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PROPERTIES OF FUZZY RELATIONS POWERS

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Properties of sup-$*$ compositions of fuzzy relations were first examined in Goguen [8] and next discussed by many authors. Power sequence of fuzzy relations was mainly considered in the case of matrices of fuzzy relation on a finite set. We consider sup-$*$ powers of fuzzy relations under diverse assumptions about $*$ operation. At first, we remind fundamental properties of sup-$*$ composition. Then, we introduce some manipulations on relation powers. Next, the closure and interior of fuzzy relations are examined. Finally, particular properties of fuzzy relations on a finite set are presented.

Keywords: fuzzy relation, binary operation, relation composition, sup-$*$ composition, relation powers, relation closure, relation interior

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1. INTRODUCTION

Main notions on fuzzy relations were introduced by L. A. Zadeh [19, 20] and developed by A. Kaufmann [9]. Simultaneously, J. A. Goguen [8] introduced L-fuzzy relations and their sup-$*$ composition applying the notion of complete lattice ordered semigroup (closg, l-monoid) described by G. Birkhoff [1]. Powers of fuzzy relations were introduced by A. Kaufmann [9] and examined by M. G. Thomason [16].

Our experience with fuzzy relations suffers from some misunderstanding about assumptions necessary in their algebra. At first, G. Birkhoff [1] distinguished conditions of infinite distributivity (conditions (1), (1'), p. 118 self-dual in Boolean lattices) and complete distributivity of lattices (conditions (4), (4'), p. 119 self-dual in complete lattices; cf. also G. Szász [17], section 31, conditions (3), (4) and (7), (8)). However, J. A. Goguen [8] p. 151 referred to the condition of infinite distributivity as to complete distributivity, which leads to false interpretation of assumptions. Next, the results of the paper [8] can have false applications if they are copied literally (without precise assumptions on L).

For example, in [8] Proposition 1, p. 162 we read 'Composition of L-relations is associative', while assumptions used in the proof were explained on pp. 154–155 and not repeated in Proposition 1. As a result we obtain a common conviction that sup-$*$ composition with associative $*$ operation is also associative. Such false statement (cf. [4], Example 6) is widely used for relation compositions with triangular norms (cf. e. g. [7], Proposition 2.2; [11], formula (5.14) or [14], formula (7.1)).

In this situation the exact examination of dependence between algebraic properties of $*$ operation and induced sup-$*$ composition was necessary (cf. e.g. [4]). First, we remind fundamental properties of sup-$*$ composition (Section 2). Then, we introduce some manipulations on relation powers and relation closures (Section 3). Next, particular properties of fuzzy relations on a finite set are presented (Section 4). Finally, we discuss powers in classes of fuzzy relations (Section 5).

2. FUZZY RELATIONS

We begin with a set $X \neq \emptyset$ and a binary operation $* : [0,1]^2 \to [0,1]$.

Definition 1. (Zadeh [19]) A fuzzy relation in a set X is an arbitrary function $R : X \times X \to [0,1]$. The family of all fuzzy relations in X is denoted by $FR(X)$.

As important examples of $R \in FR(X)$ we consider the identity relation $I = I_X$ and constant relations $c_{X\times X}$ for $c \in [0,1]$, where $c_{X\times X}(x,y) = c$ for $x, y \in X$. In particular one has empty relation $0_{X\times X}$ and total relation $1_{X\times X}$. We use set theoretical operations on fuzzy relations as complement $R' = 1-R$, inclusion $R \leqslant S$, sum $R \vee S$ and intersection $R \wedge S$, which are defined pointwise for $x, y \in X$:

$$R \leqslant S \Leftrightarrow R(x,y) \leqslant S(x,y),$$

$$(R \vee S)\,(x,y) = \max(R(x,y), S(x,y)),\ (R \wedge S)\,(x,y) = \min(R(x,y), S(x,y)).$$

By analogy, for arbitrary set T of indexes, $T \neq \emptyset$ we use

$$\left(\bigvee\nolimits_{t\in T} R_t\right)(x,y) = \sup\nolimits_{t\in T} R_t(x,y),\ \left(\bigwedge\nolimits_{t\in T} R_t\right)(x,y) = \inf_{t\in T} R_t(x,y) \quad \text{for} \quad x,\, y \in X.$$

