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IDEMPOTENT VERSIONS OF HAAR'S LEMMA: LINKS BETWEEN COMPARISON OF DISCRETE EVENT SYSTEMS WITH DIFFERENT STATE SPACES AND CONTROL

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Haar's Lemma (1918) deals with the algebraic characterization of the inclusion of polyhedral sets. This Lemma has been involved many times in automatic control of linear dynamical systems via positive invariance of polyhedrons. More recently, it has been used to characterize stochastic comparison w.r.t. linear/integral ordering of Markov (reward) chains.

In this paper we develop a state space oriented approach to the control of Discrete Event Systems (DES) based on the remark that most of control constraints of practical interest are naturally expressed as the inclusion of two systems of linear (w.r.t. idempotent semiring or semifield operations) inequalities. Thus, we establish tropical version of Haar's Lemma to obtain the algebraic characterization of such inclusion. As in the linear case this Lemma exhibits the links between two apparently different problems: comparison of DES and control via positive invariance. Our approach to the control differs from the ones based on formal series and is a kind of dual approach of the geometric one recently developed.

Control oriented applications of the main results of the paper are given. One of these applications concerns the study of transportation networks which evolve according to a time table. Although complexity of calculus is discussed the algorithmic implementation needs further work and is beyond the scope of this paper.

Keywords: Max-plus algebra, control, monotonicity, positive invariance, residuation, duality

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1. INTRODUCTION

In 1918, Haar demonstrated the following result which provides an algebraic characterization of the inclusion of two polyhedral sets. For more details see e.g. [20]. Let $\leq_n$ denotes the component-wise order or product order on $\mathcal{X}^n$ where $(\mathcal{X}, \leq)$ is a poset, i.e. $x \leq_n y \overset{\text{def}}{\Leftrightarrow} \forall\, i,\ x_i \leq y_i$. Let $(\mathbb{R}, \leq)$ be the naturally ordered set of real numbers, we have.

Result 1. (Lemma of Haar) Let P (resp. Q) be an $m \times d$ (resp. $m' \times d$) matrix. Let p (resp. q) be a m (resp. m') dimensional column vector. The following assertion

$$\emptyset \neq \{\mathbf{x} \in \mathbb{R}^d : Px \leq_m p\} \subseteq \{x \in \mathbb{R}^d : Qx \leq_{m'} q\},$$

is true if and only if (iff) there exists an $m' \times m$ matrix H which entries are non-negative and such that the following conditions hold:

$$\text{(i)} \quad Q = HP, \qquad \text{(ii)} \quad Hp \leq_{m'} q.$$

In the case where p and q are equal to the null vector (i.e. the homogeneous case), Haar's Lemma reduces to Farkas' lemma [16]. A recent reference for such material is [22, p. 58–61]. Recall that Haar's Lemma has been involved many times in automatic control of linear dynamical systems when the constraint domains (state and/or control) are polyhedrons (see e.g. [21, 29]).

From Haar's Lemma, we deduce our main reference result dealing with the inclusion of polyhedrons included in the non-negative orthant.

Result 2. The following assertion

$$\emptyset \neq \{\mathbf{x} \in \mathbb{R}^d : Px \leq_m p\} \cap \mathbb{R}^d_+ \subseteq \{x \in \mathbb{R}^d : Qx \leq_{m'} q\} \cap \mathbb{R}^d_+,$$

is true iff there exists an $m' \times m$ matrix H which entries are non-negative and such that the following conditions hold:

$$\text{(i)} \quad Q \leq HP, \qquad \text{(ii)} \quad Hp \leq_{m'} q.$$

Up to a change of operations, it will be noticed that Result 2 is very similar to Corollary 1 (see also Remark 3).

In [1, 2] it has been shown that the algebraic characterization of the inclusion of two polyhedral sets provided by Haar's Lemma is the fundamental notion for the characterization of the comparison and the positive invariance of discrete-time Markov (reward) chains in particular and linear systems in general. Here this idea is investigated in the context of the comparison and the control of DES.

The main results of this paper are as follows. In the first part we establish two idempotent forms of Haar's Lemma corresponding to two important idempotent structures, that are complete idempotent semirings and complete idempotent semi-fields. The second formulation (i.e. Corollary 1) is a specification of the first one (i.e. Theorem 1) and is very similar to Result 2. The proof of the two idempotent forms of Haar's Lemma is based on the residuation theory and therefore stresses the relationship between residuation and duality early noticed by Wagneur (see e.g. [34]) in a very simple way. This relation seems to be very natural in quantales theory or Heyting algebra but not so familiar in the context of DES.

In a second part we study two apparently different problems: (A) the comparison of DES and (B) the control of DES.

A: Comparison of DES. The arguments for the relevance of the development of a theory of the comparison of DES are the same as the ones for comparison of stochastic processes (see e. g. the preface of [28]). One argument is that it is useful to establish comparison between the two systems $(d, \otimes, A)$ and $(d', \otimes, B)$ respectively defined below by (1) and (2) to obtain tractable calculus for performance criteria of DES keeping control on approximation error made. For further explanations in the context of DES see also [33]. In this paper we study a particular type of comparison called in this paper $(\mathbb{K}, \mathbb{K}')$-comparison. The $(\mathbb{K}, \mathbb{K}')$-comparison is defined as follows. Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ be a complete idempotent semiring or complete idempotent semifield (see definitions in Subsection 2.3). Let us now consider the two linear systems over $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ specified by $(d, \otimes, A)$ and $(d', \otimes, B)$ and respectively defined by:

$$(d, \otimes, A) : \begin{cases} x(0) \in \mathbb{S}^d, \\ x(n) = A \otimes x(n-1), \quad n \geq 1, \end{cases} \tag{1}$$

and

$$(d', \otimes, B) : \begin{cases} y(0) \in \mathbb{S}^{d'}, \\ y(n) = B \otimes y(n-1), \quad n \geq 1, \end{cases} \tag{2}$$

with $d, d' \in \mathbb{N}$, $\mathbb{N}$ denotes the set of non-negative integers, and A (resp. B) is a $d \times d$ (resp. $d' \times d'$) matrix. The ith (resp. i'th) component of vector $A \otimes x$, $i = 1, \ldots, d$, (resp. $B \otimes y$, $i' = 1, \ldots d'$) is defined by: $\bigoplus_{j=1}^{d} a_{i,j} \otimes x_j$ (resp. $\bigoplus_{j'=1}^{d'} b_{i',j'} \otimes y_{j'}$). Let $\mathbb{K}$ (resp. $\mathbb{K}'$) be an $m \times d$ (resp. $m \times d'$) matrix. Now, let us consider the binary relation denoted $\leq_{\mathbb{K},\mathbb{K}'}$ and defined by:

$$\forall\, x \in \mathbb{S}^d,\ \forall, y \in \mathbb{S}^{d'} : \quad x \leq_{\mathbb{K},\mathbb{K}'} y \overset{\text{def}}{\Leftrightarrow} \mathbb{K} \otimes x \leq_m \mathbb{K}' \otimes y. \tag{3}$$

The choice of the matrices $\mathbb{K}$ and $\mathbb{K}'$ is generally guided by the nature of the problem (see e. g. the example of Subsection 5.2 and also the discussion in [33, Introduction]). The systems $(d, \otimes, A)$ and $(d', \otimes, B)$ are said to be $(\mathbb{K}, \mathbb{K}')$-comparable if the following assertion

$$\forall\, x(0) \in \mathbb{S}^d, \forall\, y(0) \in \mathbb{S}^{d'},\ \big(x(0) \leq_{\mathbb{K},\mathbb{K}'} y(0) \Rightarrow \forall\, n \in \mathbb{N}, x(n) \leq_{\mathbb{K},\mathbb{K}'} y(n)\ \big) \tag{4}$$

is true. Based on Haar's Lemma we characterize the assertion (4). The main results are Theorem 2 (semiring case) and Corollaries 2 and 3 (semifield case). We generalize a pioneer paper on this subject [31], and also [25] and [33]. When the assertion (4, with $d = d'$ and $A = B$) is true we say that the system $(d, \otimes, A)$ is $(\mathbb{K}, \mathbb{K}')$-monotone. The characterization of the $(\mathbb{K}, \mathbb{K}')$-monotonicity is given in Theorem 3 and Corollary 4. As mentioned in Remark 4 the Corollary 4 generalizes [30, Theorem 4.2]. Note that when $d = d'$, $\mathbb{K} = \mathbb{K}' = I$, where I denotes the identity matrix over $\mathbb{S}$, $\leq_{\mathbb{K},\mathbb{K}'}$ is reduced to $\leq_d$, (4) is true iff $A \leq B$ (entry-wise comparison) and the (I, I)-monotonicity is verified by any matrix over $\mathbb{S}$. Monotonicity is of particular importance because part of the results concerning monotonicity will be used to solve some control problems studied in this paper. In the case of complete idempotent semifield we comment on the complexity of the computations involved as follows. The computation of the optimal (in the sense of the entry-wise order between matrices) lower bound, i. e. the matrix A associated to the system (1) only requires product of the given matrices $\mathbb{K}$, $\mathbb{K}'$ and B. The computation of an upper bound, i.e. a matrix B associated with system (2) for given matrices $\mathbb{K}$, $\mathbb{K}'$ and A, seems to be tractable but mainly depends on the structure of the matrices $\mathbb{K}$ and $\mathbb{K}'$. Partial answer to these problems can be found elsewhere in e.g. [33] but a detailed study needs further work. The characterization of a $(\mathbb{K}, \mathbb{K}')$-monotone operator for given matrices $\mathbb{K}$ and $\mathbb{K}'$ is naturally expressed as a sub-fixed point of a Min-Max type function for which there exist computational algorithms (see e.g. [11, 13, 17]) which behave well in practice although no precise information about their complexities is known this time.

