ON THE STRUCTURE OF CONTINUOUS UNINORMS

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Uninorms were introduced by Yager and Rybalov [13] as a generalization of triangular norms and conorms. We ask about properties of increasing, associative, continuous binary operation $U$ in the unit interval with the neutral element $e \in [0, 1]$. If operation $U$ is continuous, then $e = 0$ or $e = 1$. So, we consider operations which are continuous in the open unit square. As a result every associative, increasing binary operation with the neutral element $e \in (0, 1)$, which is continuous in the open unit square may be given in $[0, 1)^2$ or $(0, 1]^2$ as an ordinal sum of a semigroup and a group. This group is isomorphic to the positive real numbers with multiplication. As a corollary we obtain the results of Hu, Li [7].

Keywords: uninorms, continuity, $t$-norms, $t$-conorms, ordinal sum of semigroups

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1. INTRODUCTION

Uninorms were introduced by Yager and Rybalov [13] as a generalization of triangular norms and conorms. However similar operations were considered in [3] and [4]. In [6] Fodor, Yager and Rybalov examined a general structure of uninorms. For example, the frame structure of uninorms and characterization of representable uninorms are presented.

In this paper we consider a more general class of operations than uninorms, i.e. operations from the class $\mathcal{U}(e) = \{U : [0, 1]^2 \to [0, 1] : U$ is an increasing, associative binary operation with the neutral element $e\}$ for $e \in [0, 1]$, where we omit the assumption about the commutativity. We ask about properties of continuous operation $U$ in $\mathcal{U}(e)$ where $e \in [0, 1]$. If operation $U$ is continuous then $e = 0$ or $e = 1$ (cf. [3]). So, we consider operations which are continuous in the open unit square. The structure of operations continuous on another subset of unit square we can find in [6, 11, 12].

First, in the Section 2 we present the notion of uninorms and the frame structure of uninorms. Next we present the construction of ordinal sum of semigroups. In Section 4 we present properties of the operation which is continuous in $(0, 1)^2$.

As a result every operation in $\mathcal{U}(e)$ with $e \in (0, 1)$, which is continuous in the open unit square may be given in $[0, 1)^2$ or $(0, 1]^2$ as an ordinal sum of a semigroup and a group. This group is isomorphic to the positive real numbers with multiplication.
Moreover this operation is commutative beyond from two points at the most. As a corollary we obtain results of Hu, Li [7] and Fodor, Yager, Rybalov [6].

2. NOTION OF UNINORMS

We discuss the structure of binary operations $U : [0, 1]^2 \to [0, 1]$.

**Definition 1.** (Yager and Rybalov [13]) An operation $U$ is called a uninorm if it is commutative, associative, increasing and has the neutral element $e \in [0, 1]$.

Uninorms are generalizations of triangular norms (case $e = 1$) and triangular conorms (case $e = 0$). In the case $e \in (0, 1)$ a uninorm $U$ is composed by using a triangular norm and a triangular conorm.

**Theorem 1.** (Fodor, Yager and Rybalov [6]) If a uninorm $U$ has the neutral element $e \in (0, 1)$, then there exist a triangular norm $T$ and a triangular conorm $S$ such that

$$U = \begin{cases} T^* \text{ in } [0, e]^2, \\ S^* \text{ in } [e, 1]^2, \end{cases}$$

where

$$\begin{align*}
T^*(x, y) &= \varphi^{-1}(T(\varphi(x), \varphi(y))), \quad \varphi(x) = x/e, \\
S^*(x, y) &= \psi^{-1}(S(\psi(x), \psi(y))), \quad \psi(x) = (x - e)/(1 - e),
\end{align*}$$

where $x, y \in [0, e]$. (2)

**Lemma 1.** (Fodor, Yager and Rybalov [6]) If $U$ is increasing and has the neutral element $e \in (0, 1)$ then

$$\min \leq U \leq \max \text{ in } A(e) = [0, e) \times (e, 1] \cup (e, 1] \times [0, e).$$

Furthermore, if $U$ is associative, then $U(0, 1), U(1, 0) \in \{0, 1\}$. (3)

**Theorem 2.** (Li and Shi [10]) Let $e \in (0, 1)$. If $T$ is an arbitrary triangular norm and $S$ is an arbitrary triangular conorm then formula (1) with $U = \min$ or $U = \max$ in $A(e)$ gives uninorms.

**Remark 1.** Uninorms from Theorem 2 are not continuous in some points such that one of the variables is equal to the neutral element.

