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TESTING A HOMOGENEITY OF STOCHASTIC PROCESSES

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The paper concentrates on modeling the data that can be described by a homogeneous or non-homogeneous Poisson process. The goal is to decide whether the intensity of the process is constant or not. In technical practice, e.g., it means to decide whether the reliability of the system remains the same or if it is improving or deteriorating. We assume two situations. First, when only the counts of events are known and, second, when the times between the events are available. Several statistical tests for a detection of a change in an intensity of the Poisson process are described and illustrated by an example. We cover both the case when the time of the change is assumed to be known or unknown.

Keywords: homogeneous and non-homogeneous Poisson process, counting process, change point detection

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1. INTRODUCTION

The paper concentrates on modeling data that describe occurrences of certain events in time. The reliability data where the events correspond to a failures of a product is a typical example. In the scope of mathematical statistics a counting process, often a homogeneous or non-homogeneous Poisson process, serves as a successful model.

We consider two situations. In the first one all we know are numbers of observed events $\{N_i\},\ i = 1, \ldots, q$, in consecutive non-overlapping intervals of the length $\{t_i\},\ i = 1, \ldots, q,\ t_1 + \cdots + t_q = T$. In the second one, a sequence of times of event occurrences, say $0 < \tau_1 < \tau_2 < \ldots$, is observed. It is clear that in such a case we also know lengths of intervals between events $\{Y_i\}$.

The basic question that arises when trying to find a good model is the following: *Is the observed process stationary or does its intensity change?* In the case of reliability data changes in the intensity may be caused by improvement or deterioration of the system. For detection of change(s) the hypotheses testing may be applied, where the null hypothesis claims that the process is stationary. According to an alternative under which a certain type of change is considered a test statistic may be developed. To decide whether the null or the alternative hypothesis holds the distribution of the suggested test statistic under the null hypothesis has to be derived. As the exact distribution is often complex an asymptotic distribution (for a large T or a large number of observations n) may be applied. In our paper several test statistics are proposed and their distribution under the null distribution is studied.

After the problem of stationarity is solved and an appropriate parametric model is chosen, an estimation of parameters concludes the statistical inference. Therefore, we show how to estimate parameters of several simple models usually applied to data described above.

2. COUNTING PROCESS

We suppose that we observe a stream of events, failures, e. g., that occurred in times $0 < \tau_1 < \tau_2 < \dots$ Such a sequence of variables $\{\tau_i\}$ is called a *simple point process*. At any time t we can count the number of events $N(t)$ that occurred before or at the time t. The process $\{N(t),\, t \geq 0\}$ is called a *counting process*. Between the point process $\{\tau_i\}$ and its counting process the following relation holds, i. e.,

$$\mathrm{P}\big(N(t) \geq n\big) = \mathrm{P}\big(\tau_n \leq t\big).$$

A counting process $\big\{N(t),\, t \geq 0\big\}$ is a renewal process if:

1. $N\big(0\big) = 0$.
2. The variable $Y_1 = \tau_1$ (the time to the occurrence of the first event from $t = 0$) and the variables $Y_j = \tau_j - \tau_{j-1}$, $j \geq 2$, Y_j being time between the $\big(j-1\big)$st and jth event, form a sequence of i.i.d. random variables with a distribution function $F(x)$.
3. $N(t) = \sup\big\{n : S_n \leq t\big\}$, where $S_0 = 0, \quad S_n = \sum_{i=1}^{n} Y_i,\ n \geq 1$.

If times between events $\{Y_i\}$ are i.i.d. with an exponential distribution then the process is called a homogeneous Poisson process.

Renewal processes do not allow to model improving and deteriorating systems. For modeling non-stationary systems *trend renewal processes* may be applied. For more detailed information about trend renewal processes see [14]. An example of a trend renewal process is a *non-homogeneous Poisson process*. Non-homogeneous as well as homogeneous Poisson processes with intensity $\{\lambda(s) > 0,\, s \geq 0\}$ may be defined by the following set of properties:

1. For every n and every $0 < t_1 < t_2 < \cdots < t_n$ the variables $N(t_1)$, $N(t_2) - N(t_1), \dots,\ N(t_n) - N(t_{n-1})$ are independent.
2. For every $0 \leq s < t$ the variable $N(t) - N(s)$ has a Poisson distribution with the parameter $\int_s^t \lambda(z)\,\mathrm{d}z$. Denoting $\{\Lambda(t), t \geq 0\}$ a cumulative intensity $\Lambda(t) = \int_0^t \lambda(z)\,\mathrm{d}z$, it holds

$$\mathrm{P}\Big(\big(N(t) - N(s)\big) = k\Big) = \exp\Big(-\big(\Lambda(t)-\Lambda(s)\big)\Big)\,\frac{\big(\Lambda(t)-\Lambda(s)\big)^k}{k!}, \quad k=0,1,\dots.$$

Suppose that the interval $[0,T]$ is divided into smaller time intervals of the lengths $t_1, t_2, \dots, t_q$ such that $t_1 + t_2 + \cdots + t_q = T$ and N_1 denotes the number of events in the interval $[0, t_1]$, N_2 the number of events in the interval $(t_1, t_1 + t_2]$ etc. Denote by $N(T)$ the total number of events in the time interval $[0,T]$. Clearly, $N(T) = N_1 + \cdots + N_q$. The variables $\{N_i,\ i = 1,\dots,q\}$ are independent and each has a Poisson distribution with the parameter Λ_i, where

- $\Lambda_1 = \Lambda(t_1)$;
- $\Lambda_i = \Lambda(t_1 + \cdots + t_i) - \Lambda(t_1 + \cdots + t_{i-1}), \quad i = 2,\dots,q.$