B: Control of DES. Our approach to the control of DES is based on a state space representation and positive invariance of some domain. So, it differs from the ones based on transfer series methods [6] which have been successfully applied to solve control problems via positive invariance (see e.g. [26, Chapter 5], [27]). Because transfer series are particular cases of formal series we mention that similar concepts have been developed and applied in language theory (see e.g. [24]). We study a particular case of geometric approach applied to the control of DES which concerns the problem of the algebraically (A, B)-invariance also known as $(A \oplus B \otimes F)$-invariance of some domain to be defined in the sequel. It seems that the closest referenced work on the subject we study is [23, Section 5]. For a more complete overview of methods we refer the reader to [23, and references therein]. More precisely our problem is formalized as follows. Let us consider the linear system over a complete idempotent semiring or semifield $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ defined by:

$$\begin{cases} x(0) \in \mathbb{S}^d, \\ x(n) = A \otimes x(n-1) \oplus B \otimes u(n), \quad \forall n \geq 1 \\ u(n) = F \otimes x(n-1), \end{cases} \tag{5}$$

with A a $d \times d$ matrix, B a $d \times m$ matrix, and F is an $m \times d$ matrix to be determined. Any F is called a linear feedback. Note that (5) can be rewritten as the resulting closed loop system:

$$\begin{cases} x(0) \in \mathbb{S}^d, \\ x(n) = \Omega \otimes x(n-1) \end{cases} \tag{6}$$

with $\Omega := A \oplus B \otimes F$.

We study two kind of domains. The first one is inspired by [7, Sections 3 and 4] which is related to the viability theory [5]. We study the property that the entire state trajectory for all possible trajectories of the closed loop system (5) is kept in a given domain called *target tube* modelled by a series of subsets of $\mathbb{S}^d$ $(\mathcal{T}_n)_{n \in \mathbb{T}}$, $\mathbb{T} = \{0, 1, 2, \ldots\} \subseteq \mathbb{N}$. It means that we require the following assertion:

$$\forall n \in \mathbb{T} \setminus \{0\}, \forall x \in \mathbb{S}^d, \ (x \in \mathcal{T}_{n-1} \Rightarrow \Omega \otimes x \in \mathcal{T}_n) \tag{7}$$

to be true. When the sets $\mathcal{T}_n$ are idempotent polyhedral sets (as defined by (11)) the results are as follows. We derive necessary and sufficient condition for the existence of the optimal linear feedback such that (7) is true (see Theorem 4). For finite target tube of idempotent polyhedral sets the computation of F can be done in polynomial time. When $\mathbb{T} = \mathbb{N}$ we only give a partial answer to the problem of computation in Proposition 3 of Section 5. But this problem needs further work.

The second kind of domains concerns the idempotent semimodules of the form:

$$\mathcal{S}_{G,G'} \stackrel{\text{def}}{=} \{x \in \mathbb{S}^d : G \otimes x \leq_m G' \otimes x\}, \tag{8}$$

with G and G' $m \times d$ matrices. The approach of the control is based on the $(\Omega := A \oplus B \otimes F)$-invariance of the semimodule $\mathcal{S}_{G,G'}$. Note that the Ω-invariance of a domain $\mathcal{V} \subseteq \mathbb{S}^d$ corresponds to (7, with $\mathbb{T} = \mathbb{N}$, $\forall\, n \in \mathbb{T}$, $\mathcal{T}_n = \mathcal{V}$), i. e.:

$$\mathcal{V} \text{ is } \Omega\text{-invariant} \stackrel{\text{def}}{\Leftrightarrow} \Big(\forall\, x \in \mathbb{S}^d,\ (x \in \mathcal{V} \Rightarrow \Omega \otimes x \in \mathcal{V})\Big). \tag{9}$$

This kind of domains is involved when the constraints that the controlled system $(d, \otimes, \Omega)$ must satisfy are of the form:

$$\forall\, n \in \mathbb{N},\ G \otimes x(n) \leq_m G' \otimes x(n). \tag{10}$$

We remark that the (G, G')-monotonicity of Ω implies the Ω-invariance of $\mathcal{S}_{G,G'}$ (see Proposition 2). Thus, if $G \otimes x(0) \leq_m G' \otimes x(0)$ and Ω is (G, G')-monotone then (10) is true. This latter argument proves Theorem 5 and Corollary 5 which provide sufficient condition for the existence of a linear feedback. The computation of such feedback over idempotent semifield is naturally expressed as a sub-fixed point of a Min-Max type function and previous remarks apply for computational algorithms. Let us note that in this paper the conditions that the idempotent structure must satisfy are weaker than the one of [23] and [18] which imposed sufficient condition on $\mathbb{S}$ for semimodules defined on $\mathbb{S}$ to be finitely generated or rational and then derive necessary and sufficient condition for (A, B)-invariance. Although the (A, B)-invariance is a more general approach to the control than the $(A \oplus B \otimes F)$-invariance (which directly concerns our work) the restrictions on the underlying algebraic structure $\mathbb{S}$ are still required for the control via $(A \oplus B \otimes F)$-invariance in [23]. The theoretical results based on monotonicity are applied to the study of transportation networks which evolve according to a time table. The numerical example is borrowed from [23, Section 6] which corresponds to the particular case: $G := \mathbb{K} \otimes \Omega$ and $G' := I$. In fact the author in [23, Section 5] studies the problem of the existence and the computation of a maximal algebraically (A, B)-invariant semimodule of the form $\mathrm{Im}(Q)$ which denotes the set of all linear combinations of the columns of Q, where Q is a given matrix with a finite number of columns. When $G' = I$ we have $\mathcal{S}_{G,I} = \mathrm{Im}(G^*)$, where $G^* \stackrel{\text{def}}{=} I \oplus G \oplus G^{\otimes 2} \oplus \cdots \oplus G^{\otimes n} \oplus \cdots$, if exists. Where $G^{\otimes n} \stackrel{\text{def}}{=} G \otimes \ldots \otimes G$ (n-fold) if $n \geq 1$, and I if $n = 0$. Thus, our approach could be considered as a kind of dual approach of the one developed in [23]. A numerical example on network transportation illustrates this discussion in Subsection 5.2.

This paper is organized as follows. In Section 2, we introduce the main notations used in this paper and we present the main definitions and concepts concerning idempotent algebra. Our main references are ([6, 8, 19]). In Section 3, we prove two idempotent versions of Haar's Lemma over complete idempotent semirings and complete idempotent semifields. In Section 4, we apply Haar's Lemma in Subsection 4.1

to characterize the $(\mathbb{K}, \mathbb{K}')$-comparison of DES. Characterization of monotonicity of idempotent operator is also derived in Subsection 4.2 as a particular case of comparison. In the Subsection 4.3 we study the existence and the computation of a linear feedback (possibly optimal) such that the resulting closed loop system has its entire trajectory in a specified domain for all or a part of the trajectories of the closed loop system. We illustrate theoretical results of the paper in Section 5. Finally, let us mention that the numerical computations are done by hand and that the implementation of the algorithms needs further work which is beyond the scope of this paper. It is also expected that the algorithms provide more accurate numerical results.

2. PRELIMINARIES

In this section we introduce the notations and definitions used throughout the paper and we recall basic statements on max-plus algebra and residuation theory needed in the paper. More details can be found in e. g. ([6, 8] and [19]).

2.1. Notations and definitions

- Recall that $\leq_n$ denotes the component-wise order or product order on $\mathcal{X}^n$ where $(\mathcal{X}, \leq)$ is a poset (i. e. $x \leq_n y \overset{\text{def}}{\Leftrightarrow} \forall\, i,\, x_i \leq y_i$).
- All vectors are column vectors.
- The set of $(m \times d)$-dimensional matrices with entries in a semiring or semifield $\mathbb{S}$ will be denoted by $\mathcal{M}_{m,d}(\mathbb{S})$.
- For all matrix A, $a_{i,j}$, $a_{l,\cdot}$ and $a_{\cdot,k}$ respectively denote the entry (i, j), the lth row and the kth column of A.
- If $A, B \in \mathcal{M}_{m,n}(\mathbb{S})$ then $A \leq B$ denotes the entry-wise comparison of the matrices A and B.
- Let $P \in \mathcal{M}_{m,d}(\mathbb{S})$ and p be an m-dimensional vector. The set $\mathcal{P}(P, p)$ associated with (P, p) is defined by:

$$\mathcal{P}(P,p) \overset{\text{def}}{=} \{x \in \mathbb{S}^d : \; P \otimes x \leq_m p\}. \tag{11}$$

  In the usual linear algebra, $\mathcal{P}(P, p)$ is called a polyhedral set.

