KYBERNETIKA — VOLUME *43* (2007), NUMBER 3, PAGES 343 – 346

AN ATOMIC MV–EFFECT ALGEBRA WITH NON–ATOMIC CENTER

Vladimír Olejček

Does there exist an atomic lattice effect algebra with non-atomic subalgebra of sharp elements? An affirmative answer to this question (and slightly more) is given: An example of an atomic MV-effect algebra with a non-atomic Boolean subalgebra of sharp or central elements is presented.

Keywords: lattice effect algebra, MV-effect algebra, Archimedean effect algebra, sharp element, central element, atom

AMS Subject Classification: 06D35, 06C15, 03G12, 81P10

1. INTRODUCTION

A set E equipped with a partial, commutative and associative operation $\oplus$, containing elements 0 and 1, in which the existence of a unique inverse element x' to any $x \in E$ is guaranteed, and $a \oplus 1$ is admitted only if $a = 0$ is well known as *effect algebra.* Effect algebras were introduced by Foulis and Bennet [3] and simultaneously by Kôpka and Chovanec [7] as D-posets. An effect algebra equipped with partial order $\leqslant$ can form a lattice called a *lattice effect algebra.* This structure generalizes both orthomodular lattices, i. e. the effect algebras in which $x \oplus x$ is not defined for any non-zero $x \in E$, and MV-effect algebras, i. e. effect algebras with all pairs of elements being compatible [6], and it is applied as a carrier of probability of unsharp or fuzzy events.

In connection with existence of states on lattice effect algebras properties of the subalgebra of sharp elements and the subalgebra of central elements are studied. The following definitions are consistent with the ones in [2].

Definition 1. An element $x \in E$ is called a *sharp* element, if $x \wedge x' = 0$. An element $x \in E$ is called a *central* element if for every $y \in E$, $y = (x \wedge y) \vee (x' \wedge y)$.

Let us denote $\mathcal{S}(E)$ and $\mathcal{C}(E)$ the sets of all sharp and central elements, respectively. It is known that $\mathcal{S}(E)$ is an orthomodular lattice [5] and $\mathcal{C}(E)$, called the *center* of E, is a Boolean algebra [4].

Definition 2. An effect algebra is called *Archimedian effect algebra* if for every $x \in E$ there is a positive integer n_x such that

$$n_x x = \overbrace{x \oplus x \oplus x \oplus \ldots \oplus x}^{n_x\text{-times}}$$

is defined and $(n_x + 1)x$ is not defined. The integer n_x is called the *isotropic index* of x.

Definition 3. An element $x \in E$ is called an *atom* if for every $y \in E$ such that $y \leq x$ we have either $y = 0$ or $y = x$. An effect algebra E is called *atomic* if for every non-zero element $x \in E$ there is an atom $a \in E$ such that $a \leq x$. An effect algebra E is called *non-atomic* if there is no atom in E.

Z. Riečanová studied atomicity of lattice effect algebras regarding the atomicity of $\mathcal{S}(E)$ and $\mathcal{C}(E)$. For example in [9] she proved that if E is an atomic Archimedean lattice effect algebra such that $\mathcal{S}(E) = \mathcal{C}(E)$ then $\mathcal{S}(E)$ is atomic and every block of $\mathcal{S}(E)$ is atomic. Another family of Archimedean atomic lattice effect algebras with atomic $\mathcal{S}(E)$ are modular atomic effect algebras [8].

In [10] Z. Riečanová formulated the following open problem: Does there exist an atomic lattice effect algebra with non-atomic subalgebra $\mathcal{S}(E)$ or with non-atomic $\mathcal{C}(E)$? We have found an affirmative answer to this question. In particular, we have found an example of atomic (non-Archimedean) MV-effect algebra with non-atomic subalgebra of $\mathcal{S}(E) = \mathcal{C}(E)$ of central (or sharp) elements.

2. THE MAIN RESULT

Example. Denote

$$T = \{\mathbf{0},\, a,\, 2a,\, \ldots,\, \ldots,\, \mathbf{1} - 2a,\, \mathbf{1} - a,\, \mathbf{1}\}$$

the MV-effect algebra introduced in [1], and $\mathbb{N} = \{1, 2, \ldots\}$, the set of strictly positive integers. Let $\mathcal{B}$ be the Boolean algebra of finite unions of disjoint half-open intervals in the real interval $S = [0, 1)$, i. e. an element of $\mathcal{B}$ is of the form

$$D = \bigcup_{i=1}^{n} [a_i, b_i)$$

for some positive integer n and some finite subset of elements a_i, b_i in S, satisfying the condition $0 \leq a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n \leq 1$ or $D = \emptyset$. The set

$$M = \{(x_1, y_1),\, (x_2, y_2),\, \ldots,\, (x_k, y_k)\}$$

is defined as any finite (including the empty) subset of $S \times \mathbb{N}$ with $x_i \neq x_j$ for $i \neq j$ and $D \in \mathcal{B}$. Denote M_1 the set of first coordinates of M. Consider E to be the set of all functions $f \in T^S$ defined by the pair of sets D and M, i. e. $f = f_{D,M}$ such that

$$f_{D,M}(x) = \begin{cases} \chi_D(x) & \text{if} \quad x \notin M_1, \\ y_i a & \text{if} \quad x = x_i \in M_1 \cap D^c, \\ \mathbf{1} - y_i a & \text{if} \quad x = x_i \in M_1 \cap D. \end{cases}$$

The operations $\oplus, \vee, \wedge$ are defined as the restricted ones from "component-wise" operations on T^S. For any $f \in E$ let $D(f)$, $M(f)$ be the pair of sets defining f, i. e. $f = f_{D(f),M(f)}$.