**Example 1.** (Fodor, Yager and Rybalov [6]) Formula

$$U(x, y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ \frac{xy}{(1-x)(1-y)+xy}, & \text{if } x > 0 \text{ and } y > 0 \end{cases}$$

gives a uninorm with $e = \frac{1}{2}$, $T(x, y) = \frac{xy}{x - (x+y-xy)}$, $S(x, y) = \frac{x+y}{1+xy}$, $x, y \in [0, 1]$. This uninorm is continuous apart from the points $(0, 1)$ and $(1, 0)$. 
Theorem 3. (Czogała and Drewniak [3]) If a uninorm is continuous then \( e = 0 \) or \( e = 1 \).

3. REMARK ABOUT THE ORDINAL SUM THEOREM

In this section we consider the ordinal sum and dual ordinal sum of semigroups. Next we present the characterization of continuous \( t \)-norms and \( t \)-conorms by using the ordinal sum theorem. Additional information about the ordinal sum of semigroups one may find in [1, 2, 5, 8, 9, 12].

Theorem 4. (Clifford [1], Climescu [2]) If \((X, F), (Y, G)\) are disjoint semigroups then \((X \cup Y, H)\) is a semigroup, where \( H \) is given by

\[
H(x, y) = \begin{cases} 
F(x, y), & \text{if } x, y \in X, \\
G(x, y), & \text{if } x, y \in Y, \\
x, & \text{if } x \in X, y \in Y, \\
y, & \text{if } x \in Y, y \in X.
\end{cases}
\]

(4)

By duality we obtain

Theorem 5. (Drewniak and Drygaś [5]) If \((X, F), (Y, G)\) are disjoint semigroups, then \((X \cup Y, H)\) is a semigroup, where \( H \) is given by

\[
H(x, y) = \begin{cases} 
F(x, y), & \text{if } x, y \in X, \\
G(x, y), & \text{if } x, y \in Y, \\
y, & \text{if } x \in X, y \in Y, \\
x, & \text{if } x \in Y, y \in X.
\end{cases}
\]

(5)
For our consideration it will be useful to remember the characterization of continuous \(t\)-norms or \(t\)-conorms by using ordinal sum theorems.

**Theorem 6.** (Klement, Mesiar and Pap [9], p. 128, Sander [12]) Operation \(T : [0, 1]^2 \to [0, 1]\) is continuous, associative, increasing, with the neutral element \(e = 1\) iff there exists a family \(\{(a_k, b_k)\}_{k \in A}\) (where \(A \subset \mathbb{Q} \cap [0, 1]\)) of nonempty, pairwise disjoint, open subintervals of \([0, 1]\) such that the operations \(T_k = T|_{[a_k, b_k]^2}\) are continuous, increasing, associative with Archimedean property, neutral element \(b_k\) and \(T\) is given by

\[
T(x, y) = \begin{cases} 
T_k(x, y), & \text{for } (x, y) \in (a_k, b_k]^2, \\
\min(x, y), & \text{otherwise}.
\end{cases}
\] (6)

Moreover, the operation \(T\) is commutative.

**Theorem 7.** (Klement, Mesier and Pap [9], p. 130) Operation \(S : [0, 1]^2 \to [0, 1]\) is continuous, associative, increasing, with the neutral element \(e = 0\) iff there exists a family \(\{(a_k, b_k)\}_{k \in A}\) (where \(A \subset \mathbb{Q} \cap [0, 1]\)) of nonempty, pairwise disjoint, open subintervals of \([0, 1]\) such that the operations \(S_k = S|_{[a_k, b_k]^2}\) are continuous, increasing, associative with Archimedean property, neutral element \(a_k\) and \(S\) is given by

\[
S(x, y) = \begin{cases} 
S_k(x, y), & \text{for } (x, y) \in [a_k, b_k]^2, \\
\max(x, y), & \text{otherwise}.
\end{cases}
\] (7)

Moreover, the operation \(S\) is commutative.

4. MAIN RESULTS

In Theorems 6 and 7 a characterization of continuous operations in the class \(\mathcal{U}(1)\) and \(\mathcal{U}(0)\) respectively is given. Moreover, if operation in the class \(\mathcal{U}(e)\) is continuous,
then \( e = 0 \) or \( e = 1 \) (see Theorem 3). Thus, we ask about the structure of operations in the class \( \mathcal{U}(e) \) which are continuous in the open unit square for \( e \in (0, 1) \).