If the intensity of the underlying Poisson process does not change in time then the variable N_i follows a Poisson distribution with parameter $\lambda \cdot t_i$, $i = 1,\dots,q$, and the distribution of N_i, given the number of all observations $N(T) = n$, is binomial with parameters n and $p_i = \Lambda_i/\Lambda(T)$. For a homogeneous Poisson process the distribution does not depend on the intensity λ and is binomial with parameters n and $p_i = t_i/T$, i. e.,

$$\mathrm{P}\big(N_i = x \,|\, N(T) = n\big) = \tbinom{n}{x}\big(\tfrac{t_i}{T}\big)^x \big(1 - \tfrac{t_i}{T}\big)^{n-x}.$$

For large $N(T)$ an approximation of binomial distribution by a normal distribution yields

$$\mathcal{L}\big(N_i \,|\, N(T) = n\big) \sim N\Big(n\,\tfrac{t_i}{T}, n\,\tfrac{t_i}{T}\big(1 - \tfrac{t_i}{T}\big)\Big)$$

or

$$\mathcal{L}\left(\frac{\frac{N_i}{N(T)} - \frac{t_i}{T}}{\sqrt{\frac{t_i}{T}(1 - \frac{t_i}{T})}} \sqrt{N(T)} \,\middle|\, N(T) = n \right) \sim N(0,1),$$

where $N(\mu, \sigma^2)$ denotes a normal distribution with a mean μ and a variance σ^2.

Similarly, the distribution of the vector $(N_1, \dots, N_q)$, given $N(T) = n$, is multinomial with parameters $p_1, \dots, p_q$ and n. For a homogeneous Poisson process we have

$$\begin{aligned} \mathrm{P}\big(N_1 = x_1, \dots, N_q = x_q\big) &= \frac{n!}{x_1! \dots x_q!}\big(\tfrac{t_1}{T}\big)^{x_1} \dots \big(\tfrac{t_q}{T}\big)^{x_q} \quad \text{if} \quad \textstyle\sum_{i=1}^q x_i = n,\ x_i \geq 0, \\ &= 0 \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{aligned}$$

3. STATISTICAL ANALYSIS OF TREND IN INTENSITY

For the proper statistical analysis of a system it is important to detect possible changes in the length of intervals between events. For example, reliability growth corresponds to times between failures becoming longer as time goes on (improving system), whereas aging effects often lead to decreasing inter-failure times (deteriorating system). The intensity may change in many different ways. The change may be sudden or gradual. For the decision whether the intensity is constant one may use statistical tests. We describe here statistical tests suitable for testing that the process is stationary against alternatives depending on the kind of trend to detect.

3.1. Detection of a change in an intensity of a Poisson process

3.1.1. Inspections at several discrete time points

Suppose that the interval $[0,T]$ is divided into smaller intervals of the lengths $t_i, i = 1,\dots,q$, and that all we know are the corresponding numbers of events $N_i, i = 1,\dots,q$, in the respective intervals. The decision whether a change in an intensity occurred may be based on hypotheses testing. The null hypothesis claims that N_i, $i = 1,\dots,q$, are independent random variables distributed according to a Poisson distribution $Po(\lambda\, t_i)$. The alternative claims that there exists an index k such that N_k is not distributed according to the Poisson distribution with the parameter $\lambda\, t_k$.

First, we deal with a simple situation when the interval $[0,T]$ is divided into two intervals $[0,t_1]$ and $(t_1,T]$, where N_1 denotes a number of events in the interval $[0,t_1]$ and $N_2 = N(T) - N_1$ a number of events in the interval $(t_1,T]$. We decide that the intensity changed, i. e., we reject the null hypothesis of no change at the significance level α, either if

$$\sum_{i=0}^{N_1} \binom{N(T)}{i} \left(\frac{t_1}{T}\right)^i \left(1-\frac{t_1}{T}\right)^{N(T)-i} \le \frac{\alpha}{2}$$

or if

$$\sum_{i=N_1}^{N(T)} \binom{N(T)}{i} \left(\frac{t_1}{T}\right)^i \left(1-\frac{t_1}{T}\right)^{N(T)-i} \le \frac{\alpha}{2}.$$

For $N(T)$ large the rule for rejecting the null hypothesis of no change has the form

$$\frac{\left|\frac{N_1}{N(T)} - \frac{t_1}{T}\right|}{\sqrt{\frac{t_1}{T}\left(1-\frac{t_1}{T}\right)}}\sqrt{N(T)} > u_{1-\alpha/2}$$

where $u_{1-\alpha/2}$ is a $(1-\alpha/2)100\,\%$ quantile of $N(0,1)$.

In the case that all we know are numbers of observed events $\{N_i\}$, $i{=}1,\dots,q$, in consecutive non-overlapping intervals of the length $\{t_i\}$, $i{=}1,\dots,q$, $t_1{+}\cdots{+}t_q{=}T$, and if the number of all events $N(T)$ is large, we decide that the intensity changed if

$$\sum_{i=1}^{q} \frac{\left(N_i - \frac{t_i}{T}\,N(T)\right)^2}{\frac{t_i}{T}\,N(T)} > \chi^2_{1-\alpha}(q-1), \tag{1}$$

where $\chi^2_{1-\alpha}(q-1)$ is a $(1-\alpha)100\,\%$ quantile of χ^2 distribution with $q-1$ degrees of freedom.