Similarly, we consider the inverse R^{-1} of R, where

$$R^{-1}(x,y) = R(y,x) \text{ for } x, y \in X.$$

Definition 2. (Goguen [8]) By sup-$*$ composition of fuzzy relations $R, S \in FR(X)$ we understand a fuzzy relation $R \circ S$, where

$$(R \circ S)(x,z) = \sup\nolimits_{y\in X}(R(x,y) * S(y,z)), \quad x, z \in X. \tag{1}$$

In the case $* = \min$ we simply say 'relation composition'. By inf-$*$ composition of R and S we call $R \circ' S$, where

$$(R \circ' S)(x,z) = \inf\nolimits_{y\in X}(R(x,y) * S(y,z)), \quad x, z \in X. \tag{2}$$

By direct verification we get (cf. [5], Theorem 2)

Theorem 1. (Duality principle) Let $N : [0,1] \to [0,1]$ be an involutory negation and $a *' b = N(N(a) * N(b))$ for $a, b \in [0,1]$. Compositions sup-$*$ and inf-$*'$ are connected by the formula

$$\inf\nolimits_{y\in X}(R(x,y) *' S(y,z)) = N(\sup\nolimits_{y\in X}(N(R(x,y)) * N(S(y,z)))), \quad x, z \in X. \tag{3}$$

Example 1. Using matrix representation $R = [r_{i,k}]$ for fuzzy relations on a finite set $X = \{x_1, x_2, \ldots, x_n\}$, where $r_{i,k} = R(x_i, x_k)$, $i, k = 1, 2, \ldots, n$, we have ($n = 3$):

$$R = \begin{bmatrix} 0.1 & 1 & 0.3 \\ 0.8 & 0.2 & 0.4 \\ 0.5 & 0.6 & 0.7 \end{bmatrix}, \quad S = \begin{bmatrix} 0.2 & 0.9 & 1 \\ 0.7 & 0.8 & 0.3 \\ 0 & 0.4 & 0.6 \end{bmatrix},$$

$$N(R) = \begin{bmatrix} 0.9 & 0 & 0.7 \\ 0.2 & 0.8 & 0.6 \\ 0.5 & 0.4 & 0.3 \end{bmatrix}, \quad N(S) = \begin{bmatrix} 0.8 & 0.1 & 0 \\ 0.3 & 0.2 & 0.7 \\ 1 & 0.6 & 0.4 \end{bmatrix},$$

$$R \circ' S = \begin{bmatrix} 0.2 & 0.4 & 0.6 \\ 0.4 & 0.4 & 0.3 \\ 0.5 & 0.7 & 0.6 \end{bmatrix}, \quad N(R) \circ N(S) = \begin{bmatrix} 0.8 & 0.6 & 0.4 \\ 0.6 & 0.6 & 0.7 \\ 0.5 & 0.3 & 0.4 \end{bmatrix},$$

with $* = \min$, $*' = \max$, $N(x) = 1 - x$, $x \in [0, 1]$.

Because of formula (3) we obtain the direct dependence between properties of sup-$*$ composition and inf-$*'$ composition (duality). Thus, we can omit detail considerations of dual properties. However, $\min - \max$ composition and inf-$*$ composition are still examined independently on sup-$*$ composition (cf. e. g. [12] and [15]).

Properties of the above compositions depend on additional assumptions about the operation $*$. Our assumptions on binary operations as associativity, neutral element or zero element are based on [1], Chapter XIV.

Definition 3. (Drewniak and Kula [4]) Operation $* : [0, 1]^2 \to [0, 1]$ is infinitely sup-distributive if

$$\sup_{t \in T}(x_t * y) = (\sup_{t \in T} x_t) * y, \quad \sup_{t \in T}(y * x_t) = y * (\sup_{t \in T} x_t)\,. \tag{4}$$

Operation $*$ is infinitely inf-distributive if

$$\inf_{t \in T}(x_t * y) = (\inf_{t \in T} x_t) * y, \quad \inf_{t \in T}(y * x_t) = y * (\inf_{t \in T} x_t)\,. \tag{5}$$

We are interested in a few particular properties of the relation compositions.

