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TEST FOR EXPONENTIALITY AGAINST WEIBULL AND GAMMA DECREASING HAZARD RATE ALTERNATIVES

Simos G. Meintanis

A sub-exponential Weibull random variable may be expressed as a quotient of a unit exponential to an independent strictly positive stable random variable. Based on this property, we propose a test for exponentiality which is consistent against Weibull and Gamma distributions with shape parameter less than unity. A comparison with other procedures is also included.

Keywords: goodness-of-fit test, empirical Laplace transform, likelihood test

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1. INTRODUCTION

Let E and S be independent non-negative random variables, the first following an exponential distribution, while the second following a strictly positive stable distribution with shape parameter γ ($\gamma < 1$), both with unit scale. Also let X be an arbitrary non-negative random variable, and recall that X is said to follow the Weibull distribution if it has density

$$f_X(x) = \frac{\vartheta x^{\vartheta-1}}{c^\vartheta} \mathrm{e}^{-(x/c)^\vartheta},\ x > 0, \tag{1}$$

where $\vartheta > 0$ (resp. $c > 0$) denotes a shape parameter (resp. scale parameter). With the aid of the Laplace transform (LT) of S,

$$\Lambda(t) = \mathrm{E}(\mathrm{e}^{-tS}) = \exp(-t^\gamma), \tag{2}$$

we can prove the following lemma:

Lemma 1.1. Assume that W follows a Weibull distribution with shape parameter $\vartheta < 1$, and unit scale. Then W admits the representation

$$W = \frac{E}{S},$$

with S following a strictly positive stable law with $\gamma = \vartheta$.

Proof. Let $F_W(\cdot)$ denote the distribution function of W, and $f_S(s)$ denote the density of S. Then by conditioning on S we have

$$F_W(w) = \mathrm{P}(W \le w) = \int_0^\infty \mathrm{P}(W \le w|s) f_S(s)\,\mathrm{d}s = \int_0^\infty \mathrm{P}(E \le ws) f_S(s)\,\mathrm{d}s \tag{3}$$
$$= \int_0^\infty (1 - \mathrm{e}^{-ws}) f_S(s)\,\mathrm{d}s.$$

The density $f_W(\cdot)$ of W results by differentiating with respect to $w > 0$ in the last integral in (3). Then,

$$f_W(w) = \int_0^\infty s\mathrm{e}^{-ws} f_S(s)\,\mathrm{d}s.$$

Hence $f_W(w) = \mathrm{E}(S\mathrm{e}^{-wS})$, which is equal to $-\Lambda'(w)$, with $\Lambda(\cdot)$ given by (2) with $\gamma = \vartheta$. Consequently, $f_W(w) = \vartheta w^{\vartheta-1}\exp(-w^\vartheta)$, and the proof of the lemma is complete by comparing f_W with the density figuring in (1). □

Let f_E and f_V denote the densities of E and $V = 1/S$, respectively. The following lemma provides the basis for our test.

Lemma 1.2. Let $l(t) = \mathrm{E}(\mathrm{e}^{-tW})$ denote the LT of W, where W is as in Lemma 1.1. Then

$$l(t) \ge \frac{1}{t\,\Gamma(1 + \vartheta^{-1}) + 1}, \tag{4}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Proof. Use Lemma 1.1, to get

$$\begin{aligned} l(t) &= \int_0^\infty \int_0^\infty \mathrm{e}^{-t\varepsilon v} f_E(\varepsilon) f_V(v)\,\mathrm{d}\varepsilon\mathrm{d}v \\ &= \int_0^\infty \left(\int_0^\infty \mathrm{e}^{-t\varepsilon v}\mathrm{e}^{-\varepsilon}\,\mathrm{d}\varepsilon\right) f_V(v)\,\mathrm{d}v \\ &= \int_0^\infty \frac{1}{tv+1} f_V(v)\,\mathrm{d}v = \mathrm{E}\left(\frac{1}{tV+1}\right). \end{aligned}$$

