbb{R}^{2n} | Sx = x\}$ (see Figure 1).

Under this situation, we explain each step of the algorithm in detail.

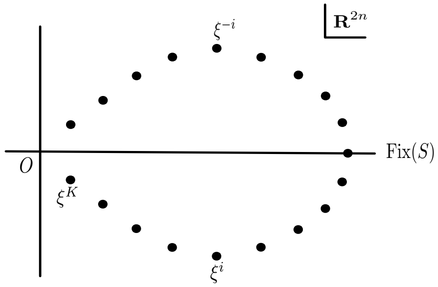

Fig. 1. Numerical homoclinic solution.

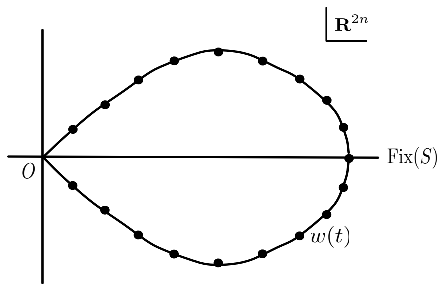

Fig. 2. Approximate solution $w(t)$.

2.1. Step 1: Construction of an approximate solution

In this step, we construct an approximate solution $w(t) \in \mathbb{R}^{2n}, t \in \mathbb{R}$, as a continuous curve by a given numerical homoclinic solution (3). A basic strategy for the construction is given as follows (see Figure 2):

- $w(t_i) := \xi^i$.
- Polynomial interpolation for each time interval $[t_i, t_{i+1}],\ i = 0, \ldots, K-1$.
- $w(t) := \xi^K e^{-\lambda_1(t-t_K)}, t \geq t_K$.
- $w(t) := Sw(-t), t \leq 0$.

Namely, we adopt a polynomial interpolation for each time interval in finite time region $[0, t_K]$, and we put an exponential decay property for $t \in [t_K, \infty)$. Here, let us note that the decay rate is determined by λ_1. This is because a homoclinic orbit generically decays along the stable subspace given by the eigenvector of $-\lambda_1$[4].

In practical numerical verifications, we shall put some additional information on coefficients of polynomial interpolations. For example, we can determine a polynomial interpolation by specifying its differential coefficients at each end point $t = t_i$. These derivative information will be given in such a way that an operator introduced in Step 3 becomes contractive. We will discuss this subject in Section 2.3 in detail.

2.2. Step 2: Enclosure of a fundamental matrix solution

First of all, let us recall an exponential dichotomy property [3] on an ordinary differential equation
$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in I, \tag{4}$$
where I is an interval in $\mathbb{R}$. Let $X(t)$ be its fundamental matrix solution.

Definition 1. The equation (4) is said to have an exponential dichotomy on I if there exist positive constants M, α, and a projection matrix P such that the following inequalities
$$\begin{aligned} &|X(t)PX(s)^{-1}| \leq Me^{-\alpha(t-s)}, && \text{if } s \leq t \text{ and } s,t \in I \\ &|X(t)(I-P)X(s)^{-1}| \leq Me^{-\alpha(s-t)}, && \text{if } t \leq s \text{ and } s,t \in I \end{aligned} \tag{5}$$
are satisfied.

We consider the variational equation
$$\dot{x} = A(t)x, \quad A(t) = f_x(w(t)) \tag{6}$$
with respect to the approximate solution $w(t)$. Then, due to [2] and [9], the following property holds for (6).

Lemma 2. The variational equation (6) has an exponential dichotomy on $\mathbb{R}_+ = [0, \infty)$ with the projection matrix
$$P = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}. \tag{7}$$

In this step, we explicitly construct an enclosure of the fundamental matrix solution which satisfies the exponential dichotomy property on $\mathbb{R}_+$ with the projection matrix (7).