Theorem 2. (Drewniak and Kula [4]) Let $T \neq \emptyset$, $R, S_t \in FR(X)$, $t \in T$. If operation $*$ is increasing, then

$$R \circ \Big(\bigvee_{t \in T} S_t\Big) \geqslant \bigvee_{t \in T}(R \circ S_t), \quad R \circ \Big(\bigwedge_{t \in T} S_t\Big) \leqslant \bigwedge_{t \in T}(R \circ S_t), \tag{6}$$
$$R \circ' \Big(\bigvee_{t \in T} S_t\Big) \geqslant \bigvee_{t \in T}(R \circ' S_t), \quad R \circ' \Big(\bigwedge_{t \in T} S_t\Big) \leqslant \bigwedge_{t \in T}(R \circ' S_t).$$

If operation $*$ is associative and infinitely sup-distributive, then sup-$*$ composition is associative and infinitely sup-distributive. Thus

$$R \circ \Big(\bigvee_{t \in T} S_t\Big) = \bigvee_{t \in T}(R \circ S_t), \quad \Big(\bigvee_{t \in T} S_t\Big) \circ R = \bigvee_{t \in T}(S_t \circ R). \tag{7}$$

Dually, if operation $*$ is associative and infinitely inf-distributive, then inf-$*$ composition is associative and infinitely inf-distributive. Thus

$$R \circ' \Big(\bigwedge_{t \in T} S_t\Big) = \bigwedge_{t \in T}(R \circ' S_t), \quad \Big(\bigwedge_{t \in T} S_t\Big) \circ' R = \bigwedge_{t \in T}(S_t \circ' R). \tag{8}$$

Theorem 3. (Drewniak and Kula [4]) If operation $*$ has zero element $z = 0$, neutral element $e = 1$ and is associative, infinitely sup-distributive, then $(FR(X), \circ)$ is an ordered semigroup with identity I.
Dually, if operation $*$ has zero element $z = 1$, neutral element $e = 0$ and is associative, infinitely inf-distributive, then $(FR(X), \circ')$ is an ordered semigroup with identity I'.

As examples of the above semigroups we can consider left-continuous triangular norms (case e=1) (cf. [10], p. 4). Unfortunately, property (7) is usually stated for triangular norm $*$ without additional assumptions (cf. e. g. [11], formula (5.15) or [14], formula (7.2)). Moreover, formula (7.3) in [14] wrongly states that sup-$*$ composition is infinitely inf-distributive. Thus we need some examples.

Example 2. Let $T = (0, 1)$, $c \in (0, 1)$, card $X = 2$,

$$R = \begin{bmatrix} c & 0 \\ c & c \end{bmatrix}, \quad S_t = \begin{bmatrix} t & t \\ 0 & t \end{bmatrix}.$$

Operation (cf. [10], Example 1.2)

$$x * y = \begin{cases} \min(x, y), & \max(x, y) = 1 \\ 0, & \max(x, y) < 1 \end{cases}$$

is a triangular norm but it is not left-continuous. Using sup-$*$ composition we get

$$R \circ \left(\bigvee_{t \in T} S_t\right) = c_{X\times X} > 0_{X\times X} = \bigvee_{t\in T} (R \circ S_t)$$

contradictory to (7).

Example 3. Let us consider $* = \min$, card $X = 2$, $c \in (0, 1)$,

$$R = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}, \quad S = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}, \quad U = \begin{bmatrix} c & c \\ c & c \end{bmatrix} = c_{X\times X}.$$

Since $R \wedge S = 0_{X\times X}$, $(R \wedge S) \circ U = 0_{X\times X}$, $R \circ U = S \circ U = c_{X\times X}$, then

$$(R \circ U) \wedge (S \circ U) = c_{X\times X} > 0_{X\times X} = (R \wedge S) \circ U.$$

Therefore, sup-$*$ composition is not distributive even with respect to the lattice product. So it is not infinitely inf-distributive.