3.1.2. Decision based on all events

In the preceding paragraph we supposed that we observed a counting process of a underlying Poisson process in several discrete time points t_1, t_1+t_2, t_1+t_2+

$t_3, \ldots, t_1+t_2+\cdots+t_q = T$. Therefore, our decision whether the intensity of the process is constant was based only on the values of the counting process $\{N(t_1), N(t_1+t_2), \ldots, N(T)\}$ or on the values $\{N_1, \ldots, N_q\}$, where $N(t_1+\cdots+t_i) = N_1+\cdots+N_i$. However, it can happen that we follow the process continuously so that we know all time points $0 < \tau_1 < \tau_2 < \cdots < \tau_n \leq T$ in which the events occurred. As we have already explained, if the intensity is constant then for every fixed $t \in [0,T]$ the distribution of $N(t)$, given $N(T) = n$, is binomial $Bi\big(n, t/T\big)$.

The process has a non-constant intensity if at least for one $0 < t < T$ the variable

$$\frac{\left|\frac{N(t)}{N(T)} - \frac{t}{T}\right|}{\sqrt{\frac{t}{T}(1-\frac{t}{T})}}\sqrt{N(T)}$$

is large. Therefore, we may base our decision on the statistic

$$\mathrm{CP}_1 = \sup_{0<t<T}\left\{\frac{\left|\frac{N(t)}{N(T)} - \frac{t}{T}\right|}{\sqrt{\frac{t}{T}(1-\frac{t}{T})}}\sqrt{N(T)}\right\}. \tag{2}$$

If the intensity is constant and the number of observations $N(T)$ large, one may replace t/T in the denominator of (2) by $N(t)/N(T)$ and apply the statistic

$$\mathrm{CP}_2 = \sup_{0<t<T}\left\{\frac{\left|\frac{N(t)}{N(T)} - \frac{t}{T}\right|}{\sqrt{\frac{N(t)}{N(T)}(1-\frac{N(t)}{N(T)})}}\sqrt{N(T)}\right\}. \tag{3}$$

For large T it holds under the null hypothesis that

$$\mathrm{P}\left(\sup_{0<t<T}\left\{\frac{\left|\frac{N(t)}{N(T)} - \frac{t}{T}\right|}{\sqrt{\frac{t}{T}(1-\frac{t}{T})}}\sqrt{N(T)}\right\} \leq \frac{x+b_T}{a_T}\right) \approx \exp(-2\,e^{-x}) \tag{4}$$

and

$$\mathrm{P}\left(\sup_{0<t<T}\left\{\frac{\left|\frac{N(t)}{N(T)} - \frac{t}{T}\right|}{\sqrt{\frac{N(t)}{N(T)}(1-\frac{N(t)}{N(T)})}}\sqrt{N(T)}\right\} \leq \frac{x+b_T}{a_T}\right) \approx \exp(-2\,e^{-x}), \tag{5}$$

where $\approx$ denotes that the distribution of the LHS can be for large values of T approximated by the RHS, $a_T = \sqrt{2\log\log T}$ and $b_T = 2\log\log T + \frac{1}{2}\log\log\log T - \frac{1}{2}\log\pi$. The approximations (4) and (5) provide us with approximate critical values. For more details see [6]. Another interesting approach is described in [17].

3.2. Detection of a change in mean time between events

We suppose again that we register a sequence of times of events $0 < \tau_1 < \tau_2 < \ldots$ Until now we were interested in a counting process $\{N(t)\}$, where $N(t)$ denotes the number of occurrences of an event in the interval $[0,t]$. However, a different approach for decision whether a change in the process occurred may be based on times between events $\{Y_i = \tau_i - \tau_{i-1}\}$. If studied Poisson process is homogeneous then all the variables $\{Y_i\}$ are i.i.d. with an exponential distribution. We will discuss this situation more profoundly in the following paragraphs.

3.2.1. Testing for a change with a known change point

It can happen that after having observed kth event the conditions that rule the process changed and we ask whether this change affected the mean time between events. In the case of a constant intensity the times between events are exponentially distributed so that the problem is to decide whether all variables $Y_1, \dots, Y_n$ have the same mean δ or whether the mean δ_1 of the variables $Y_1, \dots, Y_k$ differs from the mean δ_2 of the variables $Y_{k+1}, \dots, Y_n$. If we formulate the problem within the theory of hypotheses testing, we are to test the null hypothesis H_0 against the alternative A:

$$\begin{array}{lcll} H_0 & : & Y_i \sim \mathit{Exp}\,(\delta), & i = 1, \dots, n, \\ A & : & Y_i \sim \mathit{Exp}\,(\delta_1), & i = 1, \dots, k, \\ & & Y_i \sim \mathit{Exp}\,(\delta_2), & i = k+1, \dots, n, \end{array} \tag{6}$$

where $\delta > 0$, $\delta_1 > 0$ and $\delta_2 > 0$ are unknown. It depends on our prior knowledge about the sign of $\delta_2 - \delta_1$ whether we consider a one-sided or a two-sided alternative. We consider a two-sided alternative $\delta_1 \neq \delta_2$ if we do not know a priori whether the change causes an increase or decrease of the mean time between events. We consider a one-sided alternative $\delta_1 < \delta_2$ (resp. $\delta_1 > \delta_2$) if we expect that the change might increase (resp. decrease) the mean time between events. The problem is so called two–samples problem for exponentially distributed random variables. It is often used if we compare the life–times of two sets of products. The most frequently applied test statistics are equivalent to the log-likelihood ratio

$$\frac{\sup_{\delta_1,\delta_2} \prod_{i=1}^k f(y_i;\delta_1) \prod_{i=k+1}^n f(y_i;\delta_2)}{\sup_\delta \prod_{i=1}^n f(y_i;\delta)}.$$