2.2. Ordered sets and elements of residuation theory

Let $(\Omega, \leq)$ be a partially ordered set. $(\Omega, \leq)$ is a sup semilattice (resp. inf semilattice) iff any set $\{\omega_1, \omega_2\} \subset \Omega$ has a supremum $\bigvee\{\omega_1, \omega_2\}$ (resp. an infimum $\bigwedge\{\omega_1, \omega_2\}$). $(\Omega, \leq)$ is a lattice iff $(\Omega, \leq)$ is a sup and inf semilattice. $(\Omega, \leq)$ is complete iff any set $A \subset \Omega$ has a supremum $\bigvee A$. A complete ordered set is also a complete lattice because $\bigwedge A \overset{\text{def}}{=} \bigvee\{\omega \in \Omega : \; \forall\, a \in A,\; \omega \leq a\}$. A lattice is distributive iff $\wedge$ and $\vee$ are distributive with respect to (w.r.t.) one another.

A map $f : (\Omega, \leq) \to (\Omega', \preceq)$, where $(\Omega, \leq)$ and $(\Omega', \preceq)$ are two ordered sets, is $(\leq, \preceq)$-monotone if it is a compatible morphism with respect to $\leq$ and $\preceq$. The map $f : (\Omega, \leq) \to (\Omega', \preceq)$ is residuated iff there exists a map $f^\natural : (\Omega', \preceq) \to (\Omega, \leq)$ such

that: $\forall\,\omega \in \Omega, \forall\,\omega' \in \Omega'$, $f(\omega) \preceq \omega' \Leftrightarrow \omega \leq f^\natural(\omega')$. This map can be defined as follows:
$$f^\natural(\cdot) \overset{\text{def}}{=} \bigvee\{\omega \in \Omega :\ f(\omega) \leq\ \cdot\ \}.$$

A monotone map $f : (\Omega, \leq) \to (\Omega', \preceq)$, where $(\Omega, \leq)$ and $(\Omega', \preceq)$ are complete sets, is said to be continuous iff $\forall\,A \subset \Omega$, $f(\bigvee_{\leq} A) = \bigvee_{\preceq} f(A)$, $\bigvee_{\leq}$ (resp. $\bigvee_{\preceq}$) denotes the supremum w.r.t. $\leq$ (resp. $\preceq$); $f(A) \overset{\text{def}}{=} \{f(a) :\ a \in A\}$. The following result (see e.g. [8, Th. 5.2] or [6, Th. 4.50]) provides a characterization for a residuable function over two complete ordered sets.

Result 3. Let $(\Omega, \leq)$ and $(\Omega', \preceq)$ two complete sets. A map $f : (\Omega, \leq) \to (\Omega', \preceq)$ is residuated iff f is continuous and $f(\bigwedge \Omega) = \bigwedge \Omega'$.

2.3. Basic algebraic structures

For any set $\mathbb{S}$, $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a semiring if $(\mathbb{S}, \oplus, \varepsilon)$ is a commutative monoid, $(\mathbb{S}, \otimes, e)$ is a monoid, $\otimes$ distributes over $\oplus$, the neutral element ε for $\oplus$ is also absorbing element for $\otimes$, i. e. $\forall\,a \in \mathbb{S}$, $\varepsilon \otimes a = a \otimes \varepsilon = \varepsilon$, and e is the neutral element for $\otimes$.

$(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is an idempotent semiring (called also dioid) if $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a semiring, the internal law $\oplus$ is idempotent, i. e. $\forall\,a \in \mathbb{S}$, $a \oplus a = a$. If $(\mathbb{S}, \otimes, e)$ is a commutative monoid, then the idempotent semiring $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is said to be commutative.

$(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a an idempotent semifield if $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is an idempotent semiring and $(\mathbb{S} \setminus \{\varepsilon\}, \otimes, e)$ is a group, i. e. $(\mathbb{S} \setminus \{\varepsilon\}, \otimes, e)$ is a monoid such that all its elements are invertible ($\forall\,a \in \mathbb{S}\setminus\{\varepsilon\}$, $\exists\,a^{\otimes -1}\ :\ a \otimes a^{\otimes -1} = a^{\otimes -1} \otimes a = e$). Also if $(\mathbb{S}\setminus\{\varepsilon\}, \otimes, e)$ is a commutative monoid, then the idempotent semifield $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is said to be commutative.

Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ be an idempotent semiring. Each element of $\mathbb{S}^n$ is a n-dimensional column vector. We equip $\mathbb{S}^n$ with the two laws $\oplus$ and $\cdot$ as follows: $\forall\,x, y \in \mathbb{S}^n$, $(x \oplus y)_i = x_i \oplus y_i, \forall\,s \in \mathbb{S}$, $(s \cdot x)_i \overset{\text{not.}}{=} (s \otimes x)_i \overset{\text{def}}{=} s \otimes x_i$, $i = 1, \ldots, n$. The addition $\oplus$ and the multiplication $\otimes$ are naturally extended to matrices with compatible dimension. Any $n \times p$ matrix A is associated with a $(\oplus, \otimes)$-linear map $A : \mathbb{S}^p \to \mathbb{S}^n$. The (i, j) entry, the lth row-vector and the kth column-vector of matrix A, are respectively denoted $a_{i,j}$, $a_{l,\cdot}$ and $a_{\cdot,k}$. Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ be an idempotent semiring or an idempotent semifield, then $(\mathbb{S}, \oplus, \varepsilon)$ is an idempotent monoid, which can be equipped with the natural order relation $\leq$ defined by:

$$\forall\,a, b \in \mathbb{S},\ a \leq b \overset{\text{def}}{\Leftrightarrow} a \oplus b = b. \tag{12}$$

We say that $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is complete if it is complete as a naturally ordered set and if the respective left and right multiplications, $\lambda_a, \rho_a : \mathbb{S} \to \mathbb{S}$, $\lambda_a(x) = a \otimes x$, $\rho_a(x) = x \otimes a$ are continuous for all $a \in \mathbb{S}$. In such case we adopt the following notations for all $a, b \in \mathbb{S}$:

$$\lambda_a^\natural(b) \overset{\text{not.}}{=} a\backslash b \overset{\text{def}}{=} \bigvee\{x \in \mathbb{S} :\ a \otimes x \leq b\},$$
$$\rho_a^\natural(b) \overset{\text{not.}}{=} b/a \overset{\text{def}}{=} \bigvee\{x \in \mathbb{S} :\ x \otimes a \leq b\}.$$

A typical example of complete dioid is the top completion of an idempotent semifield. Let us note that if $a \in \mathbb{S}$ is invertible then: $a \backslash b = a^{\otimes -1} \otimes b$ and $b/a = b \otimes a^{\otimes -1}$. And if $\otimes$ is commutative then: $a \backslash b = b/a \overset{\text{not.}}{=} \frac{a}{b}$. Let us note also that as $\mathbb{S}$ is complete it possesses a top element $\bigvee \mathbb{S} \overset{\text{not.}}{=} \top$. We have by convention the following identities:

$$\varepsilon \otimes \top = \top \otimes \varepsilon = \varepsilon, \quad \text{and } \forall\, a \in \mathbb{S},\ a \oplus \top = \top, a \wedge \top = \top \wedge a = a. \tag{13}$$

We suppose besides that (for a discussion to this topic, see e. g. ([6, pp. 163–164]):

$$\forall\, a \neq \varepsilon,\ a \otimes \top = \top \otimes a = \top. \tag{14}$$

By definition of / (idem for \) and properties of $\top$, ε and of [14] we have for all $a \in \mathbb{S}$:

$$a/\varepsilon = \top,\ \top/a = \top, \tag{15}$$

$$a/\top = \begin{cases} \varepsilon & \text{if } a \neq \top \\ \top & \text{if } a = \top \end{cases} \quad \varepsilon/a = \begin{cases} \varepsilon & \text{if } a \neq \varepsilon \\ \top & \text{if } a = \varepsilon \end{cases}. \tag{16}$$

We mention below (Table 1) the list of remarkable identities of the operator / (similar relations for \) that will serve us thereafter. The left column gives valid formulas for idempotent semirings (see [6, p. 183]) and the right column valid formulas for idempotent semifields whose elements possess the reflexive property in the sense of ([12, Section 4.3]).