**Lemma 2.** Let \( e \in (0, 1) \). If operation \( U \in \mathcal{U}(e) \) is continuous in \((0, 1)^2 \) then operation \( U|_{[0,e]^2} \) is isomorphic to a continuous \( t \)-norm and \( U|_{(e,1]^2} \) is isomorphic to a continuous \( t \)-conorm.

**Proof.** First we prove that operation \( U|_{[e,1]^2} \) is continuous. The operator \( U \) is continuous in \((0, 1)^2 \). From this we obtain the continuity of the operation \( U|_{[e,1]^2} \) in \([e,1]^2 \). Moreover \( U(x,y) \geq \max(x,y) \) for \( x,y \in [e,1] \) and \( U(x,1) = U(1,x) = 1 \) for \( x \in [e,1] \). Let \( x,y \in [e,1] \), then \( 1 \geq U(x,y) \geq \max(x,y) \), \( \lim_{x \to 1} \max(x,y) = 1 \) and \( \lim_{y \to 1} \max(x,y) = 1 \). It means that \( U(x,y) \) is continuous in \((0, 1)^2 \), i.e. functions \( U(x,t) \) and \( U(t,y) \), \( t \in [e,1] \) are continuous for all \( x,y \in [e,1] \). This implies continuity of the operation \( U|_{[e,1]^2} \). It means, that \( U|_{[e,1]^2} \) is a continuous, associative, increasing operation with neutral element \( e \), then it is isomorphic to a continuous \( t \)-conorm.

In similar way we obtain that the operation \( U|_{[0,e]^2} \) is isomorphic to a continuous \( t \)-norm.

**Lemma 3.** Let \( e \in (0, 1) \) and \( U \in \mathcal{U}(e) \). If there exists \( a \in [0,e) \) such that \( U(x,y) = x \) for \( x \in (a,e) \), \( y \in (e,1) \) or \( U(x,y) = y \) for \( x \in (e,1) \), \( y \in (a,e) \) then \( U \) is not continuous in \((0,1)^2 \).

**Proof.** Let \( U(x,y) = x \) for \( x \in (a,e) \), \( y \in (e,1) \). Take \( s \in (e,1) \) and let \( f(t) = U(t,s) \), \( t \in [0,1] \). We have \( f(t) = U(t,s) = t < e \) for \( t \in (a,e) \) and \( f(e) = s > e \). It means, that the function \( f \) is not continuous at the point \( e \). This implies, that \( U \) is not continuous in \((0,1)^2 \).

In similar way as above we obtain the second part of Lemma.

In the next part of this paper we need the following lemmas

**Lemma 4.** (Klement, Mesiar and Pap [9]) Let \( J = [a,b] \) and \( F : J^2 \to J \) be associative, increasing operation with the neutral element \( b \). If \( x \in J \) is an idempotent element of operation \( F \) and functions \( f(t) = F(x,t) \), \( h(t) = F(t,x) \), \( t \in J \) are continuous in \( J \) then \( F(x,y) = F(y,x) = \min(x,y) \) for \( y \in J \).

**Lemma 5.** Let \( J = [a,b] \) and \( F : J^2 \to J \) be associative, increasing operation with the neutral element \( a \). If \( x \in J \) is an idempotent element of operation \( F \) and functions \( f(t) = F(x,t) \), \( h(t) = F(t,x) \), \( t \in J \) are continuous in \( J \) then \( F(x,y) = F(y,x) = \max(x,y) \) for \( y \in J \).

**Lemma 6.** Let \( e \in (0, 1) \) and \( U \in \mathcal{U}(e) \) be continuous in \((0,1)^2 \). If there exists \( b \in (0,e) \) such that \( U(b,y) = b \) for \( y \in (b,e) \) or \( U(x,b) = b \) for \( x \in (b,e) \) then \( U(x,y) = U(y,x) = \min(x,y) \) for \( x \in [0,b] \) and \( y \in [b,1] \).
Proof. Let $x \in [0, b]$ and $y \in (e, 1)$. For all $t \in (b, e)$ we have $U(b, t) = b$. By the continuity of the operation $U$ we have $U(b, b) = b$. This means that $b$ is an idempotent element of the continuous operation $U|_{[0, e]}$ and by Lemma 4 we have $U(b, t) = U(t, b) = \min(t, b)$ for $t \in [0, e]$. Hence, by monotonicity of $U$ we have $U(b, t) = U(t, b) = \min(t, b)$ for $t \in [0, e]$.