Denoting

$$S_k = \sum\nolimits_{i=1}^k Y_i, \quad S_k^0 = \sum\nolimits_{i=k+1}^n Y_i, \quad \overline{Y}_k = S_k/k \quad \text{and} \quad \overline{Y}_k^{\,0} = S_k^0/(n-k),$$

the maximum likelihood estimates of δ, δ_1 and δ_2 are

$$\widehat{\delta} = \overline{Y}_n, \quad \widehat{\delta}_1 = \overline{Y}_k \quad \text{and} \quad \widehat{\delta}_2 = \overline{Y}_k^{\,0},$$

and the logarithm of the likelihood ratio is equal to

$$\begin{aligned} Z_k^2 &= -k \log \frac{\overline{Y}_k}{\overline{Y}_n} - (n-k) \log \frac{\overline{Y}_k^{\,0}}{\overline{Y}_n} \\ &= -k \log \Big(\frac{n}{k} \cdot \frac{S_k}{S_n} \Big) - (n-k) \log \Big(\frac{n}{n-k} \cdot \frac{S_k^0}{S_n} \Big) \\ &= -k \log \Big(\frac{n}{k} \cdot \frac{S_k/S_k^0}{1 + S_k/S_k^0} \Big) - (n-k) \log \Big(\frac{n}{n-k} \cdot \frac{1}{1 + S_k/S_k^0} \Big). \end{aligned} \tag{7}$$

Clearly, we reject H_0 against the two-sided alternative A with $\delta_1 \neq \delta_2$ if the value of Z_k^2 is larger than a value C which corresponds to a chosen significance level. Moreover,

$$\mathrm{P}\big(Z_k^2 < C\big) = \mathrm{P}\big(a_k(C) < S_k/S_n < b_k(C)\big),$$

where $a_k(C)$ and $b_k(C)$ are solutions of the equation

$$-k\log\left(\frac{n}{k}\cdot x\right) - (n-k)\log\left(\frac{n}{n-k}\cdot(1-x)\right) = C. \tag{8}$$

Further, note that

$$\mathrm{P}\big(a_k(C) < S_k/S_n < b_k(C)\big) = \mathrm{P}\big(d_k(C) < \overline{Y}_k/\overline{Y}_k^{\,0} < h_k(C)\big),$$

where

$$d_k(C) = \frac{a_k(C)}{1-a_k(C)}\,\frac{n-k}{k} \quad \text{and} \quad h_k(C) = \frac{b_k(C)}{1-b_k(C)}\,\frac{n-k}{k}.$$

Under H_0 the statistic S_k/S_n has a beta distribution $Be(k, n-k)$ with the density

$$f_{Be}(y; k, n-k) = \frac{y^{k-1}(1-y)^{n-k-1}}{B(k, n-k)},$$

while $\overline{Y}_k/\overline{Y}_k^{\,0}$ has a F distribution with $2k$ and $2(n-k)$ degrees of freedom. Thus

$$\mathrm{P}\left(\frac{1}{F_{1-\alpha/2}\big(2(n-k), 2k\big)} < \frac{\overline{Y}_k}{\overline{Y}_k^{\,0}} < F_{1-\alpha/2}\big(2k, 2(n-k)\big)\right) = 1-\alpha,$$

where $F_{1-\beta}(p,q)$ is a $(1-\beta)\,100\,\%$ quantile of the F distribution with p and q degrees of freedom. Therefore, we reject the null hypothesis H_0 against the two-sided alternative $\delta_1 \neq \delta_2$ at the significance level α if either $\overline{Y}_k/\overline{Y}_k^{\,0}$ is larger than $F_{1-\alpha/2}\big(2k, 2(n-k)\big)$ or if it is smaller than $1/F_{1-\alpha/2}\big(2(n-k), 2k\big)$. Analogously we reject H_0 against the one-sided alternative $\delta_1 > \delta_2$ if $\overline{Y}_k/\overline{Y}_k^{\,0}$ is larger than $F_{1-\alpha}\big(2k, 2(n-k)\big)$, and similarly against the one-sided alternative $\delta_1 < \delta_2$ if $\overline{Y}_k/\overline{Y}_k^{\,0}$ is smaller than $1/F_{1-\alpha}\big(2(n-k), 2k\big)$.

3.2.2. Testing for a change with an unknown change point – maximum type test statistics

In the preceding paragraph we suspected that after we observed the kth event the change in the mean time between events might occur. The problem was to decide whether it really happened. The situation is more complicated if we would like to know whether the mean time between events remains the same all the time or whether it changed but we have no idea where the change might occur. In the scope of hypotheses testing the problem may be specified as follows:

$$\begin{array}{lll} H_0 & : & Y_i \sim Exp\,(\delta), \qquad\qquad i = 1,\dots,n, \\ A & : & \exists\, k \in \{1,\dots,n-1\} \ \text{ such that} \\ & & Y_i \sim Exp\,(\delta_1), \qquad\qquad i = 1,\dots,k, \\ & & Y_i \sim Exp\,(\delta_2), \qquad\qquad i = k+1,\dots,n, \end{array} \tag{9}$$

where $\delta > 0$, $\delta_1 > 0$ and $\delta_2 > 0$. We reject the null hypothesis against the two-sided alternative with $\delta_1 \neq \delta_2$ if at least one of the statistics $\{Z_k^2,\ k = 1, \ldots, n-1\}$ given by (7) is large or, more precisely, if $\max_{1\le k\le n-1}\{Z_k^2\}$ is larger than an appropriate critical value. Unfortunately, the distribution of $\max_{1\le k\le n-1}\{Z_k^2\}$ under H_0 is so complex that it is not possible to use it for finding critical values and this is why some approximations are needed.