Table 1. Formulas using the division /.

	semiring	semifield	
(e.1)	$x/(a \oplus b) = x/a \wedge x/b$	$x/(a \oplus b) = x/a \wedge x/b$	(f.1)
(e.2)	$x/(a \wedge b) \geq x/a \oplus x/b$	$x/(a \wedge b) = x/a \oplus x/b$	(f.2)
(e.3)	$(x \oplus y)/a \geq x/a \oplus y/a$	$(x \oplus y)/a = x/a \oplus y/a$	(f.3)
(e.4)	$(x \otimes a)/a \geq x$	$(x \otimes a)/a \geq x$	(f.4)

The operations /, \ are extended to matrices and vectors with compatible dimensions assuming that all the elements of these matrices and vectors are in a complete set $\mathbb{S}$:

$$(A \backslash y)_i = \wedge_j\ (a_{j,i} \backslash y_j); \tag{17}$$

$$(A \backslash B)_{i,j} \overset{\text{def}}{=} (\bigvee \{X :\ A \otimes X \leq B\})_{i,j} = \wedge_k\ (a_{k,i} \backslash b_{k,j}); \tag{18}$$

$$(D/C)_{i,j} \overset{\text{def}}{=} (\bigvee \{X :\ X \otimes C \leq D\})_{i,j} = \wedge_l\ (d_{i,l}/c_{j,l}). \tag{19}$$

$$A \backslash B/C \overset{\text{def}}{=} \bigvee \{X :\ A \otimes X \otimes C \leq B\} = A \backslash (B/C) = (A \backslash B)/C \tag{20}$$

with $\leq$ denoting the entry-wise comparison of matrices.

3. IDEMPOTENT VERSIONS OF HAAR'S LEMMA

In this section we present two idempotent versions of Haar's Lemma (cf. Result 1). The first version (cf. Theorem 1) is given for complete idempotent semirings. This version is more general than the second version (cf. Corollary 1) that is valid for complete idempotent semifields. We will note that this second version is very similar to the initial form of Haar's Lemma, that was formulated in the usual linear algebra (cf. Result 2).

Theorem 1. Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ be a complete idempotent semiring and $P, Q \in \mathcal{M}_{m,d}(\mathbb{S})$. Let p, q two m-dimensional vectors. The following assertion

$$\mathcal{P}(P,p) \subseteq \mathcal{P}(Q,q),$$

is true iff

$$Q \leq q/(P\backslash p). \tag{21}$$

P r o o f. We proceed by logical equivalence.
The assertion $\mathcal{P}(P,p) \subseteq \mathcal{P}(Q,q)$ can be rewritten as follows:

$$\forall\, x \in \mathbb{S}^d,\ (P \otimes x \leq_m p \Rightarrow Q \otimes x \leq_m q),$$

which is equivalent to by definition of $\backslash$:

$$\forall\, x \in \mathbb{S}^d,\ (x \leq_d P\backslash p \Rightarrow Q \otimes x \leq_m q).$$

Because $\otimes$ is non-decreasing the implication above is true iff $Q \otimes (P\backslash p) \leq_m q$. And by definition of / this last inequality is equivalent to $Q \leq q/(P\backslash p)$ which proves the result. $\square$

Before expressing the version of Haar's Lemma for complete idempotent semifields we need the following result. For any vector v we define the following sets:

$$\mathrm{supp}(v) = \{l\ :\ v_l \neq \varepsilon\},\ T_v = \{l\ :\ v_l \neq \top\}. \tag{22}$$

We adopt the following convention:

$$\bigwedge_{\emptyset} = \top,\ \bigoplus_{\emptyset} = \varepsilon. \tag{23}$$

The set $\overline{\xi}$ denotes the complementary set of ξ.

Lemma 1. Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ be a complete idempotent semifield. Let $A \in \mathcal{M}_{m,d}(\mathbb{S})$, $b, g \in \mathbb{S}^m$, and assume that the two following conditions hold:

(H_0) : $\forall\, j \in \{1, \dots, d\}$, $\mathrm{supp}(a_{\cdot,j}) \neq \emptyset$ (A have no null columns),

(H_1) : $\forall\, j \in \{1, \dots, d\}$, $\overline{\mathrm{supp}(b)} \cap \overline{T_{a_{\cdot,j}}} = \emptyset$.

Then we have:

$$b/(A\backslash g) = (b/g) \otimes A. \tag{24}$$

P r o o f. We first calculate every term of the equality (24), using equations (18) and (19).

Define $L_j = \mathrm{supp}(a_{\cdot,j}) \cap T_g$. For all i, j, we have:

$$\begin{aligned}(b/(A\backslash g))_{i,j} &= b_i/(\wedge_{l=1}^{m}(a_{l,j}\backslash g_l)) \\ &= b_i/(\wedge_{l\in L_j}(a_{l,j}\backslash g_l)) \qquad \text{(properties of } \top \text{ and } \wedge\text{, (13)).}\end{aligned}$$

In the same way, using of the sets L_j and $\overline{L_j}$, we obtain:

$$((b/g)\otimes A)_{i,j} \quad = \oplus_{l\in L_j}((b_i/g_l)\otimes a_{l,j}) \oplus (\oplus_{l\in\overline{L_j}}((b_i/g_l)\otimes a_{l,j})) \ .$$

1st case: $L_j = \emptyset$, i. e. $\forall\, l = 1, \dots, m$, $g_l = \top$ or $a_{l,j} = \varepsilon$.
In this case, using (16), we have on the one hand:

$$(b/(A\backslash g))_{i,j} = b_i/\top = \begin{cases} \top & \text{if } b_i = \top \\ \varepsilon & \text{if } b_i \neq \top. \end{cases}$$

On the other hand, we remark that: if $b_i \neq \top$ then $\forall\, l \in \overline{L_j}$, $(b_i/g_l)\otimes a_{l,j} = \varepsilon$, and if $b_i = \top$ then (H_0) implies $\exists\, l$, $(b_i/g_l)\otimes a_{l,j} = \top$. According to these remarks we have:

$$((b/g)\otimes A)_{i,j} = \oplus_{l\in\overline{L_j}}((b_i/g_l)\otimes a_{l,j}) = \begin{cases} \top & \text{if } b_i = \top \\ \varepsilon & \text{if } b_i \neq \top \end{cases},$$

and the assertion is verified.

2nd case: $L_j \neq \emptyset$ and $(\forall\, l \in L_j,\ g_l \neq \varepsilon$ and $a_{l,j} \neq \top)$.

In this case $\forall\, l \in L_j$, $a_{l,j}$ and g_l are invertible and we have:

$$\begin{aligned}(b/(A\backslash g))_{i,j} &= b_i/(\wedge_{l\in L_j}(a_{l,j}\backslash g_l)) \\ &= \oplus_{l\in L_j}(b_i/(a_{l,j}\backslash g_l)) \qquad \text{(by (f.2))} \\ &= \oplus_{l\in L_j} b_i \otimes (a_{l,j}^{\otimes -1}\otimes g_l)^{\otimes -1} \\ &= \oplus_{l\in L_j} b_i \otimes g_l^{\otimes -1} \otimes a_{l,j} \\ &= \oplus_{l\in L_j}((b_i/g_l)\otimes a_{l,j}) \\ &= ((b/g)\otimes A)_{i,j}.\end{aligned}$$

3rd case: $L_j \neq \emptyset$ and $(\exists\, l \in L_j,\ g_l = \varepsilon$ or $a_{l,j} = \top)$.

In this case, either we have: $\exists\, l \in L_j$, $g_l = \varepsilon$ and $a_{l,j} = \top$ and in this case according to (13), (14), (15), (16), for all $b_i \in \mathbb{S}$, $(b/(A\backslash g))_{i,j} = \top$ and $((b/g)\otimes A)_{i,j} = \top$ hence the equality holds. Or, we have: $\exists\, l \in L_j$, $g_l \neq \varepsilon$ and $a_{l,j} = \top$. According to (H_1) then $b_i \neq \varepsilon$ and we have: $(b/(A\backslash g))_{i,j} = \top = ((b/g)\otimes A)_{i,j}$.

In any case the equality between the two terms of the assertion (24) is verified for all i, j, therefore the result is proved. □

Now we are able to express the following corollary.

Corollary 1. Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ be a complete idempotent semifield. Let $P, Q \in \mathcal{M}_{m,d}(\mathbb{S})$. Let p and q be two m-dimensional vectors. We assume that the hypotheses (H_0) and (H_1) of Lemma 1 are verified for P, p and q. The assertion

$$\mathcal{P}(P, p) \subseteq \mathcal{P}(Q, q),$$

is true iff

$$\exists\, H \in \mathcal{M}_{m,m}(\mathbb{S}), \quad \text{(i) } Q \leq H \otimes P, \quad \text{(ii) } H \otimes p \leq_m q. \tag{25}$$

Proof. (Necessity.) As a complete idempotent semifield is a particular case of complete idempotent semiring, Theorem 1 can be applied. Using Lemma 1, the term $q/(P \backslash p))$ on the right hand of the equation (21) is equal to $((q/p) \otimes P)$, then (21) becomes:

$$Q \leq (q/p) \otimes P. \tag{26}$$

It is sufficient to take $H = q/p$. By definition of $/$ it is clear that the condition (ii) of the corollary holds true, and from (26) it is trivial to check that (i) is also satisfied. (Sufficiency.) Obvious. □

Remark 1. Let us note that the form of this result is very close to the form of Result 2. This is due to the fact that all elements of a semifield or a semiring are positive w.r.t. the natural order defined by (12).