Suppose that there exists $z \in (e, 1)$ such that $U(b, z) \geq e$. By continuity of the operation $U$ and condition $U(b, e) = b$ there exists $w \in (e, z)$ such that $U(b, w) = e$. Then

$$b = U(b, e) = U(b, U(b, w)) = U(U(b, b), w) = U(b, w) = e,$$

which is a contradiction. Therefore $U(b, y) < e$ for all $y \in (e, 1)$. By continuity of the operation $U$ and condition $U(e, y) = y$ there exists $v \in (b, e)$ such that $U(v, y) = e$. Therefore for all $x \leq b$ we have

$$U(x, y) = U(\min(x, v), y) = U(U(x, v), y) = U(x, U(v, y)) = U(x, e) = x.$$

By commutativity of the operation $U|_{[0, e]}$ we obtain $U(y, x) = x$ for $x \in [0, b]$ and $y \in [b, e]$. In similar way as above we obtain $U(y, x) = \min(x, y)$ for $x \in [0, b]$, $y \in [b, 1)$. If we assume that $U(x, b) = b$ for $x \in (b, e)$ then the proof is analogous. □

By duality we obtain

**Lemma 7.** Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0, 1)^2$. If there exists $a \in (e, 1)$, such that $U(a, y) = a$ for $y \in (e, a)$ or $U(x, a) = a$ for $x \in (e, a)$ then $U(x, y) = U(y, x) = \max(x, y)$ for $x \in [a, 1]$ and $y \in (0, a]$.

**Lemma 8.** (cf. Hu and Li [7]) Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0, 1)^2$. Then there exist idempotent elements $a \in [0, e)$ and $b \in (e, 1]$ such that operations $U|_{[a, e]}$ and $U|_{(e, b]}$ are strictly increasing. Moreover $a = 0$ or $b = 1$. 

![Fig. 3. The operation $U$ from the Lemma 6.](image-url)
Fig. 4. The operation $U \in \mathcal{U}(e)$ from Lemma 8.

\textbf{Proof.} By Lemma 2 operation $U|_{[0,e]^2}$ is isomorphic to a continuous t-norm. By Theorem 6 there exists a countably family of intervals $(a_k, b_k) \subset [0, e)$ such that $U|_{[0,e]^2}$ is an ordinal sum of semigroups $T_k = U|_{[a_k, b_k]^2}$ with Archimedean property or $T_k = \min$.

Suppose that there does not exist such $a \in [0, e)$ that $U|_{[a,e]^2}$ is a semigroup with Archimedean property. Then there exists $r \in [0, e)$ such that $U|_{[r,e]^2} = \min$ or for every neighborhood of the point $e$ there exists $k$ such that interval $(a_k, b_k)$ is included in that neighborhood, i.e. there exists an increasing subsequence $\{b_{k_n}\}$ of sequence $\{a_k\}$, e.g. $c_n = e - \frac{1}{n+\lceil \frac{1}{r-e}\rceil} \in [r, e)$ in the first case, and $c_n = b_{k_n}$ in the second case. According to (6) we have $U(c_n, y) = c_n$ for all $y \in (c_n, e)$. By Lemma 6, $U(x, y) = x$ for $x \in [0, c_n]$ and $y \in (e, 1)$. It implies that $U(x, y) = x$ for $x \in [0, e) = \bigcup_{n=1}^{\infty} [0, c_n]$ and $y \in (e, 1)$. Now, by Lemma 3, operation $U$ is not continuous in $(0, 1)^2$, which is a contradiction. So, there exists $a \in [0, e)$ such that $U|_{[a,e]^2}$ is isomorphic to a continuous Archimedean t-norm. Moreover $a$ is an idempotent element of operation $U$ and the zero element of operation $U|_{[a,e]^2}$.

Now we show that $U|_{[a,e]^2}$ is strictly increasing. Suppose that it is not. It means that $U|_{[a,e]^2}$ is isomorphic to the Lukasiewicz t-norm $T_L$. By continuity of $U$ there exist $p \in (a, e)$ and $w \in (e, 1)$ such that $U(p, w) = e$. By the fact that $U|_{[a,e]^2}$ is isomorphic to $T_L$ (all elements from $(a, e)$ are zero divisors, where zero element is equal to $a$) it follows that $U(p, q) = U(q, p) = a$ for some $q \in (a, e)$ and by monotonicity of operation $U$ and because $U(a, a) = a$ we have $U(t, p) = a$ for all $t \in [a, q]$. Therefore $U(t, U(p, w)) = U(t, e) = t$ and $U(U(t, p), w) = U(a, w)$. By associativity of $U$ we have $U(a, w) = t$ for all $t \in [a, q]$, which leads to a contradiction. Thus $U|_{[a,e]^2}$ is strictly increasing.