The simple approximation is obtained by the Bonferroni inequality

$$\mathrm{P}\left(\max_{1\le k\le n-1}\{Z_k^2\} > C\right) = \mathrm{P}\left(\bigcup_{k=1}^{n-1}\{Z_k^2 > C\}\right) \le \sum_{k=1}^{n-1}\mathrm{P}\left(Z_k^2 > C\right) \quad (10)$$
$$= \sum_{k=1}^{n-1}\mathrm{P}\left(a_k(C) < \frac{S_k}{S_n} < b_k(C)\right) = \sum_{k=1}^{n-1}\mathrm{P}\left(d_k(C) < \frac{\overline{Y}_k}{\overline{Y}_k^0} < h_k(C)\right),$$

where $a_k(C)$, $b_k(C)$, $d_k(C)$ and $h_k(C)$ were introduced in the previous section. If we find a constant C such that $\mathrm{P}(Z_k^2 > C) \le \frac{\alpha}{n-1}$ for all k, then it is obvious that the exact $\alpha 100\,\%$ critical value is smaller than C or, in other words, C is a conservative estimate of the $\alpha 100\,\%$ critical value. The inequality (10) may also serve for finding an upper bound of the p-value.

Despite the distribution of statistics $\{Z_k^2\}$ being of the same type, $\{Z_k^2\}$ are not identically distributed (parameters depend on k), and that is why the smallest bound C that fulfills $\mathrm{P}\left(Z_k^2 > C\right) \le \frac{\alpha}{n-1}$ for all k is unnecessarily large. As the contributions of different terms of the sum (10) are for fixed C different, it is better to look for such a $C^\star$ that

$$\sum_{k=1}^{n-1}\mathrm{P}\left(a_k(C^\star) < \frac{S_k}{S_n} < b_k(C^\star)\right) = \sum_{k=1}^{n-1}\mathrm{P}\left(d_k(C^\star) < \frac{\overline{Y}_k}{\overline{Y}_k^0} < h_k(C^\star)\right) = \alpha.$$

As a result we obtain a less conservative estimate of the $\alpha 100\,\%$ critical value.

To get an idea about the magnitude of the critical values, following table presents the critical values calculated using simulations and the Bonferroni inequality. We recommend to apply the Bonferroni inequality for smaller samples with a number of observations less than 70.

Table 1. Critical values for the maximum type statistics (11) calculated using simulations, the Bonferroni inequality and asymptotic approximation for standard exponential distribution *Exp* (1).

	$\alpha = 0.1$			$\alpha = 0.05$			$\alpha = 0.01$		
n	Simul.	Bonf.	Asymp.	Simul.	Bonf.	Asymp.	Simul.	Bonf.	Asymp.
20	2.625	2.858	3.113	2.895	3.079	3.599	3.429	3.545	4.700
50	2.788	3.123	3.181	3.046	3.325	3.617	3.583	3.758	4.604
100	2.867	3.312	3.226	3.123	3.505	3.637	3.639	3.916	4.570

If the number of observations n is large, we can use the asymptotic distribution of $\max_{1\le k\le n-1}\left\{Z_k^2\right\}$ to find approximate critical values. Under H_0, the Taylor expansion yields that

$$\max_{1\le k\le n-1}\left\{2\,Z_k^2\right\}$$

is equivalent to

$$\max_{1\le k\le n-1}\left\{\frac{\frac{n}{k(n-k)}\left(S_k-\frac{k}{n}S_n\right)^2}{(S_n/n)^2}\right\}.$$

From the theory of extremes of random processes, cf. [6], we have

$$\mathrm{P}\left(\max_{1\le k\le n-1}\left\{\sqrt{2}\,|Z_k|\right\}>\frac{x+b_n}{a_n}\right)\approx 1-\exp\left(-2e^{-x}\right), \tag{11}$$

where $a_n=\sqrt{2\log\log n}$ and $b_n=2\log\log n+\frac{1}{2}\log\log\log n-\frac{1}{2}\log\pi$.

For one-sided alternative $\delta_1>\delta_2$ we may use the test statistic $\max_{1\le k\le n-1}\left\{S_k/S_n\right\}$ or equivalently $\max_{1\le k\le n-1}\left\{\overline{Y}_k/\overline{Y}_k^{\,0}\right\}$. To find approximate critical values we recommend again to use the Bonferroni inequality for small n, while for large n we recommend to use the asymptotic distribution

$$\mathrm{P}\left(\max_{1\le k\le n-1}\left\{\frac{\sqrt{\frac{n}{k(n-k)}}(S_k-\frac{k}{n}S_n)}{S_n/n}\right\}>\frac{x+b_n}{a_n}\right)\approx 1-\exp(-e^{-x}). \tag{12}$$

Another results based on the "strong invariance principle" can be found in [8].

3.2.3. Sum−type test statistics

A different approach how to derive test statistics for testing problem (6) is a so called pseudo–Bayesian approach. This approach was applied for the first time by [4] and [10] for testing a sudden change in a mean of normally distributed random variables. The test statistic is again an analogue to the log-likelihood ratio under H_0 provided a prior distribution of a change point is known and the amount of change (i. e. $|\delta_2-\delta_1|$) tends to zero. For the problem (9) with the one-sided alternative $\delta_2>\delta_1$ (resp. $\delta_2<\delta_1$) and under the assumption of a uniform prior distribution of k over $\{1,\dots,n-1\}$, the statistic

$$T_1=\frac{\sqrt{12}}{n\sqrt{n}}\sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\left(\frac{Y_j}{\overline{Y}_n}-1\right)=-\frac{\sqrt{12}}{n\sqrt{n}\,\overline{Y}_n}\sum_{k=1}^{n-1}\left(S_k-\frac{k}{n}S_n\right)$$

is obtained. The exact distribution of T_1 under H_0 is again complex but it is possible to calculate explicitly all its moments since $\left(\frac{Y_1}{\sum Y_i},\dots,\frac{Y_n}{\sum Y_i}\right)$ has a Dirichlet distribution with parameters $(1,1,\dots,1)$. Using the central limit theorem it may be shown that the statistic T_1 has asymptotically a standard normal distribution

$N(0,1)$. Hence, appropriate quantiles of a normal distribution may serve as approximate critical values. For a better approximation of critical values we may use the Edgeworth expansion.