3. FUZZY RELATIONS POWERS

We consider the family D of all binary operations $* : [0, 1]^2 \to [0, 1]$, which are associative and infinitely sup-distributive. As examples in D we can use left-continuous, associative, increasing operations (cf. [4]). In particular, arbitrary left-continuous triangular norm belongs to D. Using associative composition (1) we can consider powers of fuzzy relation and further operations on powers.

Definition 4. (Kaufmann [9]) Let $* \in D$. The powers of a relation R are defined by

$$R^1 = R, \quad R^{m+1} = R^m \circ R, \quad m = 1, 2, \ldots. \tag{9}$$

Its closure $R^\vee$ and interior $R^\wedge$ are defined by

$$R^\vee = \bigvee_{k=1}^{\infty} R^k, \quad R^\wedge = \bigwedge_{k=1}^{\infty} R^k. \tag{10}$$

If operation $*$ is monotonic (increasing or decreasing with respect to the first and to the second variable), then sup-$*$ and inf-$*$ compositions are monotonic of the same kind (cf. [4], Section 3). In particular, if operation $*$ is increasing or sup-distributive, then sup-$*$ composition is increasing. By mathematical induction relation powers are increasing and one has

Theorem 4. Let $* \in D$, $R, S \in FR(X)$. If $R \leqslant S$, then

$$R^n \leqslant S^n, \quad n = 1, 2, \ldots, \quad R^\vee \leqslant S^\vee, \quad R^\wedge \leqslant S^\wedge.$$

As a 'lattice' consequence we obtain

Theorem 5. If $* \in D$ and $R, S \in FR(X)$, then

$$(R \vee S)^n \geqslant R^n \vee S^n, \; (R \wedge S)^n \leqslant R^n \wedge S^n, \quad n = 1, 2, \ldots,$$

$$(R \vee S)^\vee \geqslant R^\vee \vee S^\vee, (R \vee S)^\wedge \geqslant R^\wedge \vee S^\wedge,$$

$$(R \wedge S)^\vee \leqslant R^\vee \wedge S^\vee, (R \wedge S)^\wedge \leqslant R^\wedge \wedge S^\wedge.$$

All the above inequalities can be strict for particular fuzzy relations.

Example 4. Let $* = \wedge$, $X = [0, 1]$. If we use projections $R = P_2$, $S = P_1$,

$$P_1(x, y) = x, \quad P_2(x, y) = y, \quad x, y \in [0, 1], \tag{11}$$

then we get $(R \vee S)(x, y) = x \vee y$, $(R \vee S)^2(x, z) = 1$, $x, y, z \in [0, 1]$. Since $R^2 = R$, $S^2 = S$, then $(R \vee S)^2 > R^2 \vee S^2$, which implies $(R \vee S)^\vee > R^\vee \vee S^\vee$.

Example 5. Let $* = \wedge$, card $X = 2$, $c \in (0, 1]$. We have

$$R = \begin{bmatrix} c & 0 \\ c & c \end{bmatrix}, \quad S = \begin{bmatrix} c & c \\ c & 0 \end{bmatrix}, \quad R \wedge S = \begin{bmatrix} c & 0 \\ c & 0 \end{bmatrix}.$$

Since $R^2 = R$, $S^2 = c_{X \times X}$, $(R \wedge S)^2 = R \wedge S$, then we obtain

$$(R \wedge S)^2 = R \wedge S < R = R^2 \wedge S^2, \qquad (R \wedge S)^\vee = R \wedge S < R = R^\vee \wedge S^\vee.$$

Theorem 6. Let $R \in FR(X)$. If operation $*$ is commutative, then

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}.$$

Moreover, if $* \in D$, then

$$(R^{-1})^n = (R^n)^{-1}, \quad n = 1, 2, \dots,$$

$$(R^{-1})^{\vee} = (R^{\vee})^{-1}, \quad (R^{-1})^{\wedge} = (R^{\wedge})^{-1}.$$