In similar way we prove that there exists idempotent element $b \in (e, 1]$, which is the zero element of $U|_{[e,b]^2}$, such that $U|_{[e,b]^2}$ is strictly increasing.

Suppose that $a > 0$ and $b < 1$. Since $U(a, y) = a$ for all $y \in (a, e)$, Lemma 6 implies that $U(x, y) = \min(x, y)$ for $x \in [0, a]$ and $y \in (e, 1)$. Similarly, since $b$ is the
zero element of $U|_{[e,b]^2}$. Lemma 7 implies that $U(x, y) = \max(x, y)$ for $x \in (0, e)$ and $y \in [b, 1]$. Therefore $U(x, y) = x$ and $U(x, y) = y$ for $x \in (0, a]$ and $y \in [b, 1)$, which is a contradiction.

Accordingly $a = 0$ or $b = 1$. □

**Lemma 9.** Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0,1)^2$. If there exists $a \in [0, e)$ such that operations $U|_{(a,e)^2}$ and $U|_{[e,1)^2}$ are strictly increasing then the operation $U|_{(a,1)^2}$ is strictly increasing.

**Proof.** To show, that $U|_{(a,1)^2}$ is strictly increasing we must show that $U$ is strictly increasing on the set $(a, e] \times [e, 1) \cup [e, 1) \times (a, e]$. By Lemma 2 operations $U|_{[0,e]^2}$ and $U|_{[e,1]^2}$ are commutative. Let $x, y \in (a, e] \times [e, 1)$ and $y \in e, 1$. Suppose that $U(x, y) = U(y, z)$. Then $z > e$ because $U(x, e) = x < y = U(y, e)$.

If $U(x, z) = U(y, z) < e$ then by continuity of $U$ and inequality $U(e, z) = z > e$ there exists $s \in (x, e)$ such that $U(s, z) = e$. Then

$$x = U(x, e) = U(U(x, s), e) = U(U(s, x), e) = U(s, U(x, z))$$

which is a contradiction.

If $U(x, z) = U(y, z) \geq e$ then, by continuity of $U$ and condition $U(x, e) = x, x < y \leq e$, there exists $c \in (e, z]$ such that $U(x, c) = y$. From $U(y, e) = y \leq e \leq U(y, z)$, there exists $d \in [e, z]$ such that $U(y, d) = e$. Thus $U(e, z) = z$ and

$$z = U(e, z) = U(U(y, d), z) = U(y, U(d, z)) = U(y, U(z, d))$$

Moreover operation $U|_{[e,1)^2}$ is strictly increasing and $z, c \in (e, 1)$. This leads to a contradiction. Therefore $U$ is strictly increasing with respect to the first variable in the $(a, e] \times [e, 1)$. Now let $x, y \in [e, 1), x < y$ and $y \in (a, e]$. Suppose that $U(z, x) = U(z, y)$. Then $z < e$ because $U(e, x) = x < y = U(e, y)$.

If $U(z, x) = U(z, y) > e$ then, by continuity of $U$ and inequality $U(z, e) = z < e$, there exists $s \in (e, x)$ such that $U(z, s) = e$. Therefore

$$x = U(e, x) = U(U(z, s), x) = U(z, U(s, x)) = U(z, U(x, s)) = U(U(z, x), s)$$

which is a contradiction.
If $U(z, x) = U(z, y) \leq e$ then, by continuity of $U$ and condition $U(e, y) = y$, $e \leq x < y$, there exists $c \in (z, e)$ such that $U(c, y) = x$. From $U(e, x) = x > e \geq U(z, x)$ there exists $d \in [z, e]$ such that $U(d, x) = e$. Therefore

$$z = U(z, e) = U(z, U(d, x)) = U(U(z, d), x) = U(d, z, x)$$

$$= U(U(d, z), U(c, y)) = U(d, U(U(z, c), y)) = U(d, U(z, c), y))$$

$$= U(U(c, d), U(z, x)) = U(U(c, d), z, x) = U(U(c, z), d, x) = U(U(c, z), d, x)$$

$$= U(U(c, z), e) = U(c, z).$$

Moreover, operation $U_{[a,e]}^2$ is strictly increasing and $z, c \in (a, e)$. This leads to a contradiction. Thus $U$ is strictly increasing with respect to second variable on $(a, e] \times [e, 1]$.