It is interesting that the statistic T_1 is asymptotically "equivalent" (for large values of n) to the *Laplace test statistic*

$$L = \frac{\frac{1}{n-1}\sum_{i=1}^{n-1}\tau_i - \frac{1}{2}\tau_n}{\tau_n\sqrt{\frac{1}{12(n-1)}}}.$$

More precisely, it holds that $T_1 = -\sqrt{\frac{n-1}{n}}\,L$. The Laplace test statistic was suggested to test the null hypothesis that a process is a homogeneous Poisson process against the alternative that it is a non-homogeneous Poisson process with an intensity $\lambda(t) = e^{\alpha+\beta\, t}$, for details see Cox and Lewis [5].

It is also worth to notice that the test statistic L is asymptotically equivalent to a standardized least squares estimator of a slope parameter b in a linear regression

$$\mathrm{E}\,(Y_i) = a + b\,i, \quad i = 1,\dots,n-1,$$

where the standard deviation is estimated by $\overline{Y}_n = n^{-1}\sum_{i=1}^{n} Y_i = \tau_n/n$. If $b = 0$ then the variables $\{Y_i\}$ have the same exponential distribution with $\mathrm{E}\,Y_i = std\,Y_i = \delta$ and $\overline{Y}_n$ is a good estimate of the standard deviation. However, if the variables $\{Y_i\}$ are distributed according to another distribution with smaller variability then it is reasonable to replace the estimator of the standard deviation by another estimator. If we estimate the variance by the standard sample variance

$$s^2_{\mathrm{LR}} = (n-1)^{-1}\sum_{i=1}^{n}\left(Y_i - \overline{Y}_n\right)^2,$$

we get the *Lewis–Robinson test statistic*

$$\mathrm{LR}_1 = \frac{\overline{Y}_n}{s_{\mathrm{LR}}}\cdot\frac{\sum_{i=1}^{n-1}\tau_i - \frac{n-1}{2}\tau_n}{\tau_n\sqrt{\frac{n-1}{12}}}.$$

Clearly, the statistic s^2_{LR} is a good estimator of the variance under the null hypothesis claiming that there is no trend. In the case where there is a trend then s^2_{LR} overestimates the true variance. Therefore some authors recommend to estimate the variance by

$$s^2_{\mathrm{LI}} = \frac{1}{2\,(n-1)}\sum_{i=1}^{n-1}\left(Y_{i+1} - Y_i\right)^2$$

and to use the test statistic

$$\mathrm{LR}_2 = \frac{\overline{Y}_n}{s_{\mathrm{LI}}}\cdot\frac{\sum_{i=1}^{n-1}\tau_i - \frac{n-1}{2}\tau_n}{\tau_n\sqrt{\frac{n-1}{12}}}.$$

For a two-sided alternative with $\delta_1 \neq \delta_2$ we get from the pseudo–Bayesian approach the statistic

$$T_2 = \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{\sqrt{n}}\sum_{j=k+1}^{n}\left(\frac{Y_j}{\overline{Y}_n}-1\right)\right)^2 = \frac{1}{n^2\,\overline{Y}_n^2}\sum_{k=1}^{n-1}\left(S_k-\frac{k}{n}S_n\right)^2.$$

Under the null hypothesis the statistic T_2 converges, as the number of observations tends to infinity, in distribution to $\int_0^1 B^2(s)\,\mathrm{d}s$, where $\{B(t),\ t \geq 0\}$ is a Brownian bridge. The distribution of $\int_0^1 B^2(s)\,\mathrm{d}s$ was studied by Kiefer in [11]. The 5 % asymptotic critical value is equal to 0.461, the 2.5 % asymptotic critical value to 0.580 and the 1 % asymptotic critical value to 0.743.

Instead of the statistic T_2 some authors recommend to use the statistic

$$T_3 = \frac{1}{n}\sum_{k=1}^{n-1}\frac{n}{k(n-k)}\left(\sum_{j=k+1}^{n}\left(\frac{Y_j}{\overline{Y}_n}-1\right)\right)^2 = \frac{1}{\overline{Y}_n^2}\sum_{k=1}^{n-1}\frac{1}{k(n-k)}\left(S_k-\frac{k}{n}S_n\right)^2.$$

The statistic T_3 is a so called Anderson–Darling statistic. Under H_0 its distribution tends (as the number of observations n increases) to the distribution of $\int_0^1 \frac{B^2(s)}{s(1-s)}\,\mathrm{d}s$. The 5 % asymptotic critical value is equal to 2.49, the 2.5 % asymptotic critical value to 3.08 and the 1 % asymptotic critical value to 3.86. In comparison with T_2 the statistic T_3 has a larger power when a change occurs in the beginning or the end of our observations. On the contrary the statistic T_2 has a larger power when a change occurs in the middle of the sample. Similarly as above, if we are not sure whether the observations $\{Y_i\}$ are distributed according to an exponential distribution we estimate the true variance by the estimator s_{LI}^2, instead of $\overline{Y}_n^2$, and we get a generalized Anderson–Darling statistic as suggested by Kvaløy in [14].