To verify the closeness of E with respect to the effect algebra operation $\oplus$ and with respect to the lattice operation $\wedge$ it is enough to determine the sets $D(f \oplus g)$, $M(f \oplus g)_1$, and $D(f \wedge g)$, $M(f \wedge g)_1$, respectively. The analogous properties of $D(f \vee g)$ and $M(f \vee g)_1$ will follow from the duality principle:

$$f \vee g = \mathbf{1} - (\mathbf{1} - f) \wedge (\mathbf{1} - g)$$

Directly from the definition of $f_{D,M}(x)$ we have

$$M(f \oplus g)_1 = \{x \in M(f)_1 \cup M(g)_1 : f(x) \oplus g(x) < \mathbf{1}\}$$

and $D(f \oplus g) = D(f) \cup D(g)$ (if $D(f) \cap D(g) = \emptyset$, otherwise $f \oplus g$ is not defined). For the lattice intersection $\wedge$ we have

$$M(f \wedge g)_1 = \{x \in M(f)_1 \cup M(g)_1 : f(x) \wedge g(x) > \mathbf{0}\}$$

and $D(f \wedge g) = D(f) \cap D(g)$.

It is easy to see that $\mathbf{0} = f_{\emptyset,\emptyset} \in E$, $\mathbf{1} = f_{S,\emptyset} \in E$. If $f = f_{D,M} \in E$ then $f' = \mathbf{1} - f = f_{D^c,M} \in E$.

Consequently, E is a sub-effect algebra and a sub-lattice of T^S.

It also follows that E is closed with respect to the operation $\ominus$, defined by

$$f \ominus g = h \text{ iff } \; g \oplus h = f$$

and since

$$f \vee g = f \oplus (g \ominus (f \wedge g))$$

on T^S, it remains to be true also on E. Thus E is an MV-effect algebra.

Finally,

$$f \wedge f' = f_{D,M} \wedge f_{D^c,M} = \mathbf{0} \text{ if and only if } \; M = \emptyset$$

whence $\mathcal{S}(E)$ is non-atomic. On the other hand $f_{D,M} \in E$ is an atom if and only if $D = \emptyset$ and $M = \{(x,1)\}$ for some $x \in S$. Therefore E is atomic.

Note 1. The cardinality of $\mathcal{S}(E)$ is $2^{\aleph_0}$. It can be reduced, for instance, by restriction of endpoints of intervals determining sets $D \subseteq S$ to the rational numbers with dyadic denominator. The cardinality of $\mathcal{S}(E)$ of the modified E is then $\aleph_0$.

Note 2. The lattice effect algebra E in the Example is evidently non-Archimedian. Therefore the problem formulated by Z. Riečanová remains open for Archimedian lattice effect algebra: Is there an atomic Archimedean lattice effect algebra with non-atomic set $\mathcal{S}(E)$ of sharp elements?

ACKNOWLEDGEMENT

The research is supported by the grant G-1/3025/06 of the Ministry of Education of the Slovak Republic.

(Received October 3, 2006.)

REFERENCES

[1] C. C. Chang: Algebraic analysis of many-valued logics. Trans. Amer. Math. Math. Soc. *88* (1958), 467–490.
[2] A. Dvurečenskij and S. Pulmannová: New Trends in Quantum Structures Theory. Kluwer Academic Publications, Dordrecht 2000.
[3] D. J. Foulis and M. K. Bennett: Effect algebras and unsharp quantum logics. Found. Phys. *24* (1994), 1325–1346.
[4] R. J. Greechie, D. Foulis, and S. Pulmannová: The center of an effect algebra. Order *12* (1995), 91–106.
[5] G. Jenča and Z. Riečanová: On sharp elements in lattice effect algebras. BUSEFAL *80* (1999), 24–29.
[6] F. Kôpka: Compatibility in D-posets. Internat. J. Theor. Phys. *34* (1995), 1525–1531.
[7] F. Kôpka and F. Chovanec: D-posets. Math. Slovaca *44* (1994), 21–34.
[8] Z. Riečanová: Continuous lattice effect algebras admitting order-continuous states. Fuzzy Sets and Systems *136* (2003), 41–54.
[9] Z. Riečanová: Archimedean atomic lattice effect algebras in which all sharp elements are central. Kybernetika *42* (2006), 2, 143–150.
[10] Z. Riečanová, I. Marinová, and M. Zajac: Some aspects of generalized prelattice effect algebras. In: Theory and Application of Relational Structures as Knowledge Instruments II (H. de Swart et al., eds., Lecture Notes in Artificial Intelligence 4342), Springer–Verlag, Berlin 2006, pp. 290–317.

Vladimír Olejček, Department of Mathematics, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, Ilkovičova 3, 812 19 Bratislava. Slovak Republic.
e-mail: vladimir.olejcek@stuba.sk