Since $g(V) = 1/(tV + 1)$ is convex in $V > 0$, for each $t > 0$, we have from Jensen's inequality that $l(t) = \mathrm{E}[g(V)] \ge g[\mathrm{E}(V)]$, and the result follows since the first moment of negative order of a strictly positive stable law is equal to $\Gamma(1+\vartheta^{-1})$ (see for instance Meintanis [2]). □

Notice that equality is obtained in (4) by taking the limit as $\vartheta \to 1$ from below. Consequently, on the basis of independent copies X_j, $j = 1, 2, \ldots, n$, of X, a test of exponentiality against decreasing hazard rate Weibull alternatives may rely on

$$D_n(t; \hat\vartheta_n) = \left(\hat\vartheta_n + t\Gamma(1/\hat\vartheta_n)\right) l_n(t) - \hat\vartheta_n, \tag{5}$$

where

$$l_n(t) = \frac{1}{n}\sum_{j=1}^{n} \mathrm{e}^{-tY_j},$$

is the empirical LT of the standardized observations $Y_j = X_j/\hat{c}_n,\ j = 1, 2, \ldots, n$, and $\hat{\vartheta}_n$ (resp. $\hat{c}_n$) denotes a consistent estimator of shape parameter ϑ (resp. the scale parameter c). Especially we propose to reject the null hypothesis of exponentiality for large values of the LT of $\sqrt{n}D_n(t; \hat{\vartheta}_n)$

$$T_{n,\lambda} = \sqrt{n}\int_0^\infty D_n(t; \hat{\vartheta}_n)\mathrm{e}^{-\lambda t}\,\mathrm{d}t,$$

where $\lambda > 0$. Straightforward algebra yields,

$$T_{n,\lambda} = \hat{\vartheta}_n \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{\lambda + Y_j} + \Gamma\left(\frac{1}{\hat{\vartheta}_n}\right)\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\frac{1}{(\lambda + Y_j)^2} - \sqrt{n}\,\frac{\hat{\vartheta}_n}{\lambda}.$$

From the Glivenko–Cantelli lemma for the empirical LT, it follows that the test based on $T_{n,\lambda}$ will be consistent, for the null hypothesis

$$H_0 : \text{the law of } X \text{ is } W(c, \vartheta) \text{ with } \vartheta = 1 \text{ and } c > 0,$$

against the alternative

$$H_{1W} : \text{the law of } X \text{ is } W(c, \vartheta) \text{ with } \vartheta < 1 \text{ and } c > 0,$$

where $W(c, \vartheta)$ denotes a Weibull distribution with scale parameter c, and shape parameter ϑ. Notice that $\vartheta = 1$ in H_0 corresponds to the exponential distribution; see also (1). We will indicate, that at least under moment estimation, this test is also consistent against the hypothesis

$$H_{1\Gamma} : \text{ the law of } X \text{ is } \Gamma(c, \vartheta) \text{ with } \vartheta < 1, \text{ and } c > 0,$$

where $\Gamma(c, \vartheta)$ denotes the Gamma distribution with scale parameter c and shape parameter ϑ. To see this we will use the following lemma, but first recall that the moment estimator (MOE) $(\hat{c}_n, \hat{\vartheta}_n)$ of (c, ϑ) satisfies, the equations

$$g(\hat{\vartheta}_n) = \frac{\overline{X_n^2}}{(\overline{X}_n)^2}, \qquad \hat{c}_n = \frac{\overline{X}_n}{\Gamma(1 + \hat{\vartheta}_n^{-1})}, \tag{6}$$

where $\overline{X}_n = n^{-1}\sum_{j=1}^{n} X_j$, $\overline{X_n^2} = n^{-1}\sum_{j=1}^{n} X_j^2$, and $g(u) = 2u\Gamma(2u^{-1})/(\Gamma(u^{-1}))^2$.