P r o o f. Let $x, z \in X$. We obtain

$$\forall_{x,z \in X} \ (R \circ S)^{-1}(z, x) = (R \circ S)(x, z) = \bigvee_{y \in X} R(x, y) * S(y, z)$$

$$= \bigvee_{y \in X} R^{-1}(y, x) * S^{-1}(z, y) = \bigvee_{y \in X} S^{-1}(z, y) * R^{-1}(y, x) = (S^{-1} \circ R^{-1})(z, x).$$

Now, by mathematical induction we get the formula with powers and properties of supremum and infimum in $[0, 1]$ finishes the proof. □

Example 6. Let $* = P_1$ (cf. (11)), $X = [0, 1]$, $R(x, y) = x \wedge y$, $x, y \in [0, 1]$. We have $R^{-1} = R$, $(R^{-1})^2 = R^2$, where $R^2 = P_1$, while $(R^2)^{-1} = P_2$. Therefore $(R^2)^{-1} \neq (R^{-1})^2$, which shows that we need a commutative operation $*$ in the above theorem.

Since the relation composition is not commutative, we must restrict some considerations to commuting pairs of relations, i. e. $R, S \in FR(X)$, with property $R \circ S = S \circ R$. By mathematical induction we get

Theorem 7. Let $* \in D$. If $R, S \in FR(X)$ are commuting, then all their powers also commute, i. e.

$$R^k \circ S^p = S^p \circ R^k, \quad k, p = 1, 2, \dots.$$

In particular, arbitrary two powers of R commute and

$$R^k \circ R^p = R^p \circ R^k = R^{k+p}, \quad (R^k)^p = (R^p)^k = R^{kp}, \quad k, p = 1, 2, \dots.$$

Now, using mathematical induction and properties (7), (6) we get

Theorem 8. Let $* \in D$. If R, $S \in FR(X)$ commute, then

$$(R \circ S)^n = R^n \circ S^n,\ n = 1, 2, \dots, \quad (R \circ S)^{\vee} \leqslant R^{\vee} \circ S^{\vee}, \quad (R \circ S)^{\wedge} \geqslant R^{\wedge} \circ S^{\wedge}.$$

Example 7. Let $* = \wedge$, card $X = 2$, $c \in (0, 1]$. We have

$$R = S = \begin{bmatrix} 0 & c \\ c & 0 \end{bmatrix}, \quad R^2 = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix},$$

$R \circ S = (R \circ S)^2 = (R \circ S)^{\vee} = (R \circ S)^{\wedge} = R^2$. Since $R^{\vee} = c_{X \times X}$, $R^{\wedge} = 0_{X \times X}$, then $R^{\vee} \circ S^{\vee} = R^{\vee} > (R \circ S)^{\vee}$ and $R^{\wedge} \circ S^{\wedge} = R^{\wedge} < (R \circ S)^{\wedge}$. Thus, the inequalities in Theorem 8 can be strong.

Theorem 9. If $* \in D$ and $R \in FR(X)$, then

$$R^n \circ R^\vee = R^\vee \circ R^n = (R^\vee)^{n+1}, (R^\vee)^n = \bigvee_{k=n}^{\infty} R^k \geqslant (R^n)^\vee, \tag{12}$$

$$(R^\wedge)^{n+1} \leqslant \left\{ \begin{array}{l} R^n \circ R^\wedge \\ R^\wedge \circ R^n \end{array} \right. \leqslant \bigwedge_{k=n+1}^{\infty} R^k \leqslant (R^{n+1})^\wedge, \quad n = 1, 2, \ldots . \tag{13}$$

P r o o f. By infinite sup-distributivity (7) we obtain

$$R \circ R^\vee = R \circ \left(\bigvee_k R^k\right) = \bigvee_k R^{k+1} = \bigvee_{k=2}^{\infty} R^k,$$

$$R^\vee \circ R = \left(\bigvee_k R^k\right) \circ R = \bigvee_k (R^k \circ R) = \bigvee_{k=2}^{\infty} R^k,$$

which proves that R and $R^\vee$ are commuting and by Theorem 7 we get the first part of (12). The second part is obtained by mathematical induction using also the infinite sup-distributivity (7). Inequalities (13) can be obtained in a similar way using the sub-distributivity from (6). □