In a similar way we prove that $U$ is strictly increasing on $[e, 1] \times (a, e]$. \qed

Theorem 8. Let $e \in (0, 1)$ and $U \in U(e)$ be continuous in $(0, 1)^2$. If there exists an idempotent element $a \in [0, e)$ of $U$ such that operations $U_{[a,e]}^2$ and $U_{[e,1]}^2$ are strictly increasing, then operation $U_{[0,1]}^2$ is an ordinal sum of continuous semigroup $U_{[0,a]}^2$ with the neutral element $a$ and continuous group $U_{[e,1]}^2$ with Archimedean property and the neutral element $e$.

Proof. By Lemma 2, the operation $U_{[0,e]}^2$ is isomorphic to a continuous $t$-norm and, since $a$ is an idempotent element of this operation, $U_{[0,a]}^2$ is also isomorphic to a continuous $t$-norm. By Lemma 9, operation $U_{[a,1]}^2$ is strictly increasing and therefore it is isomorphic to the real numbers with addition. Now, taking into account Lemma 6 we have that $U_{[0,1]}^2$ is an ordinal sum of the semigroup $U_{[0,a]}^2$ and the group $U_{[a,1]}^2$. \qed

Similarly, we obtain the following results:

Lemma 10. Let $e \in (0, 1)$ and $U \in U(e)$ be continuous in $(0, 1)^2$. If there exists $b \in (e, 1]$ such that operations $U_{[0,e]}^2$ and $U_{[e,b]}^2$ are strictly increasing then the operation $U_{[0,b]}^2$ is strictly increasing.

Theorem 9. Let $e \in (0, 1)$ and $U \in U(e)$ be continuous in $(0, 1)^2$. If there exists an idempotent element $b \in (e, 1]$ of $U$ such that operations $U_{[0,e]}^2$ and $U_{[e,b]}^2$ are strictly increasing then operation $U_{[0,1]}^2$ is a dual ordinal sum of continuous group $U_{[0,b]}^2$ with Archimedean property and the neutral element $e$ and continuous semigroup $U_{[b,1]}^2$ with the neutral element $b$.

So, we have the characterization of this operation in the open unit square. Now we ask about it’s structure on the boundary.
Lemma 11. Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0, 1)^2$. If there exists an idempotent element $a \in [0, e)$ of $U$ such that operations $U|_{(a,e)^2}$ and $U|_{(e,1)^2}$ are strictly increasing then there exist idempotent elements $c, d \in [0, a]$ of operation $U$ such that

$$U(x, 1) = \begin{cases} x, & \text{if } x \in [0, c), \\ 1, & \text{if } x \in (c, 1], \\ x \text{ or } 1, & \text{if } x = c, \end{cases}$$

(8)

$$U(1, x) = \begin{cases} x, & \text{if } x \in [0, d), \\ 1, & \text{if } x \in (d, 1], \\ x \text{ or } 1, & \text{if } x = d. \end{cases}$$

(9)

Moreover $c = d$.

Proof. By the Lemma 1, $U(0, 1) = 0$ or $U(0, 1) = 1$. If $U(0, 1) = 1$ then by monotonicity of $U$ we have $U(x, 1) = 1$ for $x \in [0, 1]$. Therefore we obtain (8) for $c = 0$. Moreover 0 is an idempotent element of the operation $U$.

If $U(0, 1) = 0$ then by Theorem 9 the semigroup $U|_{(a,1)^2}$ is isomorphic to the real numbers with addition. Thus we have $\lim_{y \to 1} U(x, y) = 1$ for $x \in (a,1)$ and by monotonicity of the operation $U$ we obtain $U(x, 1) = 1$ for $x \in (a,1]$. Let $x \in (0, a]$. First we will prove that $U(x, 1) = x$ or $U(x, 1) = 1$. Suppose that there exists $z \in (0, a]$ such that $z < U(z, 1) < 1$ and let $w = U(z, 1)$.

If $w \in (a, 1)$ then for $y \in (e, 1)$, by associativity of $U$ and strictly monotonicity of $U|_{(a, 1)^2}$, we obtain

$$w = U(z, 1) = U(z, U(y, 1)) = U(z, U(1, y))$$

$$= U(U(z, 1), y) = U(w, y) > U(w, e) = w,$$

which is a contradiction.

If $w \in (z, a]$ then by the conditions $U(0, w) = 0$, $U(e, w) = w$ and continuity of $U|_{(0,e)^2}$ there exists $v \in (0, e)$ such that $U(v, w) = z$ and by associativity of $U$, we obtain

$$w = U(z, 1) = U(U(v, w), 1) = U(U(v, U(z, 1)), 1)$$

$$= U(U(v, z), U(1, 1)) = U(U(v, z), 1) = U(v, U(z, 1)) = U(v, U(z, 1)) = U(v, w) = z,$$

which is a contradiction. Therefore $U(x, 1) = x$ or $U(x, 1) = 1$ for $x \in [0, 1]$.