It should be noted that, from the asymptotic behavior of $A(t)$, there exist fundamental solutions $\varphi_i(t), i = \pm 1, \pm 2, \ldots, \pm n$, of (6) such that the following property holds (e. g., [2]):
$$\lim_{t\to\infty} \varphi_i(t)e^{-\lambda_i(t-t_K)} = p_i \qquad \lim_{t\to\infty} \varphi_{-i}(t)e^{\lambda_i(t-t_K)} = Sp_i. \tag{8}$$
Here p_i is an eigenvector for the eigenvalue λ_i and, from the reversibility, Sp_i corresponds to an eigenvector for the eigenvalue $-\lambda_i$. Then, it is easily shown that the fundamental matrix solution determined by $X(t) = [\varphi_{-1}(t) \cdots \varphi_{-n}(t)\varphi_1(t) \cdots \varphi_n(t)]$ attains the exponential dichotomy property on $\mathbb{R}_+$ with (7). Hence, for the enclosure of $X(t)$, it suffices to enclose all fundamental solutions $\varphi_i(t), i = \pm 1, \ldots, \pm n$, which satisfy (8) on $\mathbb{R}_+$. In what follows, by dividing into the asymptotic part $[t_K, \infty)$ and the finite interval part $[0, t_K]$, we construct these enclosures on both parts by different ways, respectively.

We first explicitly construct the successive approximations $\varphi_i^{(j)}(t), j = 0, 1, \ldots,$ of the fundamental solutions $\varphi_i(t), i = \pm 1, \ldots, \pm n$, for $t \in [t_K, \infty)$ by using the similar manner discussed in Chapter 3 of [2]. Namely, we successively derive approximate solutions $\varphi_i^{(j)}(t)$ by taking $\varphi_i^{(0)}(t) = p_i e^{\lambda_i(t-t_K)}$ and $\varphi_{-i}^{(0)}(t) = Sp_i e^{-\lambda_i(t-t_K)}$ as the initial approximate solutions. It can be proved that, for each eigenvalue, there exist functions $\varphi_i(t), i = \pm 1, \ldots, \pm n$, to which the sequences of the approximate solutions converge, and these functions satisfy (6) and (8). Moreover, it is important that the error bound between the approximate solution $\varphi_i^{(j)}(t)$ and $\varphi_i(t)$ can be explicitly derived for each j. Therefore, it enables us to enclose the fundamental solution $\varphi_i(t)$ for $t \in [t_K, \infty)$ by using the approximate solution $\varphi_i^{(j)}(t)$ and its error bound.

Next, let us enclose the fundamental solutions in the finite interval part by using Lohner's method [11], which is one of the numerical verification techniques to enclose solutions of initial value problems for a finite time interval in ordinary differential equations. We take the enclosure of $\varphi_i(t_K)$ as the set of initial values and solve (6) from $t = t_K$ to $t = 0$ by Lohner's method.

2.3. Step 3: Construction of the stable manifold

In this step we characterize the stable manifold of the origin in a neighborhood of $w(t)$. For this purpose, let us introduce a new variable $v := x - w$. Then the differential equation (1) is transformed into

$$\begin{aligned} \dot{v} &= A(t)v + g(t,v) \\ g(t,v) &:= -\dot{w}(t) + f(w(t)+v) - A(t)v. \end{aligned} \tag{9}$$

It should be noted that, due to the hyperbolicity of the origin and the asymptotic behavior of $w(t)$, if $v(t)$ is a solution of (9) such that $\sup_{t\in\mathbb{R}_+} v(t) < \epsilon$ for a sufficiently small ϵ, then $x(t) = w(t) + v(t)$ stays on the stable manifold of the origin.

Let $B(\mathbb{R}_+)$ be the set of all continuous and bounded functions from $\mathbb{R}_+$ to $\mathbb{R}^{2n}$. This function space becomes a Banach space under the norm defined by $||v|| := \sup_{t\in\mathbb{R}_+} |v(t)|$. Then, the following lemma holds due to the exponential dichotomy.

Lemma 3. (cf. Kukubu [9]) The differential equation (9) is equivalent to

$$v(t) = X(t)P\left\{X(0)^{-1}v(0) + \int_0^t X(s)^{-1}g(s,v)\,\mathrm{d}s\right\} - \int_t^\infty X(t)(I-P)X(s)^{-1}g(s,v)\,\mathrm{d}s \tag{10}$$

on $B(\mathbb{R}_+)$.

We now define an operator on $B(\mathbb{R}_+)$ which depends on a parameter $\eta \in \mathbb{R}^{2n}$ as follows:

$$(T_\eta(v))(t) := X(t)P\left\{X(0)^{-1}\eta + \int_0^t X(s)^{-1}g(s,v)\,\mathrm{d}s\right\} - \int_t^\infty X(t)(I-P)X(s)^{-1}g(s,v)\,\mathrm{d}s \tag{11}$$

Note that a fixed point $v = T_\eta(v)$ becomes a solution of (9) and η controls the initial value of the fixed point. In addition, we can also show that $T_\eta : B(\mathbb{R}_+) \to B(\mathbb{R}_+)$ from the exponential dichotomy.