4. APPLICATIONS OF IDEMPOTENT VERSIONS OF HAAR'S LEMMA

The main aim of this section is to show as indicated in the introduction that the two idempotent versions of Haar's Lemma over the two complete idempotent structures (semiring and semifield) play a central role in two different problems. The first one is the characterization of the $(\mathbb{K}, \mathbb{K}')$-comparison of DES which is mainly motivated by obtaining models for which performance evaluation is simpler. The second one is the control of DES using the properties of positive invariance of some domains to be specified. Control of DES based on positive invariance of transfer series has been successfully applied in e. g. [26] and [27]. Positive invariance has also been used as a corollary of the geometric approach developed in [23]. Here, we explore a third way. We also mention a fourth way which is being explored. It is inspired by the second method of Lyapunov (see [3, 4] and generalized a previous work (see [32]).

4.1. $(\mathbb{K}, \mathbb{K}')$-comparison of discrete event systems

Consider now the two linear systems $(d, \otimes, A)$ and $(d', \otimes, B)$ over a complete idempotent semiring or a complete idempotent semifield $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ respectively defined by (1) and (2). In this subsection, we use the two idempotent versions of Haar's Lemma to give necessary and sufficient conditions for the $(\mathbb{K}, \mathbb{K}')$-comparison of these two linear systems.

We assume that the matrices $\mathbb{K} \in \mathcal{M}_{m,d}(\mathbb{S} \setminus \{\top\})$ and $\mathbb{K}' \in \mathcal{M}_{m,d'}(\mathbb{S} \setminus \{\top\})$ have no null columns and matrix $\mathbb{K}'$ has no null rows. In this case the hypotheses (H_0) and (H_1) of Lemma 1 are verified.

Noticing that the $(\mathbb{K}, \mathbb{K}')$ comparison (4) of systems $(d, \otimes, A)$ and $(d, \otimes, A)$ is equivalent to:

$$\forall x \in \mathbb{S}^d, \forall y \in \mathbb{S}^{d'} : x \leq_{\mathbb{K},\mathbb{K}'} y \Rightarrow A \otimes x \leq_{\mathbb{K},\mathbb{K}'} B \otimes y, \tag{27}$$

we have the following theorem.

Theorem 2. (NSC of $(\mathbb{K}, \mathbb{K}')$-comparison) The two systems $(d, \otimes, A)$ and $(d', \otimes, B)$ respectively defined by (1) and (2) over a complete idempotent semiring $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ are $(\mathbb{K}, \mathbb{K}')$-comparable iff the following condition holds:

$$\forall y \in \mathbb{S}^{d'} : \mathbb{K} \otimes A \leq (\mathbb{K}' \otimes B \otimes y)/(\mathbb{K} \backslash (\mathbb{K}' \otimes y)). \tag{28}$$

P r o o f. By definition of the $(\mathbb{K}, \mathbb{K}')$-comparison (3), the assertion (27) can be rewritten as follows:

$$\forall x \in \mathbb{S}^d, \forall y \in \mathbb{S}^{d'} : \mathbb{K} \otimes x \leq_m \mathbb{K}' \otimes y \Rightarrow \mathbb{K} \otimes A \otimes x \leq_m \mathbb{K}' \otimes B \otimes y$$

which is equivalent to:

$$\forall y \in \mathbb{S}^{d'} : \mathcal{P}(\mathbb{K}, \mathbb{K}' \otimes y) \subseteq \mathcal{P}(\mathbb{K} \otimes A, \mathbb{K}' \otimes B \otimes y).$$

Applying the first idempotent version of Haar's Lemma (cf. Theorem 1) with

$$\begin{cases} P := \mathbb{K}, \quad p := \mathbb{K}' \otimes y \\ Q := \mathbb{K} \otimes A, \quad q := \mathbb{K}' \otimes B \otimes y, \end{cases}$$

this last assertion is true iff

$$\forall y \in \mathbb{S}^{d'} : \mathbb{K} \otimes A \leq (\mathbb{K}' \otimes B \otimes y)/(\mathbb{K} \backslash (\mathbb{K}' \otimes y)). \qquad \square$$

In the specific case of complete idempotent semifield, we get the following corollary.

Corollary 2. Let $(d, \otimes, A)$ and $(d', \otimes, B)$ be the two systems defined by (1) and (2), respectively, and assume that $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a complete idempotent semifield. Then, the two systems are $(\mathbb{K}, \mathbb{K}')$-comparable iff

$$\forall y \in \mathbb{S}^{d'} : \mathbb{K} \otimes A \leq (\mathbb{K}' \otimes B \otimes y/\mathbb{K}' \otimes y) \otimes \mathbb{K}. \tag{29}$$

P r o o f. As a complete idempotent semifield is a particular case of complete idempotent semiring, Theorem 2 can be applied, hence (28) holds. The result is then obtained thanks to Lemma 1. $\square$

A second characterization for the $(\mathbb{K}, \mathbb{K}')$-comparison of linear systems over a complete idempotent semifield can be deduced from the following lemma.

Lemma 2. Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ be a complete idempotent semifield and $D, C \in \mathcal{M}_{m,d'}(\mathbb{S})$. The following equality holds true:

$$\bigwedge_{y \in \mathbb{S}^{d'}} ((D \otimes y)/(C \otimes y)) = D/C. \tag{30}$$

P r o o f. For all i, j we have $(D/C)_{i,j} = \wedge_{l=1}^{d'}(d_{i,l}/c_{j,l})$. Now we focus our attention on $((D \otimes y)/(C \otimes y))_{i,j}$. And

$$\begin{aligned} ((D \otimes y)/(C \otimes y))_{i,j} &= (d_{i,\cdot} \otimes y)/(c_{j,\cdot} \otimes y) \\ &= (d_{i,\cdot} \otimes y)/(\oplus_{l=1}^{d'}(c_{j,l} \otimes y_l)) \\ &= \wedge_{l=1}^{d'}(d_{i,\cdot} \otimes y)/(c_{j,l} \otimes y_l). \qquad \text{(by (f.2))} \end{aligned}$$

For all l we have:

$$\begin{aligned} (d_{i,\cdot} \otimes y)/(c_{j,l} \otimes y_l) &= \oplus_k (d_{i,k} \otimes y_k)/(c_{j,l} \otimes y_l) \\ &\geq (d_{i,l} \otimes y_l)/(c_{j,l} \otimes y_l) \end{aligned} .$$

Since $\otimes$ is associative, we notice that for all index l we have:

$$(((d_{i,l} \otimes y_l)/y_l)/c_{j,l}) \otimes c_{j,l} \otimes y_l \leq d_{i,l} \otimes y_l.$$

Therefore:

$$\forall\, l : (d_{i,l} \otimes y_l)/(c_{j,l} \otimes y_l) \geq ((d_{i,l} \otimes y_l)/y_l)/c_{j,l}.$$

Since $(f.4) : (d_{i,l} \otimes y_l)/y_l \geq d_{i,l}$, and since $x \mapsto x/c_{j,l}$ is increasing $\forall\, l$, then

$$(d_{i,l} \otimes y_l)/(c_{j,l} \otimes y_l) \geq (d_{i,l}/c_{j,l}).$$

It remains to show that for all i, j, we can find a vector which reaches $(D/C)_{i,j}$. Since $\{1, \ldots, d'\}$ is a finite set, $\exists\, l^* : (D/C)_{i,j} = d_{i,l^*}/c_{j,l^*}$. In this case we take $y^* \in \mathbb{S}^{d'}$ defined by:

$$y^* \in \mathbb{S}^{d'} : \begin{cases} y_l^* = e \text{ if } l = l^*, \\ y_l^* = \varepsilon \text{ otherwise.} \end{cases}$$

Then it is easy to check that:

$$((D \otimes y^*)/(C \otimes y^*))_{i,j} = d_{i,l^*}/c_{j,l^*}. \qquad \square$$

From Lemma 2 we deduce an algebraic characterization of the $(\mathbb{K}, \mathbb{K}')$-comparison of linear systems over a complete idempotent semifields as follows.

Corollary 3. Let $(d, \otimes, A)$ and $(d, \otimes, B)$ be the two systems respectively defined by (1) and (2) over a complete idempotent semifield $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$. A characterization of the $(\mathbb{K}, \mathbb{K}')$-comparison is

$$\mathbb{K} \otimes A \leq ((\mathbb{K}' \otimes B)/\mathbb{K}') \otimes \mathbb{K}. \tag{31}$$

Proof. Thanks to Corollary 4.1, the assertion (29) is equivalent to:

$$\mathbb{K} \otimes A \leq \bigwedge_{y \in \mathbb{S}^{d'}} \left[((\mathbb{K}' \otimes B \otimes y)/(\mathbb{K}' \otimes y)) \otimes \mathbb{K} \right]$$

$$= \left[\bigwedge_{y \in \mathbb{S}^{d'}} ((\mathbb{K}' \otimes B \otimes y)/(\mathbb{K}' \otimes y)) \right] \otimes \mathbb{K} \quad \text{by distributivity of } \otimes \text{ over } \wedge.$$

Applying Lemma 2, with $D := \mathbb{K}' \otimes B$ and $C := \mathbb{K}'$, the result is obtained. □

Remark 2. In the case where $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a complete idempotent semiring, the condition (31) is a sufficient condition for the $(\mathbb{K}, \mathbb{K}')$-comparison of the two systems $(d, \otimes, A)$ and $(d', \otimes, B)$.