Thus, for $c = \inf\{x \in [0, a] : U(x, 1) = 1\}$ we obtain (8), moreover $c \in [0, a]$.

Let $x \in (0, c)$, $y \in (c, e]$ then we have

$$U(x, y) = U(y, x) = U(y, U(x, 1)) = U(U(y, x), 1)$$

$$= (U(x, y), 1) = U(x, U(y, 1)) = U(x, 1) = x = \min(x, y).$$

By monotonicity of $U$ and inequality $U|_{(0,e)^2} \leq \min$ we obtain $U(c, y) = c$ for $y \in (c, e)$. By above and continuity of $U$ we have $U(c, c) = c$, i.e. $c$ is an idempotent element of operation $U$. Similarly we prove (9).
To prove that $c = d$ suppose that $d < c$. Then there exists $y \in (d, c)$ such that $U(1, y) = 1$ and $U(y, 1) = y$. Taking $z \in (d, y)$ we have $U(1, z) = 1$ and

$$y = U(y, 1) = U(y, U(1, z)) = U(U(y, 1), z) = U(y, z) \leq U(e, z) = z < y,$$

which is a contradiction, thus $d \geq c$.

If we suppose that $d > c$ then there exists $y \in (c, d)$ such that $U(1, y) = y$ and $U(y, 1) = 1$. Taking $z \in (y, d)$ we have

$$z = U(1, z) = U(U(y, 1), z) = U(y, U(1, z)) = U(y, z) \leq U(y, e) = y < z,$$

which is a contradiction. Thus $c = d$. □

**Lemma 12.** Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0, 1)^2$. If there exists an idempotent element $b \in (e, 1]$ of $U$ such that operations $U|_{(0,e)^2}$ and $U|_{[e,b]^2}$ are strictly increasing then there exist idempotent elements $p, q \in [b, 1]$ of operation $U$ such that

$$U(x, 0) = \begin{cases} 0, & \text{if } x \in [0, p), \\ x, & \text{if } x \in (p, 1], \\ 0 \text{ or } x, & \text{if } x = p, \end{cases}$$

(10)

$$U(0, x) = \begin{cases} 0, & \text{if } x \in [0, q), \\ x, & \text{if } x \in (q, 1], \\ 0 \text{ or } x, & \text{if } x = q. \end{cases}$$

(11)

Moreover $p = q$.

![Fig. 5. Operation $U \in \mathcal{U}(e)$ continuous in the open unit square with $a > 0$.](image)

As a results of our considerations we obtain

**Theorem 10.** Let $e \in (0, 1)$ and $U \in \mathcal{U}(e)$ be continuous in $(0, 1)^2$. Then one of the following two cases holds:
(i) There exist idempotent elements \( a \in [0, e) \) and \( c \in [0, a] \) of operation \( U \) such that \( U|_{[0,1]^2} \) is an ordinal sum of continuous semigroup \( U|_{[0,a]^2} \) with the neutral element \( a \) and continuous group \( U|_{(a,1)^2} \) with Archimedean property and the neutral element \( e \) and conditions (8) and (9) hold.

(ii) There exist idempotent elements \( b \in (e, 1] \) and \( p \in [b, 1] \) of operation \( U \), such that \( U|_{(0,1)^2} \) is a dual ordinal sum of continuous semigroup \( U|_{[b,1]^2} \) with the neutral element \( b \) and continuous group \( U|_{(0,b)^2} \) with Archimedean property and the neutral element \( e \) and conditions (10) and (11) hold.

\[ \begin{array}{|c|c|c|c|}
\hline
 & & & \\
\hline
 & \max & \max & U|_{[1,1]^2} \\
\hline
q & & & \\
\hline
b & & & \\
\hline
 & U|_{(b,b)^2} & & \\
\hline
 & & \max & \\
\hline
e & & & \\
\hline
0 & e & b & p \ 1 \\
\hline
\end{array} \]

Fig. 6. Operation \( U \in \mathcal{U}(e) \) continuous in the open unit square with \( b < 1 \).

Proof. By Lemma 8 there exist \( a \in [0, e) \) and \( b \in (e, 1] \) \((a = 0 \text{ or } b = 1)\) such that \( U|_{(a,b)^2} \) is strictly increasing (Lemma 9 and 10).

If \( b = 1 \) then by Theorem 8 and Lemma 11 we obtain (i).