If nothing is known about the distribution of the variables $\{Y_i = \tau_{i+1} - \tau_i\}$, the *Mann test* may be applied. The Mann test is a rank test for testing the null hypothesis that all variables $\{Y_i\}$ are independent identically distributed (i.e a renewal process is observed) against an alternative of a monotone trend. The test statistic M is computed by counting the number of reverse arrangements among the interarrival times $\{Y_i\}$. We speak about a reverse arrangement of Y_i and Y_j if $Y_i < Y_j$ for $i < j$. Thus

$$M = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} I\big(Y_i < Y_j\big),$$

where $I(\cdot)$ is an indicator function. Under the null hypothesis M is approximately normally distributed with the expectation $n(n-1)/4$ and the variance $(2n^3 + 3n^2 - 5n)/72$. If the variables $\{Y_i\}$ are stochastically increasing (the intensity of the underlying processes decreases), then the statistic M attains large values. On the other hand, if the variables $\{Y_i\}$ are stochastically decreasing (the intensity increases) the statistic M attains small values. For more details see, e. g., classical monograph [9].

4. ESTIMATION OF INTENSITY OF A POISSON PROCESS

For a homogeneous Poisson process on the interval $[0,T]$, the maximum likelihood estimate of λ is of the form

$$\widehat{\lambda} = \frac{N(T)}{T}.$$

Notice that all the information about the intensity λ is contained in the value of the counting process $N(T)$ so that it is not necessary to know times when events (failures) occurred. For large T we have

$$\frac{N(T)}{T} \to \lambda \quad \text{and} \quad \sqrt{\frac{T}{\lambda}}\left(\frac{N(T)}{T} - \lambda\right) \sim N(0,1).$$

The maximum likelihood estimate of the mean time between events $\delta = 1/\lambda$ is

$$\widehat{\delta} = \frac{T}{N(T)}.$$

The exact $(1-\alpha)\,100\,\%$ confidence interval for λ has the form

$$\left(\frac{\chi^2_{\alpha/2}(2\,N(T)+2)}{2\,T}, \frac{\chi^2_{1-\alpha/2}(2\,N(T)+2)}{2\,T}\right). \tag{13}$$

For large T a normal approximation may be used, giving the interval

$$\left(\frac{N(T)+\frac{1}{2}u^2_{\alpha/2}-u_{1-\alpha/2}\sqrt{\frac{u^2_{\alpha/2}}{4}+N(T)}}{T}, \frac{N(T)+\frac{1}{2}u^2_{\alpha/2}+u_{1-\alpha/2}\sqrt{\frac{u^2_{\alpha/2}}{4}+N(T)}}{T}\right). \tag{14}$$

Now we will turn to the estimation of a trend in intensity. We start with a situation that the times $0 < \tau_1 < \cdots < \tau_n \le t$ when the events occurred are known. Let $Y_1 = \tau_1$ and $Y_i = \tau_i - \tau_{i-1}$ denote the times between events. We are especially interested in models with a monotone intensity to model the reliability growth or decrease. The goal of statistical inference is to estimate the unknown parameters.

A class of models which have been widely used because of its simplicity is the non-homogeneous Poisson process described earlier. We focus here on two models.

In the first one the intensity is given by $\lambda(t) = \alpha\,\beta\,t^{\beta-1}$ and the cumulative intensity function is thus $\Lambda(t) = \alpha\,t^{\beta}$. The model is called the Weibull process or the power-law process. It describes either a reliability growth when $\beta > 1$ or a reliability decrease when $\beta < 1$. When $\beta = 1$, it reduces to a homogeneous Poisson process. The maximum likelihood estimators of the parameters based on the observation of the n event times $\tau_1, \ldots, \tau_n$, have a very simple closed form

$$\widehat{\alpha} = \frac{n}{\tau_n^{\widehat{\beta}}} \quad \text{and} \quad \widehat{\beta} = \frac{n}{\sum_{i=1}^{n} \ln \frac{\tau_n}{\tau_i}}.$$

In the second model, so called *exponential-law model*, the intensity is given by $\lambda(t) = e^{\alpha+\beta\, t}$ and the cumulative intensity is thus $\Lambda(t) = \frac{e^{\alpha}}{\beta}\left(e^{\beta t} - 1\right)$. It describes a reliability growth if $\beta > 0$ or a reliability decrease if $\beta < 0$. If $\beta < 0$ then $\Lambda(+\infty) < \infty$ and it means that with a positive probability the process dies out and we do not get more information about the intensity if we observe the process for a longer time. To obtain the maximum likelihood estimators of the parameters based on the observation of the n event times $\tau_1, \ldots, \tau_n$, we have to solve the equation for β

$$\frac{n}{\beta} - \frac{n\tau_n}{1 - e^{\beta\tau_n}} + \sum_{i=1}^{n} \tau_i \;=\; 0.$$

After obtaining the estimator $\widehat{\beta}$ as the solution of the preceding equation the estimator $\widehat{\alpha}$ of α has a simple form

$$\widehat{\alpha} = \log\left(n\widehat{\beta}\,\right) - \log\left(e^{\widehat{\beta}\tau_n} - 1\right).$$

Finally we consider the situation that all we know are numbers of events $\{N_i,\, i = 1, \ldots, q\}$ in consecutive intervals $\{t_i,\, i = 1, \ldots, q\}$. As $\mathrm{E}\, N(t) = \Lambda(t)$ a regression analysis can be done directly in terms of the $\{N_i\}$. Supposing that $\{N_i\}$ have a Poisson distribution, their variance increases with their mean. Therefore, [5] suggested to apply one of the following transformations either $\log(N_i + 1/2)$ or $\sqrt{N_i + 1/2}$.