Lemma 1.3. Assume that the law of X is $\Gamma(c, \vartheta)$, with $\vartheta < 1$ and $c > 0$. Then $\hat{\vartheta}_n \xrightarrow{\mathrm{P}} \tilde{\vartheta} < 1$, and $D_n(t; \hat{\vartheta}_n) \xrightarrow{\mathrm{P}} D(t; \vartheta, \tilde{\vartheta})$, where

$$D(t; \vartheta, \tilde{\vartheta}) = \frac{\tilde{\vartheta} + t\Gamma(1/\tilde{\vartheta})}{(1 + \rho t)^{\vartheta}} - \tilde{\vartheta}, \tag{7}$$

and

$$\rho = \vartheta^{-1}\Gamma(1 + \tilde{\vartheta}^{-1}) > 1. \tag{8}$$

P r o o f. To find the stochastic limit of $\hat{\vartheta}_n$, notice that the stochastic limit of the first moment equation in (6) is $g(u) = a$ where $a = 1 + (1/\vartheta) > 2$. The function g is continuous in (0,1) and satisfies

(i) $\lim_{u\to 0^+} g(u) = \infty$ and $\lim_{u\to 1^-} g(u) = 2$,

(ii) $g'(u) < 0$.

Therefore, a solution of $g(u) = a$, in $(0, 1)$ exists. Moreover, since $g(u) < 2$ for $u > 1$, it follows that this solution, say $u = \tilde{\vartheta}$, is unique. Hence

$$\hat{\vartheta}_n \xrightarrow{\mathrm{P}} \tilde{\vartheta} < 1. \tag{9}$$

In turn, by recalling that $\mathrm{E}(X) = c\vartheta$, one has from the second moment equation in (6),

$$\hat{c}_n = \frac{\overline{X}_n}{\Gamma(1+\hat{\vartheta}_n^{-1})} \xrightarrow{\mathrm{P}} \tilde{c} = \frac{\mathrm{E}(X)}{\Gamma(1+\tilde{\vartheta}^{-1})} = \frac{c\vartheta}{\Gamma(1+\tilde{\vartheta}^{-1})} < c. \tag{10}$$

On the other hand, by the uniform consistency of the empirical LT we have that,

$$l_n(t) \xrightarrow{\mathrm{P}} L(t/\tilde{c}), \tag{11}$$

where $L(t) = (1+ct)^{-\vartheta}$, denotes the LT of $\Gamma(c,\vartheta)$. Hence by using (10) we obtain $\rho = c/\tilde{c}$ in the form given in (8). Finally, by taking the stochastic limit in (5) and inserting (9) and (11) one arrives at (7).

Naturally $\tilde{\vartheta}$ is a function, say $\tilde{\vartheta} = \phi(\vartheta)$, of ϑ. In fact it follows from (i) above that ϕ satisfies

$$\lim_{\vartheta\to 1^-} \phi(\vartheta) = 1, \qquad \lim_{\vartheta\to 0^+} \phi(\vartheta) = 0. \tag{12}$$

Hence let us write $D(t;\vartheta)$ instead of $D(t;\vartheta,\tilde{\vartheta})$. Although we were not able to prove that $D(t;\vartheta) > 0$, if $X \sim \Gamma(c,\vartheta)$, for all $\vartheta < 1$, by using (12) in (7), it follows that

$$\lim_{\vartheta\to 1^-} D(t;\vartheta) = 0, \qquad \lim_{\vartheta\to 0^+} D(t;\vartheta) = \infty.$$

Moreover both numerical and simulation results support that $D(t;\vartheta)$ is monotonic in ϑ, and therefore we are inclined to believe that the test that rejects H_0 for large values of $T_{n,\lambda}$ is consistent against Gamma decreasing hazard rate alternatives. □