If \( a = 0 \) then by Theorem 9 and Lemma 9 we obtain (ii). \( \square \)

Remark 2. Operation \( U \) in the previous theorem is commutative in the set

\( (i) \ [0,1]^2 \setminus \{(c,1), (1,c)\} \),

\( (ii) \ [0,1]^2 \setminus \{(0,p), (p,0)\} \).

5. CONCLUSION

By the above consideration we obtain the following results known from the papers [6] and [7].
Theorem 11. (Hu and Li [7], Theorem 4.5) Let \( e \in (0, 1) \) and \( U \) be a uninorm which is continuous in \((0, 1)^2\). Then \( U \) can be represented as follows:

\[
(i) \quad U(x, y) = \begin{cases} 
  eT\left(\frac{x}{e}, \frac{y}{e}\right), & \text{if } x, y \in [0, a], \\
  h^{-1}(h(x) + h(y)), & \text{if } x, y \in (a, 1), \\
  x, & \text{if } x \in [0, a], y \in (a, 1) \text{ or } x \in [0, c), y = 1, \\
  y, & \text{if } x \in (a, 1), y \in [0, a] \text{ or } x = 1, y \in [0, c), \\
  1, & \text{if } x \in (c, 1), y = 1 \text{ or } x = 1, y \in (c, 1), \\
  x \text{ or } y, & \text{if } x = c, y = 1 \text{ or } x = 1, y = c,
\end{cases}
\]

where \( a \in [0, e), c \in [0, a], U(c, c) = c \), function \( h : [a, 1] \rightarrow [-\infty, +\infty) \) is strict and \( h(a) = -\infty, h(e) = 0, h(1) = +\infty; \)

\[
(ii) \quad U(x, y) = \begin{cases} 
  e + (1 - e)S\left(\frac{x}{1-e}, \frac{y}{1-e}\right), & \text{if } x, y \in [b, 1], \\
  h^{-1}(h(x) + h(y)), & \text{if } x, y \in (0, b), \\
  y, & \text{if } x \in (0, b), y \in [b, 1] \text{ or } x = 0, y \in (p, 1], \\
  x, & \text{if } x \in [b, 1], y \in (0, b) \text{ or } x \in (p, 1], y = 0, \\
  0, & \text{if } x = 0, y \in [0, p) \text{ or } x \in [0, p), y = 0, \\
  x \text{ or } y, & \text{if } x = p, y = 0; \text{or } x = 0, y = p,
\end{cases}
\]

where \( b \in (e, 1], p \in [b, 1], U(p, p) = p \), function \( h : [0, b] \rightarrow [-\infty, +\infty] \) is strict and \( h(0) = -\infty, h(e) = 0, h(b) = +\infty. \)

Theorem 12. (Fodor, Yager and Rybalkov [6]) Let \( e \in (0, 1) \) and \( U \) be a uninorm continuous without the points \((0, 1)\) and \((1, 0)\). Then operations \( U|_{(0,e]^2} \) and \( U|_{(e,1)^2} \) are strictly increasing and

\[
U(x, y) = \begin{cases} 
  h^{-1}(h(x) + h(y)), & \text{for } (x, y) \in [0, 1]^2 \setminus \{(0, 1), (1, 0)\}, \\
  0 \text{ or } 1, & \text{elsewhere},
\end{cases}
\]

where \( h : [0, 1] \rightarrow [-\infty, +\infty] \) is an increasing bijection such that \( h(e) = 0. \)

Proof. Operation \( U|_{(0,1)^2} \) is continuous. Suppose that in Theorem 10 the condition \((i)\) holds, i.e. there exists \( a \in [0, e) \), such that operation \( U|_{(a,1)^2} \) is strictly increasing. By Lemma 11 there exists \( c \in [0, a] \) such that \((8)\) holds.

Suppose that \( c < a \), then for \( x \in (c, a) \) and \( y \in (e, 1) \) we have \( U(x, y) = \min(x, y) = x \) and \( U(x, 1) = 1 \). It means that \( U \) is not continuous at the points \((x, 1), x \in (c, a)\). Therefore \( c = a. \)

Suppose now, that \( a > 0 \). By Lemma 11 we have \( U(x, 1) = x \) for \( x \in [0, a) \) and \( U(x, 1) = 1 \) for \( x \in (a, 1] \). It means that the point \((a, 1)\) is a point of discontinuity of the operation \( U \), which leads to a contradiction. Thus \( a = 0. \) Now, directly by the above theorem, we obtain \((12)\). 

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