If $\Lambda(t) = \alpha t^{\beta}$, then $\log\left(\Lambda(t)\right) = \log\alpha + \beta\log t$. Considering $\log t$ to be independent and $\log\left(N(t)\right.$ dependent variable, $\log\alpha$ and β may be obtained by the least squares method. As the values $\{\log\left(N(t_i)\right)\}$ are neither independent nor identically distributed the regression technique has to be applied cautiously.

5. EXAMPLE

Sigma Re insurance company published in [16] a list of most costly insurance losses in 1970 − 1995 due to the hurricanes, tornados, earthquakes, floods, etc., worth of $851 \cdot 10^6 - 16 \cdot 10^9$ US \$. These data were republished and analyzed from the point of view of large deviations by [7]. Let us look on these data from the point of view of the theory described in this paper.

The times of events, in days starting January 1st, 1970, are following: 215, 1210, 1552, 1721, 2192, 3532, 3698, 4976, 5098, 6761, 6829, 7197, 7229, 7235, 7329, 7338, 7361, 7899, 7939, 7962, 8271, 8289, 8469, 8646, 8699, 8782, 9147, 9151, 9256, 9407. Moreover, the end of observations, i. e. December 31st, 1995, corresponds to the value 9495. It follows that times between the catastrophes (in days) are 995, 342, 169, 471, 1340, 166, 1278, 122, 1663, 68, 368, 32, 6, 94, 9, 23, 538, 40, 23, 309, 18, 180, 177, 53, 83, 365, 4, 105, 151.

Figure 1 presents lengths of intervals between the events (in days) versus index. Figure 2 presents the counting process $N(t)$ versus time t. Looking at these figures it seems that there is a change around the tenth observation, i. e., around the 7 000th day, when the frequency of events started to increase. Let us take a look whether this suspicion can be "confirmed" by the theory described above.

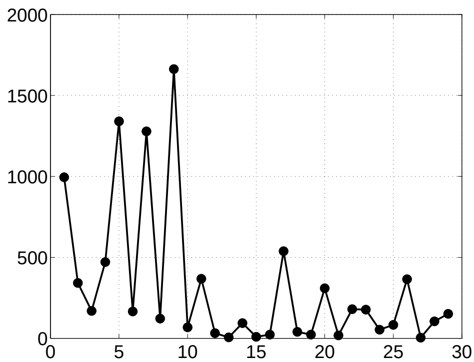

Fig. 1. Lengths of intervals between events.

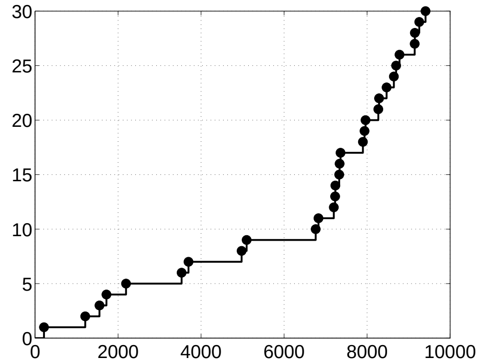

Fig. 2. Corresponding counting process.

We have calculated values of the test statistics presented in this paper. All of them were statistically significant of the level 5 %. The following table summarizes the values of them.

Test statistic	CP_1	CP_2	T_1	L	LR_1	LR_2	T_2	T_3
Value	5.00	4.93	–3.43	3.49	2.51	2.46	1.36	6.53
5 % asympt. crit.value	3.76	3.76	–1.65	1.76	1.65	1.65	0.46	2.49

In Figure 3 we present the values of statistics forming CP_1 and CP_2, for details see (2) and (3), while in Figure 4 we present the values of statistics

$$\frac{\sqrt{\frac{n}{k(n-k)}}(S_k - \frac{k}{n}S_n)}{S_n/n}$$

corresponding to the statistic (12) of the max-type considered in Subsection 3.2.2. We can see that their maximum is larger than both asymptotical critical value as well as the critical value obtained using the Bonferroni approach, so that we can reject the null hypothesis of no change in the intensity of appearances of the catastrophes.

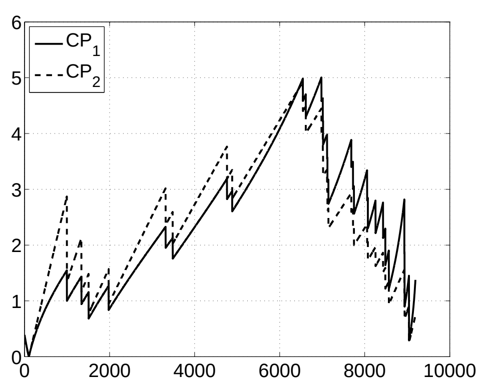

Fig. 3. Statistics forming CP_1 and CP_2.

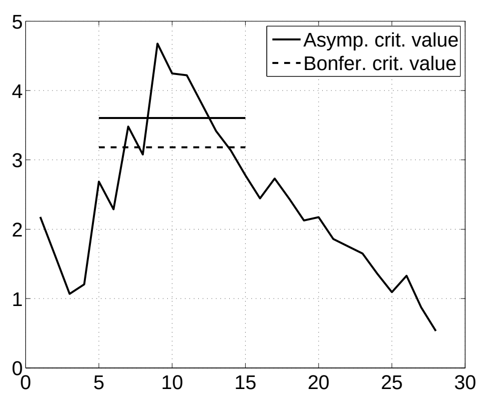

Fig. 4. Statistics forming (11).

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