2. OTHER TESTING PROCEDURES AND COMPARISON

Typically, testing the null hypothesis H_0 may be based on the likelihood principle. In fact, exact likelihood procedures concerning the scale parameter of a Gamma or a Weibull distribution have been proposed by Stehlík [4, 5], including scale homogeneity tests for several populations with common shape parameter (see also Rublík [3]). Corresponding exact results concerning the shape parameter are not known to the author. Therefore an asymptotic likelihood procedure is employed. In particular let $\mathcal{L}_W(c,\vartheta)$ denote the log-likelihood under the Weibull model, and $\mathcal{L}_E(c)$ denote the log-likelihood under the exponential model. Then the likelihood ratio test rejects H_0 in favor of H_{1W} if $L_n = l_n/n$ is 'large', where

$$l_n = \mathcal{L}_W(\tilde{c}_n, \tilde{\vartheta}_n) - \mathcal{L}_E(\overline{X}_n),$$

with $(\tilde{c}_n, \tilde{\vartheta}_n)$ the maximum likelihood estimator (MLE) of the parameter (c, ϑ) of the Weibull distribution. Straightforward calculation shows that the profile version of L_n (where $\tilde{c}_n$ has been replaced by $n^{-1}\sum_{j=1}^n X_j^{\tilde{\vartheta}_n}$) is given by

$$L_n = \log \tilde{\vartheta}_n + (\tilde{\vartheta}_n - 1)\,\overline{\log X} + \log \overline{X}_n - \log\left(\overline{X_{\tilde{\vartheta}_n}}\right), \tag{13}$$

where $\overline{\log X} = n^{-1}\sum_{j=1}^n \log X_j$, and $\overline{X_\vartheta} = n^{-1}\sum_{j=1}^n X_j^\vartheta$.

Notice that from (6) and (13), both the MOE and MLE may be computed with fairly mild computational effort. In fact one has to solve the first equation in (6) and obtain $\hat{\vartheta}_n$, and then compute the MOE $\hat{c}_n$ from the second equation in (6) in a straightforward manner. Likewise maximization of L_n in (13) yields the MLE $\tilde{\vartheta}_n$, and the corresponding estimator of c is computed as $\tilde{c}_n = \overline{X_{\tilde{\vartheta}_n}}$. Solving the first moment equation in (6) or maximizing L_n (note that $\log \overline{X}_n$ may be dropped from (13)) may be accomplished by a simple search procedure or by Newton–Raphson. In our case we used the former procedure, and encountered no problem in locating the solution to accuracy 10^{-4}.

Wong and Wong [6] proposed an alternative procedure to test H_0 that employs the ratio $R_n = X_{(n)}/X_{(1)}$, of the extreme observations, where $X_{(j)}$, $1 \leq j \leq n$, denotes the ordered X_j-value. More recently Chen [1] devised a test with utilizes

$$\xi_n(k) = \frac{n^{-1}\left[\sum_{j=1}^{k-1} X_{(j)} + (n-k+1)X_{(k)}\right]}{\left[\Pi_{j=1}^{k-1} X_{(j)} X_{(k)}^{n-k+1}\right]^{1/n}}.$$

In both cases H_0 is rejected in favor of H_{1W} for 'large' values of the corresponding test statistic.

Critical points for all test procedures were calculated by drawing $M = 100,000$ samples from the unit exponential distribution: Let T_j denote the value of the test statistic for the j sample, $j = 1, 2, \ldots, M$. Then $(1-\alpha) \times 100\,\%$ critical point is $T_{(M-aM)}$, where $T_{(j)}$ denote the order statistics corresponding to T_j, $j = 1, 2, \ldots, M$. In Table 1, the 1 %, 5 % and 10 % critical points are given for $T_{n,\lambda}$, $\lambda = 0.1, 0.2, 0.5, 1.0$, with sample size $n = 25$ and $n = 50$, and with parameters estimated by the method of moments, as well as by the method of maximum likelihood. Corresponding values for the tests based on L_n, R_n, and $\xi_n := \xi_n(n)$ are reported in Table 2.