Let $B_M(\mathbb{R}_+) := \{v \in B(\mathbb{R}_+) \mid ||v|| \leq M\}$ be the closed ball with the radius M. Then, by using the similar arguments in [16], we can derive the following proposition about the shadowing of orbits converging to the origin.

Proposition 4. Suppose $Y, Z > 0$ are taken for $\eta \in \mathbb{R}^{2n}$ and $\epsilon > 0$ such that

$$||T_\eta(0)|| \leq Y, \quad \sup_{w_1, w_2 \in B_\epsilon(\mathbb{R}_+)} ||T'_\eta(w_1)w_2|| \leq Z.$$

If $Y + Z < \epsilon$, then there exists the unique fixed point v_η of T_η in $B_{Y+Z}(\mathbb{R}_+)$.

Let us note that Y and Z appearing in the above can be explicitly calculated for each ϵ and η, and hence this proposition enables us to study the existence of the fixed point of T_η. In fact, by the explicit form of $w(t)$ and $X(t)$ treated in Step 1 and Step 2, we can estimate $T_\eta(0)$ and $T'_\eta(w_1)w_2$ for a given $B_\epsilon(\mathbb{R}_+)$. Namely, we derive these estimates by numerical verifications for $[0, t_K]$, and by the asymptotic forms of $w(t)$ and the enclosure of $X(t)$ for $[t_K, \infty)$, respectively.

In addition, let us remark that, if there exist Y, Z, and ϵ satisfying the sufficient condition $Y + Z < \epsilon$ for any η in some subset $D \subset \mathbb{R}^{2n}$, the stable manifold of the origin in a neighborhood of $w(0)$ can be described by $w(0)+v_\eta(0)$ for $\eta \in D$, where v_η expresses the unique fixed point of T_η. Thus, in the practical numerical verification, we try to construct a suitable subset $D \subset \mathbb{R}^{2n}$ given by the product of intervals such that the sufficient condition is satisfied for any $\eta \in D$, and characterize the stable manifold, which will be finally analyzed to show the existence of symmetric homoclinic orbits in the next step.

Before discussing Step 4, let us briefly summarize the relationship of the contractivity of T_η, the choice of the approximate solution $w(t)$, and ϵ. In general, it is obvious that we can not expect T_η to be contractive. One of the reasons is that, since the fundamental matrix solution $X(t)$ possesses the exponential dichotomy property, the fundamental solutions $\varphi_{-i}(t), i = 1, 2, \ldots, n$, grows exponentially as $t \to 0$. This causes Y and Z to be large unless $g(s, v)$ and $g_v(s, v)$ are sufficiently small, where $g_v(s, v)$ denotes the derivative of $g(s, v)$ with respect to v. Hence, let us here explain how we guarantee the contractivity of the operator T_η by controlling $g(s, v)$ and $g_v(s, v)$.

Let us first discuss how to make Y small. Since Y is obtained by an upper estimate of

$$(T_\eta(0))(t) = X(t)P\left\{X(0)^{-1}\eta + \int_0^t X(s)^{-1}g(s, 0)\,\mathrm{d}s\right\} - \int_t^\infty X(t)(I-P)X(s)^{-1}g(s, 0)\,\mathrm{d}s$$

for $\eta \in D$ and D is usually taken as a small subset in $\mathbb{R}^{2n}$, if we have small $g(t, 0)$, then we can derive small Y. Here $g(t, 0)$ is given from (9) as

$$g(t, 0) = f(w(t)) - \dot{w}(t). \tag{12}$$

As is mentioned right after Proposition 4, $|T_\eta(0)(t)|$ is estimated by the numerical verification for $[0,t_K]$. Especially, the rigorous calculations of the integral parts are performed for each time step $[t_i,t_{i+1}], i=0,\dots,K-1$. Hence, if $g(t,0)$ is small for each time step, then the estimates of the integrals become small.