Remark 3. Assuming $\mathbb{K}$, $\mathbb{K}'$ and B known, there exists a unique optimal upper bound on the set of matrices A satisfying (31) which is $\mathbb{K} \backslash (((\mathbb{K}' \otimes B)/\mathbb{K}') \otimes \mathbb{K})$. The computation of this bound depends on the size of the matrices $\mathbb{K}$, $\mathbb{K}'$ and B.

Assuming $\mathbb{K}$, $\mathbb{K}'$ and A known the computation of a matrix B satisfying (31) can be done by the following non optimal procedure:

Step 1. Find a matrix C such that: $\mathbb{K} \otimes A \leq C \otimes \mathbb{K}$.

Step 2. Then, find a matrix B such that: $\mathbb{K}' \otimes B \geq C \otimes \mathbb{K}'$.

Because of the assumptions on matrices $\mathbb{K}$ and $\mathbb{K}'$ the steps 1 and 2 always have solution but the existence and the computation of an optimal matrix B seems to be a more difficult problem which requires further attention.

4.2. $(\mathbb{K}, \mathbb{K}')$-monotonicity of matrix

In this subsection, the matrices $\mathbb{K}$ and $\mathbb{K}'$ are assumed to have the same dimensions, i. e. $d = d'$, have no null column vectors and are elements of $\mathcal{M}_{m,d}(\mathbb{S} \setminus \{\top\})$. In this case the hypotheses (H_0) and (H_1) of Lemma 1 are verified.

We remark that the system $(d, \otimes, A)$ is $(\mathbb{K}, \mathbb{K}')$-monotone, i. e. (4, with $d = d'$ and $A = B$) is true, is equivalent to:

$$\forall\, x, y \in \mathbb{S}^d, \ x \leq_{\mathbb{K},\mathbb{K}'} y \Rightarrow A \otimes x \leq_{\mathbb{K},\mathbb{K}'} A \otimes y. \tag{32}$$

Thus, we just have to specify Theorem 2 in the case where $A = B$. And we obtain the following criteria for the matrix A to be $(\mathbb{K}, \mathbb{K}')$-monotone.

Theorem 3. (NSC of $(\mathbb{K}, \mathbb{K}')$-monotonicity of matrix) Let $A \in \mathcal{M}_{d,d}(\mathbb{S})$ in a complete idempotent semiring $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$. The matrix A is $(\mathbb{K}, \mathbb{K}')$-monotone iff

$$\forall\, y \in \mathbb{S}^d : \ \mathbb{K} \otimes A \leq (\mathbb{K}' \otimes A \otimes y)/(\mathbb{K} \backslash \mathbb{K}' \otimes y). \tag{33}$$

In the case where $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a complete idempotent semifield, we obtain the following corollary.

Corollary 4. Let $A \in \mathcal{M}_{d,d}(\mathbb{S})$ in a complete idempotent semifield $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$. Then, the matrix A is $(\mathbb{K}, \mathbb{K}')$-monotone iff

$$\mathbb{K} \otimes A \leq (\mathbb{K}' \otimes A/\mathbb{K}') \otimes \mathbb{K}. \tag{34}$$

Remark 4. In the case where $\mathbb{K} = \mathbb{K}'$ the above-mentioned Corollary coincides with [30, Theorem 4.2] obtained by a different proof.

Remark 5. Rewriting (34) as follows: $A \leq \mathbb{K}\backslash((\mathbb{K}' \otimes A/\mathbb{K}') \otimes \mathbb{K})$ we see that the computation of a $(\mathbb{K}, \mathbb{K}')$-monotone matrix A is naturally formalized as a sub-fixed point computation problem of a Min-Max type function for which there exist computational algorithms which well behave in practice (see e. g. [11, 13, 17]).

4.3. Control of discrete event systems

In this subsection, we study the existence and the computation of a (static) linear state feedback such that the closed loop system $(d, \otimes, \Omega)$ defined by (6) enjoys the property (7): i. e. $\forall n \in \mathbb{T} \setminus \{0\}, \forall x \in \mathbb{S}^d$, $(x \in \mathcal{T}_{n-1} \Rightarrow \Omega \otimes x \in \mathcal{T}_n)$. This approach is directly inspired by [7] and also works on viability theory [5].

In Subsubsection 4.3.1 we study the case where $\mathcal{T}_n$ are idempotent polyhedral defined by (11). The main results of this part come directly from Haar's Lemma.

In Subsubsection 4.3.2 we study the case where the sets $\mathcal{T}_n$ are all equal to a given semimodule defined as the set of the solutions of a system of linear inequalities (see (8)). This formulation seems to be the natural way to define the constraints on the states of the system.

4.3.1. Reachability of a target tube

We study the property (7) to be satisfied by the closed loop system $(d, \otimes, \Omega)$ over a complete idempotent semiring $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ defined by (6) in the case where: $\forall n \in \mathbb{T}$, $\mathcal{T}_n = \mathcal{P}(P_n, p_n)$ with $\mathcal{P}(P_n, p_n)$ denoting the polyhedral set in the sense of definition (11).

Remark 6. Even if $\forall n \in \mathbb{T}$ $\mathcal{P}(P_n, p_n) = \mathcal{P}(I, P_n \backslash p_n)$ we prefer to keep the description of the target tube using the sets $\mathcal{P}(P_n, p_n)$. Our choice is motivated as follows. The matrix P_n could be interpreted as a matrix where each row is a reward function. Usually a reward function is defined as a function of the state system. The ith component of the vector p_n represents the bound of the ith reward function stored in the matrix P_n. For example if we are interested in the computation of the top Lyapunov exponent of a system which provides e. g. its cycle time, it is natural to take $P_n = (e, e, \ldots, e)^T$ and $p_n = (\gamma)$, where $\gamma \in \mathbb{S}$ is interpreted as the maximal top Lyapunov allowed for the system (which corresponds to the minimal cycle time allowed).

We assume that $\forall n \in \mathbb{T}$ the matrix P_n satisfies the hypothesis (H_0) of Lemma 1 and $\forall n \in \mathbb{T} \setminus \{0\}$ the vector p_n and the matrix P_{n-1} verify the hypothesis (H_1) of Lemma 1.

Proposition 1. The necessary and sufficient condition for the linear system $(d, \otimes, \Omega)$ over $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ to keep its entire state trajectory in the target tube $(\mathcal{P}(P_n, p_n))_{n\in\mathbb{T}}$ is:

(i) If $\mathbb{S}$ is a complete idempotent semiring: $\Omega \leq \bigwedge_{n\in\mathbb{T}\setminus\{0\}} P_n\backslash p_n/(P_{n-1}\backslash p_{n-1})$.

(ii) If $\mathbb{S}$ is a complete idempotent semifield: $\Omega \leq \bigwedge_{n\in\mathbb{T}\setminus\{0\}} P_n\backslash((p_n/p_{n-1})\otimes P_{n-1})$.

Proof. Let us prove (i). We just have to remark that assertion (7) with $\forall\, n \in \mathbb{T},\ \mathcal{T}_n = \mathcal{P}(P_n, p_n)$ is equivalent to:

$$\forall\, n \in \mathbb{T}\setminus\{0\},\ \mathcal{P}(P_{n-1}, p_{n-1}) \subseteq \mathcal{P}(P_n \otimes \Omega, p_n).$$

Thus, applying Haar's Lemma for idempotent semiring (i. e., Theorem 1 with $P := P_{n-1}$, $p := p_{n-1}$, $Q := P_n \otimes \Omega$ and $q := p_n$) we have:

$$\forall\, n \in \mathbb{T}\setminus\{0\},\ P_n \otimes \Omega \leq p_n/(P_{n-1}\backslash p_{n-1}),$$

which is equivalent to by definition of $\backslash$:

$$\forall\, n \in \mathbb{T}\setminus\{0\},\ \Omega \leq P_n\backslash p_n/(P_{n-1}\backslash p_{n-1}).$$

Thus, by definition of $\bigwedge$ the result (i) is now proved. The proof of (ii) is a direct consequence of (i) and Corollary 1. □

We now derive the main result concerning the existence and the computation of the optimal linear feedback control.