On the basis of these critical points, the power of the test statistics was simulated against Weibull and Gamma alternatives with decreasing hazard rate. We denote these distributions by $W(\vartheta)$ and $\Gamma(\vartheta)$, respectively. In Table 3, power figures are shown (percentage of rejection rounded to the nearest integer) for the proposed test statistic with moment and maximum likelihood estimation, based on 10 000 replications ($\star$ denotes power 100 %). Corresponding results for the competing tests are shown in Table 4. The new test incorporating the MLE seems to perform slightly better than the same test incorporating the MOE. Also 'middle' values of the parameter λ, such as $\lambda = 0.2$ and $\lambda = 0.5$, lead to a more competitive test, at least for the alternatives considered herein. (There is no theoretical solution to the 'optimal' value of λ, but prior Monte Carlo results indicate that it depends on the knowledge of the alternative and the parameter involved; see for instance Meintanis [2]). For such values the new test clearly outperforms the test of Wong and Wong [6]. Moreover, the MLE-based $T_{n,\lambda}$ test seems to be overall less powerful than the likelihood ratio test and the test of Chen [1], but only slightly so.

In order to further investigate the features of the new test, we have generated samples from Pareto (P) and Lognormal (LN) distributions. Table 5 gives power results for $T_{n,\lambda}$ implemented with MLE, and the competing tests. Again the test based on R_n is the least powerful. Also for the 'compromise value' $\lambda = 0.5$, the LT-test performs better than the test based on ξ_n. Overall the likelihood method L_n is the most efficient, although $T_{n,1}$ closely matches and often outperforms this test in power. Therefore we propose the use of the empirical LT test as a novel and competitive approach having substantial power not only against Weibull and Gamma, but also against other distributions which are often considered as alternatives to the exponential law.

Table 1. Critical points for $T_{n,\lambda}$ based on 100 000 replications with sample size n for $\lambda \in \{0.10, 0.20, 0.50, 1.0\}$.

$\lambda =$	0.10	0.20	0.50	1.0	0.10	0.20	0.50	1.0
MOE	$n = 25$				$n = 50$			
$\alpha = 1\,\%$	39.6	11.6	2.13	0.557	38.3	11.5	2.15	0.561
$\alpha = 5\,\%$	26.1	7.80	1.47	0.378	26.0	7.91	1.51	0.392
$\alpha = 10\,\%$	19.2	5.85	1.10	0.283	19.6	6.02	1.15	0.298
MLE	$n = 25$				$n = 50$			
$\alpha = 1\,\%$	36.2	10.9	2.08	0.551	35.9	10.9	2.11	0.556
$\alpha = 5\,\%$	24.0	7.39	1.43	0.376	24.6	7.64	1.49	0.391
$\alpha = 10\,\%$	17.96	5.59	1.09	0.283	18.8	5.84	1.14	0.298

Table 2. Critical points based on 100 000 replications with sample size $n = 25$ (upper entries), and $n = 50$ (lower entries).

Test	$\alpha = 1\,\%$	$\alpha = 5\,\%$	$\alpha = 10\,\%$
L_n	0.089	0.0405	0.023
	0.046	0.0224	0.0127
ξ_n	2.68	2.315	2.15
	2.36	2.15	2.05
R_n	9540	1829	872
	22672	4420	2109

Table 3. 5 % (left entry) and 10 % (right entry) percentage of rejection for $T_{n,\lambda}$ based on 10 000 Monte Carlo samples of size $n=25$ (upper entries), and $n=50$ (lower entries).