Let us note that, by Tayor's theorem, $g(t,0), t\in[t_i,t_{i+1}]$, can be expressed as

$$g(t,0)=g(t_i)+\frac{\mathrm{d}g}{\mathrm{d}t}(t_i,0)(t-t_i)+\dots+\frac{\mathrm{d}^{n-1}g}{\mathrm{d}t^{n-1}}(t_i,0)\frac{(t-t_i)^{n-1}}{(n-1)!}+\frac{\mathrm{d}^n g}{\mathrm{d}t^n}(t_\theta,0)\frac{(t-t_i)^n}{n!},$$

where $t_\theta\in[t_i,t_{i+1}]$. From this expression, if

$$\frac{\mathrm{d}^k g}{\mathrm{d}t^k}(t_i,0)=0,\quad k=0,1,\dots,n-1 \tag{13}$$

holds, then $g(t,0)$ satisfies

$$g(t,0)\in\frac{1}{n!}\frac{\mathrm{d}^n g}{\mathrm{d}t^n}([t_i,t_{i+1}],0)[0,(t_{i+1}-t_i)^n].$$

It means that $g(t,0)$ can be suppressed by the nth order of the time step.

Now, as is explained in Step 1, the coefficients of the polynomial interpolations for $w(t)$ are used for satisfying (13). Namely, we successively determine the differential coefficients $\frac{\mathrm{d}^{k+1}w}{\mathrm{d}t^{k+1}}(t_i)$ by (12) in such a way that (13) holds, and obtain the polynomial interpolation for each time step $[t_i,t_{i+1}]$. From this process, we can expect to obtain small Y, if we set sufficiently small time steps. Let us comment that this process corresponds to adding C^n smoothness information to approximate the true homoclinic orbit by $w(t)$.

Next we consider the estimate of Z. In this case, since $\left(T'_\eta(w_1)w_2\right)(t)$ is given as

$$\begin{aligned}\left(T'_\eta(w_1)w_2\right)(t)&=\int_0^t X(t)PX(s)^{-1}g_v(s,w_1)w_2\,\mathrm{d}s\\&\quad-\int_t^\infty X(t)(I-P)X(s)^{-1}g_v(s,w_1)w_2\,\mathrm{d}s,\quad w_1,w_2\in B_\epsilon(\mathbb{R}_+),\end{aligned}$$

we wish to have small $g_v(t,w_1)w_2$ for $w_1,w_2\in B_\epsilon(\mathbb{R}_+)$. Here, let us consider a formal expansion of $g_v(t,w_1)w_2$ at $w(t)$. Then the following estimate holds:

$$\begin{aligned}g_v(t,w_1(t))w_2(t)&=f_x(w(t)+w_1(t))w_2(t)-Aw_2(t)\\&=f_{xx}(w(t))w_2(t)w_1(t)+\dots+\frac{1}{(n-1)!}f_{nx}(w(t))w_2(t)w_1(t)^{n-1}+\cdots\\&\subset f_{xx}(w(t))[-\epsilon^2,\epsilon^2]+O(\epsilon^3)\end{aligned}$$

It means that $g_v(t,w_1)w_2$ can be estimated by the second order with respect to ϵ.

From the above argument, since the right hand side of the sufficient condition in Proposition 4 is given by ϵ, we can expect the contractivity of the operator (11) by taking small time steps and ϵ.

2.4. Step 4: Analysis for an intersection of the stable and unstable manifolds

This is the final step of the algorithm, and we investigate an intersection of the stable and unstable manifolds of the origin. Here we explicitly use the reversibility of the vector field $f(x)$, which makes easy the analysis for an intersection of the stable and unstable manifolds. Therefore, let us first briefly recall some of the fundamental properties of reversible systems (e. g., see [14]).

Suppose a dynamical system $\dot{x} = f(x)$ is S-reversible, i. e., $f(Sx) = -Sf(x)$. It is obvious to show that, if $x(t)$ is a solution, so is $Sx(-t)$. Thus, $x(0) \in \mathrm{Fix}(S)$ leads to $x(t) = Sx(-t)$ from the uniqueness of the initial values. Let $x = 0$ be a fixed point and $W^s(0), W^u(0)$ be the stable and unstable manifolds of the origin, respectively. Then, if $x(0) \in \mathrm{Fix}(S) \cap W^s(0)$, then $x(0) \in W^u(0)$ by $\lim_{t\to-\infty} x(t) = \lim_{t\to-\infty} Sx(-t) = 0$. Namely, it becomes the symmetric homoclinic orbit.