Theorem 4. Let $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ denote a complete idempotent semiring or semifield. There exists an optimal linear feedback in the sense of entry-wise comparison of matrices that forces the dynamical system $(d, \otimes, A \oplus B \otimes F)$ over $\mathbb{S}$ to lie in the region specified by the idempotent polyhedral sets $(\mathcal{T}_n = \mathcal{P}(P_n, p_n))_{n\in\mathbb{T}}$ iff:

$$\text{(E)} \quad A \leq \begin{cases} \bigwedge_{n\in\mathbb{T}\setminus\{0\}} P_n\backslash p_n/(P_{n-1}\backslash p_{n-1})) & \text{if } \mathbb{S} \text{ is a semiring,} \\ \bigwedge_{n\in\mathbb{T}\setminus\{0\}} P_n\backslash((p_n/p_{n-1})\otimes P_{n-1})) & \text{if } \mathbb{S} \text{ is a semifield.} \end{cases}$$

Under condition (E) the greatest linear feedback F^* is defined by:

$$F^* = \begin{cases} B\backslash\big(\bigwedge_{n\in\mathbb{T}\setminus\{0\}} P_n\backslash p_n/(P_{n-1}\backslash p_{n-1}\big) & \text{if } \mathbb{S} \text{ is a semiring,} \\ B\backslash\big(\bigwedge_{n\in\mathbb{T}\setminus\{0\}} P_n\backslash\big((p_n/p_{n-1})\otimes P_{n-1}\big) & \text{if } \mathbb{S} \text{ is a semifield.} \end{cases} \tag{35}$$

Proof. The proof is an immediate consequence of the fact that $\oplus = \vee$ and Proposition 1 with $\Omega := A \oplus B \otimes F$. □

4.3.2. Control based on monotonicity

Let us consider the closed loop system $(d, \otimes, \Omega)$ over a complete idempotent semiring $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ defined by (6). Let $G, G' \in \mathcal{M}_{m,d}(\mathbb{S} \setminus \{\top\})$ with no null column. Thus, the hypotheses (H_0) and (H_1) of Lemma 1 are satisfied.

Let us recall that the matrix Ω is (G, G')-monotone (see (32, with $\mathbb{K} := G$ and $\mathbb{K}' := G')$) if $\forall x, y \in \mathbb{S}^d$, $G \otimes x \leq_m G' \otimes y \Rightarrow G \otimes \Omega \otimes x \leq_m G' \otimes \Omega \otimes y$. By taking $y = x$ in the previous assertion we deduce the following obvious result which emphasizes the links between comparison and positive invariance.

Proposition 2. If $\mathcal{S}_{G,G'}$ denotes the semimodule defined by (8) then:

$$\Omega \text{ is } (G, G')\text{-monotone} \Rightarrow \mathcal{S}_{G,G'} \text{ is } \Omega\text{-invariant}.$$

From this proposition we deduce the main result of this part.

Theorem 5. If

(a) $G \otimes x(0) \leq_m G' \otimes x(0)$ and (b) the matrix Ω is (G, G')-monotone

then the dynamical system $(d, \otimes, \Omega)$ verifies:

$$\forall n \in \mathbb{N},\ G \otimes x(n) \leq_m G' \otimes x(n).$$

We then specialize the results of the previous theorem in the case where $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a complete idempotent semifield for control constraints which form is motivated by practical control problems (see e. g. [23]): $\forall n \geq 1, \mathbb{K} \otimes x(n) \leq_m \mathbb{K}' \otimes x(n-1)$, where $\mathbb{K}, \mathbb{K}' \in \mathcal{M}_{m,d}(\mathbb{S})$.

Corollary 5. Let us consider the closed loop system $(d, \otimes, \Omega := A \oplus B \otimes F)$ over a complete idempotent semifield $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ defined by (6). Under the assumption that the matrices $\mathbb{K} \otimes A$ and $\mathbb{K}'$ are elements of $\mathcal{M}_{m,d'}(\mathbb{S} \setminus \{\top\})$ and have no null column vector the following implication is true.

$$\left\{ \begin{array}{l} \Omega \leq \mathbb{K} \backslash (\mathbb{K}' \otimes x(0))/x(0) \\ \text{and} \\ \Omega \leq \mathbb{K} \backslash \big((\mathbb{K}' \otimes \Omega / \mathbb{K}') \otimes \mathbb{K} \big) \end{array} \right\} \Rightarrow \forall n \geq 1,\ \mathbb{K} \otimes x(n) \leq_m \mathbb{K}' \otimes x(n-1). \quad (36)$$

Proof. The proof follows from Theorem 5 with $G := \mathbb{K} \otimes \Omega$, $G' := \mathbb{K}'$, Corollary 4 and the definitions of / and \. □

5. ILLUSTRATIVE EXAMPLES

In this section we study two control problems which concern the computation of linear state feedback. In Subsection 5.1 we give a partial answer to the computational issue of an optimal linear state feedback when the target tube is infinite (see formula (35) of Theorem 4). In Subsection 5.2 we illustrate the relationship between comparison and control on a simple example of network transportation borrowed from the literature. The computation of a linear feedback is done by hand and is not yet implemented. It is expected that the implementation leads to better results (see Remark 7 for a more precise comment).

5.1. Example 1: infinite target tube

Assume that $(\mathbb{S}, \oplus, \otimes, \varepsilon, e)$ is a complete commutative idempotent semiring such that:

$$\exists! \lim_{n\to\infty} a^{\otimes n} = \begin{cases} \varepsilon & \text{if } a \leq e,\ a \neq e \\ e & \text{if } a = e \\ \top & \text{otherwise.} \end{cases} \tag{37}$$

Recalling that $a^{\otimes n} \overset{\text{def}}{=} a \otimes \ldots \otimes a$ (n-fold). Because $\otimes$ is commutative for all $a, b \in \mathbb{S}$: $a/b = a\backslash b \overset{\text{not.}}{=} \frac{a}{b}$.

Define the infinite target tube which corresponds to the case where $\mathbb{T} = \mathbb{N}$ as follows. For all $n \in \mathbb{T}$, $\mathcal{T}_n = \mathcal{P}(I, p_n)$ where I denotes the $d \times d$ identity matrix. In such case, because $I\backslash A = A/I = A$ for all matrix A:

$$\bigwedge_{n\in\mathbb{T}\backslash\{0\}} P_n\backslash p_n/(P_{n-1}\backslash p_{n-1}) = \bigwedge_{n\in\mathbb{T}\backslash\{0\}} p_n/p_{n-1}.$$

Now assume that for all $n \in \mathbb{T}$, $p_n = (\nu_1 \otimes \rho_1^{\otimes n}, \ldots, \nu_d \otimes \rho_d^{\otimes n})^T$ with $\forall\, i = 1, \ldots, d$ $\nu_i, \rho_i \neq \varepsilon$ and $(\cdot)^T$ denotes the transpose operator. And finally let us define for all $i, j = 1, \ldots, d$:

$$\theta_{i,j} \overset{\text{def}}{=} \begin{cases} \frac{\nu_i \otimes \rho_i}{\nu_j} & \text{if } e \leq \frac{\rho_i}{\rho_j} \\ \varepsilon & \text{otherwise.} \end{cases}$$

And $\Theta \overset{\text{def}}{=} [\theta_{i,j}]$.

Proposition 3. The linear dynamical system $(d, \otimes, A)$ over $\mathbb{S}$ has its entire trajectory in the above defined infinite target tube if:
(T) $A \leq \Theta.$

Under condition (T) there exists an optimal linear state feedback control in the sense of entry-wise comparison of matrices defined by:

$$F^* = B\backslash\Theta$$

that forces the linear dynamical system $(d, \otimes, A \oplus B \otimes F)$ to stay in the above defined region $(\mathcal{T}_n)_{n\in\mathbb{T}}$. Moreover, if $\mathbb{S}$ is a semifield the condition (T) is also necessary.

Proof. The proof is based on the formulas [6, (f.9) and (f.12), p. 183] which are respectively rewritten as follows taking into account that $\otimes$ is commutative:

$$\text{(g.1)} \quad \frac{a}{b \otimes c} = \frac{\frac{a}{c}}{b} = \frac{\frac{a}{b}}{c}, \qquad \text{(g.2)} \quad a \otimes \frac{b}{c} \leq \frac{a \otimes b}{c} = \frac{b \otimes a}{c}.$$

For all $n \in \mathbb{T} \setminus \{0\}$ and $i, j = 1, \ldots, d$:

$$
\begin{array}{rll}
(p_n/p_{n-1})_{i,j} & = \frac{\nu_i \otimes \rho_i^{\otimes n}}{\nu_j \otimes \rho_j^{\otimes (n-1)}} & \\
& = \frac{\frac{(\nu_i \otimes \rho_i) \otimes \rho_i^{\otimes (n-1)}}{\rho_j^{\otimes (n-1)}}}{\nu_j} & \text{by (g.1) and because } \otimes \text{ is associative} \\
& \overset{(a)}{\geq} \frac{(\nu_i \otimes \rho_i) \otimes \frac{\rho_i^{\otimes (n-1)}}{\rho_j^{\otimes (n-1)}}}{\nu_j} & \text{by (g.2) with } a := \nu_i \otimes \rho_i,\ b := \rho_i^{\otimes (n-1)} \\
& & \text{and } c := \rho_j^{\otimes (n-1)} \\
& \overset{(b)}{\geq} \frac{\nu_i \otimes \rho_i}{\nu_j} \otimes \frac{\rho_i^{\otimes (n-1)}}{\rho_j^{\otimes (n-1)}} & \text{by (g.2) with } a := \nu_i \otimes \rho_i,\ b := \frac{\rho_i^{\otimes (n-1)}}{\rho_j^{\otimes (n-1)}} \\
& & \text{and } c := \nu_j \\
& \overset{(c)}{\geq} \frac{\nu_i \otimes \rho_i}{\nu_j} \otimes \left(\frac{\rho_i}{\rho_j}\right)^{\otimes (n-1)} & \text{by (g.2) and recurrence on } n.
\end{array}
$$