Distribution↓ $\lambda \rightarrow$	0.1		0.2		0.5		1.0		0.1		0.2		0.5		1.0	
W(0.5)	99	⋆	99	⋆	99	⋆	99	99	99	⋆	99	⋆	99	⋆	99	99
	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆	⋆
W(0.6)	91	95	91	95	93	96	91	95	91	95	91	95	93	96	92	95
	99	⋆	99	⋆	99	⋆	99	⋆	99	⋆	99	⋆	⋆	⋆	99	⋆
W(0.7)	69	79	69	79	73	82	71	80	70	80	70	80	73	82	72	81
	91	95	91	95	92	96	94	96	91	95	91	95	93	96	92	96
W(0.8)	38	52	38	52	41	56	41	54	39	53	39	53	42	56	42	55
	59	72	59	72	63	75	63	76	60	73	60	73	63	75	64	75
W(0.9)	16	26	16	26	17	28	17	27	16	26	16	26	17	28	17	27
	23	35	23	35	25	37	25	37	24	36	24	36	25	37	25	37
Γ(0.5)	84	90	85	92	84	90	81	88	85	91	86	92	85	90	83	89
	98	99	98	99	98	99	97	98	98	99	98	99	98	99	97	99
Γ(0.6)	64	76	66	79	64	75	60	72	66	76	67	79	65	75	63	73
	87	93	89	95	87	92	84	90	88	93	88	94	87	93	85	91
Γ(0.7)	41	55	46	56	41	55	38	52	42	55	46	56	42	55	40	53
	62	74	65	77	62	74	58	71	63	75	65	77	62	74	60	72
Γ(0.8)	23	34	23	37	23	35	22	33	24	35	24	37	23	34	22	33
	34	48	37	50	35	48	32	45	35	48	37	50	35	48	33	46
Γ(0.9)	11	20	12	21	11	19	10	19	12	20	12	21	11	19	10	19
	15	25	16	25	15	25	14	23	15	25	16	25	15	25	14	24

Table 4. 5 % (left entry) and 10 % (right entry) percentage of rejection for 10 000 Monte Carlo samples.

Distribution	L_{25}		ξ_{25}		R_{25}		L_{50}		ξ_{50}		R_{50}	
W(0.5)	99	⋆	99	⋆	87	94	⋆	⋆	⋆	⋆	95	99
W(0.6)	93	96	93	96	63	78	⋆	⋆	⋆	⋆	75	88
W(0.7)	75	83	74	83	39	55	94	97	94	97	46	64
W(0.8)	44	57	43	56	20	34	66	77	64	76	23	38
W(0.9)	18	27	18	28	10	19	26	39	25	37	11	20
Γ(0.5)	87	92	88	93	65	78	99	99	99	99	77	88
Γ(0.6)	69	78	70	80	43	59	90	94	91	95	53	70
Γ(0.7)	45	56	46	59	26	40	66	77	68	78	31	47
Γ(0.8)	24	35	25	37	15	26	37	50	38	51	18	30
Γ(0.9)	11	19	12	21	9	16	16	25	16	25	10	18

Table 5. 5 % (left entry) and 10 % (right entry) percentage of rejection based on 10 000 Monte Carlo samples of size $n = 25$ (upper entries), and $n = 50$ (lower entries).

distribution	$\lambda = 0.1$	0.2	0.5	1.0	L_n	ξ_n	R_n
P(0.5)	33 45	43 55	55 66	61 70	59 69	53 64	17 30
	51 64	65 76	78 86	83 89	83 88	76 84	20 35
P(0.75)	56 68	68 78	79 85	82 88	81 87	77 84	30 47
	81 88	90 94	96 98	97 98	97 98	95 97	37 56
P(1.0)	76 85	85 91	91 94	92 95	92 95	90 93	46 64
	95 97	98 99	99 $\star$	$\star$ $\star$	$\star$ $\star$	99 $\star$	58 76
LN(1.3)	9 17	22 33	42 53	53 63	47 57	35 46	1 4
	13 22	35 48	64 74	76 84	70 78	53 65	1 3
LN(1.4)	22 34	39 52	59 69	68 77	62 71	52 63	3 11
	35 50	62 74	83 89	90 94	86 91	76 84	2 8
LN(1.5)	39 53	58 69	73 82	79 86	75 82	67 77	7 20
	62 74	82 89	94 96	96 98	94 97	90 94	5 17

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Simos G. Meintanis, Department of Economics, National and Kapodistrian University of Athens, 8 Pesmazoglou Street, 105 59 Athens. Greece.
e-mail: simosmei@econ.uoa.gr