Example 8. Let $* = \wedge$, card $X = 3$, $c \in (0, 1]$. We have

$$R = \begin{bmatrix} 0 & 0 & c \\ c & c & 0 \\ 0 & c & 0 \end{bmatrix}, \quad R^2 = \begin{bmatrix} 0 & c & 0 \\ c & c & c \\ c & c & 0 \end{bmatrix},$$

$$R^3 = \begin{bmatrix} c & c & 0 \\ c & c & c \\ c & c & c \end{bmatrix}, \quad R^\wedge = \begin{bmatrix} 0 & 0 & 0 \\ c & c & 0 \\ 0 & c & 0 \end{bmatrix},$$

$R^4 = R^\vee = c_{X \times X}$,

$$R^\wedge \circ R = \begin{bmatrix} 0 & 0 & 0 \\ c & c & c \\ c & c & 0 \end{bmatrix}, \quad R \circ R^\wedge = \begin{bmatrix} 0 & c & 0 \\ c & c & 0 \\ c & c & 0 \end{bmatrix}.$$

This shows that $R \circ R^\wedge \neq R^\wedge \circ R$ (fuzzy relations from both sides are incomparable). In a similar way we can check that inequalities from the above theorem can be strong.

Theorem 10. If $* \in D$ and $R \in FR(X)$, then

$$(R^\vee)^\vee = R^\vee, \quad (R^\wedge)^\wedge \leqslant R^\wedge, \quad (R^\wedge)^\vee \leqslant (R^\vee)^\wedge. \tag{14}$$

P r o o f. Using properties (12) and (13) we obtain

$$(R^\vee)^\vee = \bigvee_{k=1}^{\infty} (R^\vee)^k = \bigvee_{k=1}^{\infty} \left(\bigvee_{i \geqslant k} R^i\right) = \bigvee_{i=1}^{\infty} R^i = R^\vee,$$

$$(R^\wedge)^\wedge = \bigwedge_{k=1}^{\infty} (R^\wedge)^k \leqslant \bigwedge_{k=1}^{\infty} \left(\bigwedge_{i \geqslant k} R^i\right) = \bigwedge_{i=1}^{\infty} R^i = R^\wedge,$$

$$(R^{\wedge})^{\vee} = \bigvee\nolimits_{k=1}^{\infty} (R^{\wedge})^k \leqslant \bigvee\nolimits_{k=1}^{\infty} \bigwedge\nolimits_{i \geqslant k} R^i \leqslant \bigwedge\nolimits_{i \geqslant 1} \bigvee\nolimits_{k \leqslant i} R^k,$$
$$(R^{\vee})^{\wedge} = \bigwedge\nolimits_{k=1}^{\infty} (R^{\vee})^k = \bigwedge\nolimits_{k=1}^{\infty} \bigvee\nolimits_{i \geqslant k} R^i = \bigwedge\nolimits_{i=1}^{\infty} \bigvee\nolimits_{k \leqslant i} R^k,$$

which proves all parts of (14). □

Example 9. Let $* = \wedge$, card $X = 3$, $c \in (0, 1]$. Using fuzzy relation R from Example 8 we get

$$(R^{\wedge})^{\vee} = \begin{bmatrix} 0 & 0 & 0 \\ c & c & 0 \\ c & c & 0 \end{bmatrix} \quad < \quad (R^{\vee})^{\wedge} = c_{X \times X}.$$

Similarly we get

$$S = \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & c \\ c & c & 0 \end{bmatrix}, \quad S^2 = \begin{bmatrix} 0 & 0 & c \\ c & c & 0 \\ 0 & c & c \end{bmatrix}, \quad S^3 = \begin{bmatrix} c & c & 0 \\ 0 & c & c \\ c & c & c \end{bmatrix},$$
$$S^4 = \begin{bmatrix} 0 & c & c \\ c & c & c \\ c & c & c \end{bmatrix}, \quad S^5 = c_{X \times X}, \quad S^{\wedge} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{bmatrix}.$$

Thus $S^{\wedge} > 0_{X \times X} = (S^{\wedge})^{\wedge}$.