From these properties, we can verify the existence of symmetric homoclinic orbits by investigating an intersection of the stable manifold constructed in Step 3 and $\mathrm{Fix}(S)$ without explicitly deriving the unstable manifold. This is the reason that we only treated $\mathbb{R}_+$ so far. Moreover, it is known that symmetric homoclinic orbits in reversible systems are structurally stable [7]. Hence, it is not necessary to deal with the analysis as bifurcation problems.

Now we consider how to verify an intersection of the stable manifold and $\mathrm{Fix}(S)$. Suppose that we succeeded in verifying the fixed points of (11) for η belonging to some subset D. In order to show $x_\eta(0) = w(0) + v_\eta(0) \in \mathrm{Fix}(S)$, it suffices that there exists $\eta \in D$ such that $v_\eta(0) \in \mathrm{Fix}(S)$, since $w(0) = \xi^0 \in \mathrm{Fix}(S)$. Therefore, for analyzing $v_\eta(0)$, let us introduce the following decomposition

$$\mathbb{R}^{2n} = \mathrm{Fix}(S) \oplus V, \quad V := \left\{x \in \mathbb{R}^{2n} \mid Sx = -x\right\}$$

and the projection $Q : \mathbb{R}^{2n} \to \mathrm{Fix}(S)$.

Here we define the following operator

$$\begin{aligned}
&E : \mathrm{Fix}(S) \oplus V \to V,\\
&E(\eta_1, \eta_2) := (I - Q)\tilde{E}(\eta_1, \eta_2), \quad (\eta_1, \eta_2) \in \mathrm{Fix}(S) \oplus V,\\
&\tilde{E}(\eta_1, \eta_2) := v_\eta(0) = X(0)\left(PX(0)^{-1}\eta - \int_0^\infty (I-P)X(s)^{-1} g(s, v_\eta)\,\mathrm{d}s\right),
\end{aligned} \tag{14}$$

where $\eta_1 = Q\eta,\ \eta_2 = (I - Q)\eta$. From this definition, $\eta \in D$ satisfying $E(\eta_1, \eta_2) = 0$ leads to $v_\eta(0) \in \mathrm{Fix}(S)$. Therefore, we finally transform the operator E into some fixed point form on D in order to study the existence of its fixed point by numerical verifications.

In practice, from (14), let us define

$$R := \frac{\partial}{\partial \eta_2}\left\{(I - Q)X(0)PX(0)^{-1}\eta\right\}$$

as an approximate matrix to $\frac{\partial}{\partial \eta_2} E(\eta_1, \eta_2)$ and introduce the following Newton type operator as a fixed point form of $E(\eta_1, \eta_2) = 0$:

$$F(\eta_1, \eta_2) := R^{-1}\left\{R\eta_2 - E(\eta_1, \eta_2)\right\}. \tag{15}$$

It is obvious that $F(\eta_1, \eta_2) = \eta_2$ is equivalent to $E(\eta_1, \eta_2) = 0$ and the fixed point of F can be easily studied by numerical verification techniques, since F is an operator on the finite dimensional space.

3. NUMERICAL EXAMPLE

In this section, we apply the numerical verification method presented in this paper to a practical problem in order to check the validity of the algorithm. Let us consider the following two dimensional reversible system

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(u), \quad f(u) = \begin{pmatrix} u_2 \\ 4u_1 - 3u_1^2 \end{pmatrix} \tag{16}$$

as an example. Here the vector field is reversible with respect to $S(u_1, u_2) = (u_1, -u_2)$. This dynamical system is obtained from the KdV equation under a moving coordinate and the existence of 1-soliton solutions which correspond to symmetric homoclinic orbits is known. Here, by applying Newton's method to (16), we prepare a homoclinic numerical solution

$$\left\{(\xi_1^i, \xi_2^i, t_i) \mid i = 0, \pm 1, \ldots, \pm K\right\},$$

which is shown in Figure 3. Here we take $K = 6000$, $t_K = 4.0$. In addition,

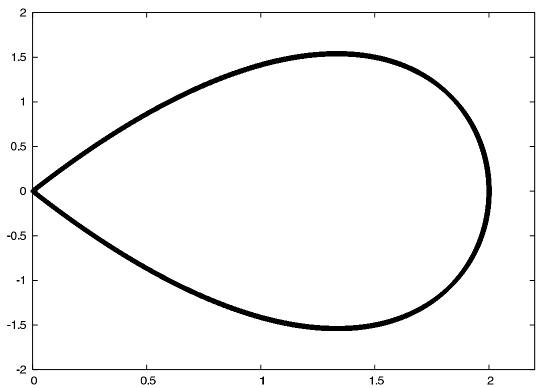

Fig. 3. Numerical homoclinic solution for (16).

we adopt cubic polynomial interpolations for the construction of the approximate solution $w(t)$.