Let us note that in the case of a semifield the inequality (g.2) becomes an equality. Thus, the above inequalities (a), (b) and (c) become equalities. Based on assumption (37) and above result one easily deduces that for all $i, j = 1, \ldots, d$:

$$
\bigwedge_{n \in \mathbb{T} \setminus \{0\}} (p_n/p_{n-1})_{i,j} \begin{cases} \geq & \theta_{i,j} \quad \text{if } \mathbb{S} \text{ semiring} \\ = & \theta_{i,j} \quad \text{if } \mathbb{S} \text{ semifield.} \end{cases}
$$

Now, the result is an immediate consequence of Theorem 4. □

5.2. Example 2: Transportation networks with time table

In this subsection we borrow the example [23, Section 6] which is a simple example of transportation network studied in e. g. [9, 15]. And we study the problem of the existence and the computation of a linear (static) feedback control. To do this we consider the linear dynamical system $(4, \otimes, \Omega := A \oplus F)$, i. e. $x(0) \in \mathbb{S}^4$, $x(n) = \Omega \otimes x(n-1), \forall\, n$, over the semifield $(\mathbb{S}, \oplus, \otimes, \varepsilon, e) := (\mathbb{Z} \cup \{-\infty, +\infty\}, \max, +, -\infty, 0)$ where:

$$
A = \begin{pmatrix} \varepsilon & 17 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 11 & 9 \\ 14 & \varepsilon & 11 & 9 \\ 14 & \varepsilon & 11 & \varepsilon \end{pmatrix}, \quad \text{and} \quad F \quad \text{is an unknown matrix to be computed.}
$$

The constraints we impose on the system $(d, \otimes, A)$ are as follows and expressed with usual operations as:

$$
\text{(c1)} \quad \forall\, i,\ x_i(n) - x_i(n-1) \leq l_i, \qquad \text{(c2)} \quad \forall\, i,\ j,\ x_j(n) + s_{i,j} - x_i(n-1) \leq m_{i,j},
$$

where l_i denotes the upper bound time between two consecutive train departures in direction i, $m_{i,j}$ denotes the upper bound on the time of passengers coming from direction i have to wait for the departure of the train which leaves in direction j.

And $s_{i,j} = a_{j,i}^{\otimes -1}$ if $a_{j,i} \neq \varepsilon$ and ε, otherwise. As in [23, Section 6] we assume that: $l_i = l \neq \varepsilon$ and $m_{i,j} = m \neq \varepsilon$. The constraints (c1) and (c2) can be rewritten as:

$$\mathbb{K} \otimes x(n) \leq_4 I \otimes x(n-1),$$

with $\mathbb{K} = m^{\otimes -1} \otimes S \oplus l^{\otimes -1} \otimes I$. Note that $\mathbb{K} \otimes A, I \in \mathcal{M}_{4,4}(\mathbb{S} \setminus \{\top\})$ and hence the hypotheses (H_0) and (H_1) of Lemma 1 are verified. Thus, the computation of a linear state feedback control can be done by applying result of (36), Theorem 5. Thus, the aim is now to construct a matrix Ω which is $(\omega.1)$: $(\mathbb{K}, I)$-monotone, i.e. $\mathbb{K} \otimes \Omega \leq \Omega \otimes \mathbb{K}$ (see (34 with $A := \Omega$, $\mathbb{K}' := I$)), and satisfies $(\omega.2)$: $A \leq \Omega \leq \mathbb{K} \backslash x(0)/x(0)$. For the numerical application we take $x(0) = (e, e, e, e)^T$. Then, $x(0)/x(0) = E = [e_{i,j} = e]$. Let us remark that $(\omega.2)$ is true iff $A \leq \mathbb{K} \backslash x(0)/x(0)$. And $A \leq \mathbb{K} \backslash x(0)/x(0) \Leftrightarrow m \geq 6,\ l \geq 17$. To simplify the calculus take: $m = 8$, $l = 17$. Then,

$$\mathbb{K} = \begin{pmatrix} 17^{\otimes -1} & \varepsilon & 22^{\otimes -1} & 22^{\otimes -1} \\ 25^{\otimes -1} & 17^{\otimes -1} & \varepsilon & \varepsilon \\ \varepsilon & 19^{\otimes -1} & 17^{\otimes -1} & 19^{\otimes -1} \\ \varepsilon & 17^{\otimes -1} & 17^{\otimes -1} & 17^{\otimes -1} \end{pmatrix}, \qquad \mathbb{K} \backslash x(0)/x(0) = 17 \otimes E.$$

Noticing that for all matrix Ω such that $\Omega \leq \mathbb{K} \backslash x(0)/x(0)$: $\mathbb{K} \otimes \Omega \leq x(0)/x(0) = E$, our problem is now to construct a matrix Ω such that $E \leq \Omega \otimes \mathbb{K}$ and $A \leq \Omega \leq 17 \otimes E$. Take

$$\Omega = \begin{pmatrix} 17 & 17 & t & 17 \\ 17 & u & v & 17 \\ 17 & w & x & 17 \\ 17 & y & z & 17 \end{pmatrix}, \quad \text{with} \;\; t, u, w, y \in [\varepsilon, 17] \;\; \text{and} \;\; v, x, z \in [11, 17].$$

Taking $t = u = w = y = \varepsilon$ and $v = x = z = 11$ a possible choice for linear feedback such that

$$A \oplus F = \Omega \;\; \text{is} \;\; F = \begin{pmatrix} 17 & \varepsilon & \varepsilon & 17 \\ 17 & \varepsilon & \varepsilon & 17 \\ 17 & \varepsilon & \varepsilon & 17 \\ 17 & \varepsilon & \varepsilon & 17 \end{pmatrix}.$$

The controlled series $(x(n))_{n \in \mathbb{N}}$ by linear (static) feedback of matrix F is defined by: $x(n) = 17^{\otimes n} \otimes x(0)$. Therefore, the timetable $(u(n) = F \otimes x(n-1))_{n \geq 1}$ is defined by: $u(n) = 17^{\otimes n} \otimes x(0)$. Noticing that $\mathbb{K} \otimes x(0) = 17^{\otimes -1} \otimes x(0)$ it is easy to see that the controlled series $x(n)$ verifies $\mathbb{K} \otimes x(n) = x(n-1), n \geq 1$, hence the constraints (c1) and (c2) are trivially satisfied. Let us note that although the matrix A is not $(\mathbb{K}, I)$-monotone (e.g. $(\mathbb{K} \otimes A)_{1,3} = 11^{\otimes -1} \not\leq (A \otimes \mathbb{K})_{1,3} = \varepsilon$) the series $x(n) = A^{\otimes n} \otimes x(0)$, $n \geq 0$:

$$x(0) = \begin{pmatrix} e \\ e \\ e \\ e \end{pmatrix}, \quad x(1) = \begin{pmatrix} 17 \\ 11 \\ 14 \\ 14 \end{pmatrix}, \quad x(2) = \begin{pmatrix} 28 \\ 25 \\ 31 \\ 31 \end{pmatrix}, \quad x(3) = 42 \otimes x(0), \; \ldots$$

satisfies the constraints (c1) and (c2). This fact confirms that the (G, G')-monotonicity of Ω is not a necessary condition for the Ω-invariance of the set $\mathcal{S}_{G,G'}$.

Remark 7. The research of a solution for this numerical example has been made by hand. However, we mention that the problem studied here can be formulated as: find x, y such that $C \otimes x = D \otimes y$, $y \oplus \nu = \nu$ (ν is a given vector) for which a pseudopolynomial algorithm exists (see e. g. [14]). Finally, let us mention a polynomial algorithm in the case where $\mathbb{S}$ is the set of rational numbers [10].

Remark 8. Following [23] the constraints (c1) and (c2) are rewritten as: $Q \otimes \overline{x}(n) \leq_8 \overline{x}(n)$, with $\overline{x}(n) = (x^T(n), x^T(n-1))^T$, $Q = \begin{pmatrix} \mathcal{E} & \mathcal{E} \\ \mathbb{K} & \mathcal{E} \end{pmatrix}$ and $\mathcal{E}$ denotes the matrix which entries are all ε. Noticing that $Q \otimes \overline{x}(n) \leq_8 \overline{x}(n) \Leftrightarrow \overline{x}(n) \in \mathrm{Im}(Q^*)$ then, the control problem is as follows: find a matrix F such that $\forall \overline{x} \in \mathbb{S}^8$, $(\overline{x} \in \mathrm{Im}(Q^*) \Rightarrow M \otimes \overline{x} \in \mathrm{Im}(Q^*))$ with $M := \begin{pmatrix} A \oplus F & \mathcal{E} \\ I & \mathcal{E} \end{pmatrix}$. This condition is equivalent to: $\exists\, H \;\; M \otimes Q^* = Q^* \otimes H$ which form coincides with the one of [23, (10) p. 17]. Thus, our approach based on monotonicity could be considered as a dual approach to the geometric approach developed in [23].

Remark 9. As a consequence of Remarks 7 and 8 and also [23, Remark 2] the computation complexity of a linear state feedback control seems to be roughly the same for both approaches, i. e. ours and the geometric one.

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