4. FUZZY RELATION POWERS ON A FINITE SET

In the case of finite set X (cf. Example 1) we can simplify the formula (10) of relation closure.

Lemma 1. (Li [13]) If $* \in D$ and $R \in FR(X)$, then

$$\underset{m}{\forall} \underset{x,z \in X}{\forall} R^m(x, z) = \bigvee\nolimits_{y_1, \ldots, y_{m-1}} (*)_{p=1}^{m} R(y_{p-1}, y_p), \tag{15}$$

where $y_0 = x$, $y_m = z$.

Lemma 2. If $* \in D$, $* \leqslant \min$, card $X = n$ and $R \in FR(X)$, then

$$\underset{m \geqslant n}{\forall} \underset{x,z \in X}{\forall} \underset{q \leqslant n}{\exists} R^m(x, z) \leqslant R^q(x, z).$$

Proof. Since operation $*$ is increasing and $* \leqslant \min$, then

$$a * b * c \leqslant a * b, \quad a, b, c \in [0, 1]. \tag{16}$$

Let $m \geqslant n$, $x, z \in X$. Using Lemma 1 we get $m + 1 > n$ elements $y_0, y_1, \ldots, y_m$ and there exist indices $i \leqslant k$ such that $y_i = y_k$. Using inequality (16) we can omit $k - i$ factors and obtain

$$\begin{aligned} & R(x, y_1) * \cdots * R(y_{i-1}, y_i) * \cdots * R(y_k, y_{k+1}) * \cdots * R(y_{m-1}, y_z) \\ \leqslant \quad & R(x, y_1) * \cdots * R(y_{i-1}, y_i) * R(y_k, y_{k+1}) * \cdots * R(y_{m-1}, y_z). \end{aligned}$$

Putting $q = m - (k - i)$ we have $R^m(x, z) \leqslant R^q(x, z)$ according to (15). After finite number of steps we obtain such inequality with $q \leqslant n$, which finishes the proof. $\square$

Directly from the above lemma we get

Lemma 3. If $* \in D$, $* \leqslant \min$ and card $X = n$, then

$$R^m \leqslant \bigvee_{k=1}^{n} R^k \text{ for } m = 1, 2, \ldots$$

Theorem 11. (cf. Kaufmann [9]) If $* \in D$, $* \leqslant \min$ and card $X = n$, then

$$R^{\vee} = \bigvee_{k=1}^{n} R^k. \tag{17}$$

P r o o f. Let

$$P = \bigvee_{k=1}^{n} R^k.$$

By Lemma 3 one has $P \leqslant R^{\vee} \leqslant P$, which proves (17). $\square$

5. CONCLUDING REMARKS

Basic properties of fuzzy relations were considered by L. A. Zadeh [20] and A. Kaufmann [9]. We consider here only three examples of such properties for a short presentation of possible results.

Definition 5. (Kaufmann [9], p. 16) Let $R \in FR(X)$. Relation R is reflexive, if $I \leqslant R$, symmetric, if $R = R^{-1}$ and $*$-transitive, if $R \circ R \leqslant R$.

Theorem 12. (Drewniak [3]) Let $* \in D$, $n \in \mathbb{N}$. If relation R is reflexive, then relations R^{-1}, R^n, $R^{\vee}$, $R^{\wedge}$ are reflexive. If R is symmetric, then R^{-1}, R^n, $R^{\vee}$, $R^{\wedge}$ are symmetric. If R is $*$-transitive, then R^{-1}, R^n, $R^{\wedge}$ are $*$-transitive.

As a consequence of property (14) we also get

Theorem 13. Let $* \in D$. Closure $R^{\vee}$ is a $*$-transitive fuzzy relation for arbitrary $R \in FR(X)$.

We have summarized some results on sup-$*$ powers of fuzzy relations. It is a presentation complementary to paper [2], where results are connected with the problem of convergence of powers of fuzzy relations on a finite set. Our aim was to complete formulas and inequalities useful in calculation on fuzzy relation powers. We have simultaneously discussed necessary assumptions with suitable counterexamples. Our examination will be continued for powers of L-fuzzy relations (cf. [8]) or matrices over residuated lattices (cf. [18]).

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