First of all, about the sufficient condition of Proposition 4, when we choose $\epsilon = 0.00005$ and $D = \left[-10^{-10}, 10^{-10}\right] \times \left[-10^{-5} \times 10^{-5}\right]$, we have obtained

$$Y = 0.000013012, \qquad Z = 0.000002167$$

for $\eta \in D$, so $Y + Z < \epsilon$ have been verified.

Next, we study an intersection of the stable manifold and $\mathrm{Fix}(S)$ by investigating the fixed point of (15) with respect to η_2. The image of D have been rigorously calculated as follows

$$F(D) \subset [-0.0000050527, 0.0000051626] \subset D_{\eta_2},$$

where $D_{\eta_2} := (I-Q)D$. Due to Brouwer's fixed point theorem, this inclusion shows the existence of the fixed point and, hence the existence of the symmetric homoclinic orbit have been verified by our method.

(Received November 30, 2006.)

REFERENCES

[1] S.-N. Chow, B. Deng, and B. Fiedler: Homoclinic bifurcation at resonant eigenvalues. J. Dyn. Differential Equations *2* (1990), 177–244.

[2] E. A. Coddington and L. Levinson: Theory of Ordinary Differential Equations. McGraw-Hill, New York 1955.

[3] W. A. Coppel: Dichotomies in Stability Theory. (Lecture Notes in Mathematics 629.), Springer-Verlag, Berlin 1978.

[4] B. Deng: The Sil'nikov problem, exponential expansion, strong λ-lemma, C^1-linearization, and homoclinic bifurcation. J. Differential Equations *79* (1989), 189–231.

[5] J. Guckenheimer and P. Holmes: Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. Springer–Verlag, Berlin 1983.

[6] Y. Hiraoka: in preparation.

[7] G. Iooss and M. C. Pérouème: Perturbed homoclinic solutions in reversible 1:1 resonance vector fields. J. Differential Equations *102* (1993), 62–88.

[8] T. Kapitula: The Evans function and generalized Melnikov integrals. SIAM J. Math. Anal. *30* (1999), 273–297.

[9] H. Kokubu: Homoclinic and heteroclinic bifurcations in vector fields. Japan J. Appl. Math. *5* (1988), 455–501.

[10] M. Kisaka, H. Kokubu, and H. Oka: Bifurcations to N-Homoclinic orbits and N-periodic orbits in vector fields. J. Dyn. Differential Equations *5* (1993), 305–357.

[11] R. J. Lohner: Einschliessung der Lösung gewonhnlicher Anfangs- and Randwertaufgaben und Anwendungen. Thesis, Universität Karlsruhe (TH) 1988.

[12] V. K. Melnikov: On the stability of center for time periodic perturbations. Trans. Moscow Math. Soc. *12* (1963) , 1–57.

[13] S. Oishi: Research Institute for Mathematical Sciences Kôkyûroku, *928* (1995), 14–19.

[14] A. Vanderbauwhede and B. Fiedler: Homoclinic period blow-up in reversible and conservative systems. Z. Angew. Math. Phys. *43* (1992), 292–318.

[15] D. Wilczak and P. Zgliczyński: Heteroclinic connections between periodic orbits in planar restricted circular three-body problem – a computer assisted proof. Comm. Math. Phys. *234* (2003), 37–75.

[16] N. Yamamoto: A numerical verification method for solutions of boundary value problems with local uniqueness by Banach's fixed-point theorem. SIAM J. Numer. Anal. *35* (1998), 2004–2013.

Yasuaki Hiraoka, Graduate School of Science, Hiroshima University, 1-7-1, Kagamiyama, Higashi–Hiroshima, 739-8521. Japan.
e-mail: hiraok@hiroshima-u.ac.jp