

A NEW CHARACTERIZATION OF GEOMETRIC DISTRIBUTION

SUDHANSU S. MAITI AND ATANU BISWAS

A characterization of geometric distribution is given, which is based on the ratio of the real and imaginary part of the characteristic function.

Keywords: discrete distribution, exponential, lack of memory

AMS Subject Classification: 62E10

1. INTRODUCTION

The geometric distribution, given by the cumulative distribution function (cdf)

$$F(x) = \Pr(X \leq x) = \begin{cases} 1 - \theta^{x+1}, & \text{if } x = 0, 1, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

and the probability mass function (pmf)

$$f(x) = (1 - \theta)\theta^x, \quad x = 0, 1, \dots, \quad (1)$$

for $0 < \theta < 1$, is the discrete analog of exponential distribution. Clearly, $F(x)$ is right continuous. If X follows an exponential distribution, $[X]$, the integer part of X , has a geometric distribution (see Kalbfleish and Prentice, [12], Chapt. 3). The exponential distribution is widely referenced probability law used in reliability and life testing for continuous data as the simplest choice. Exponential distribution has several nice properties by which the statistical analyses become simpler. When the lives of some equipment and components are being measured by the number of completed cycles of operations or strokes, or in case of periodic monitoring of continuous data, the geometric distribution is a natural choice. It possesses most of the nice properties of the exponential distribution, of course in the discrete set up.

Geometric distribution is characterized by the discrete version of the *lack of memory* and the constant hazard rate properties, which is also satisfied by the exponential distribution. Xekalaki [19], Hitha and Nair [11] and Roy and Gupta [17] have examined some characterization results for discrete models. Some characterizations of the geometric distribution are given by Rogers [16], Clawford [3], Srivastava [18],

Galambos [9], El-Newehi and Govindarajulu [4], Rao and Sreehari [15], Arnold [1], Ferguson [5, 6, 7] and Fosam and Shanbhag [8], among others. Almost all these characterizations are on the results of the order statistics and the lack of memory property. In the present paper, we give an alternative characterization in a completely different view point.

We consider the form

$$\phi(t) = C(t) + iS(t),$$

the natural expression of any characteristic function (cf) $\phi(t)$, with $S(t) = E(\sin tX)$ and $C(t) = E(\cos tX)$ are the imaginary and the real parts of $\phi(t)$. Meintanis and Iliopoulos [13] illustrated that $S(t)/C(t)$ is linear in t for exponential distribution. Here we are interested to see whether a similar result in the discrete set up characterizes the geometric distribution, the discrete analog of the exponential distribution. Then, it might be straightforward to use the geometric distribution in the discrete life testing problems. In this short note, we present a characterization of the geometric distribution based on the ratio $S(t)/C(t)$. The result is given in Section 2. Section 3 concludes.

2. THE CHARACTERIZATION

Note that, for the geometric distribution (1), the cf is given by

$$\begin{aligned} \phi(t) = E(\exp(i tX)) &= (1 - \theta)(1 - \theta \exp(it))^{-1} \\ &= (1 - \theta) \left[\sum_{j=0}^{\infty} \theta^j \cos jt + i \sum_{j=0}^{\infty} \theta^j \sin jt \right] \\ &= C(t) + iS(t). \end{aligned}$$

We first state the following Theorem from Rainville ([14], Chapt. 8, pp. 129–130).

Theorem 1. If

$$f_1(x) = \sum_{j=0}^{\infty} a_j x^j \quad \text{in } |x| < R_1,$$

and

$$f_2(x) = \sum_{j=0}^{\infty} b_j x^j \quad \text{in } |x| < R_2,$$

and if $b_0 \neq 0$, then

$$\frac{f_1(x)}{f_2(x)} = \sum_{j=0}^{\infty} q_j x^j \quad \text{in } |x| < R,$$

where $R = \min\{R_1, R_2, |z|\}$, with z being the zero of $f_2(x)$ nearest to $x = 0$.

The q_j 's are determined as follows:

$$q_0 = a_0/b_0,$$

and for $j \geq 1$,

$$b_0 q_j = a_j - \sum_{u=1}^{\infty} b_u q_{j-u}.$$

Now we state a special case of Cantor's Theorem from Bary ([2], Chapt. II, p. 193).

Lemma 1. Let $g(x)$ be a function defined at non-negative integers. Then

$$\sum_{j=0}^{\infty} \sin(jt)g(j) = 0 \text{ for all } t,$$

implies $g(j) = 0$ for $j = 1, 2, \dots$

Now we present the characterization theorem as follows.

Theorem 2. Among all distributions of nonnegative integer valued random variables, the geometric distribution is the only one for which

$$S(t) = \frac{\theta \sin t}{1 - \theta \cos t} C(t) \text{ for all } t.$$

Proof. In Theorem 1, we put $f_1 = S(t)$, $f_2 = C(t)$, $a_j = \sin jt$, $b_j = \cos jt$, $b_0 = 1$ and $R_1 = R_2 = 1$. Under this set up z comes out to be unity. Consequently, we have

$$\frac{S(t)}{C(t)} = \sum_{j=0}^{\infty} q_j \theta^j,$$

with

$$q_0 = a_0/b_0 = 0$$

and for $j \geq 1$,

$$q_j = \sin jt - \sum_{k=1}^{n-1} q_{j-k} \cos kt.$$

Clearly,

$$q_1 = \sin t,$$

$$q_2 = \sin 2t - \sin t \cos t = \sin t \cos t.$$

Suppose, for $j = 1, \dots, n$, $q_j = \sin t \cos^{j-1} t$, which holds for $j = 1, 2$.

Hence,

$$\begin{aligned}
 q_{n+1} &= \sin(n+1)t - \sin t \sum_{k=1}^n \cos kt \cos^{n-k} t \\
 &= \cos t \left\{ \sin nt - \sin t \sum_{k=1}^{n-1} \cos kt \cos^{n-1-k} t \right\} \\
 &= \dots\dots\dots \\
 &= \sin t \cos^n t.
 \end{aligned}$$

Thus, $q_j = \sin t \cos^{j-1} t$ for all j .

Consequently, we get

$$\frac{S(t)}{C(t)} = \sum_{j=1}^{\infty} \theta^j \sin t \cos^{j-1} t = \frac{\theta \sin t}{1 - \theta \cos t},$$

as $|\theta \cos t| < 1$.

Now, we prove the reverse part. If

$$\frac{S(t)}{C(t)} = \frac{\theta \sin t}{1 - \theta \cos t},$$

we immediately have

$$(1 - \theta \cos t)E(\sin tX) = (\theta \sin t) E(\cos tX),$$

which gives

$$\begin{aligned}
 0 &= E(\sin tX) - \theta E(\sin(X+1)t) \\
 &= \sum_{j=0}^{\infty} \sin jt [f(j) - \theta f(j-1)].
 \end{aligned}$$

Since this is true for all t , from special case of Cantor's Theorem stated earlier, we immediately get $f(j) - \theta f(j-1) = 0$ for $j = 1, 2, \dots$, and hence

$$f(j) = \theta f(j-1) = \theta^j f(0),$$

which, together with $\sum_{j=0}^{\infty} f(j) = 1$, gives geometric pmf (1) for $f(j)$. Hence the result follows. □

3. CONCLUDING REMARK

In this short note, a different characterization of geometric distribution, through the ratio of imaginary and real parts of the cf, is provided. Note that, for an exponential distribution with mean $1/\theta$, we have

$$S(t)/C(t) = \theta t \quad \text{for all } t.$$

(See Meintanis and Iliopoulos [13].) It is interesting to note that, unlike the case of exponential, the characterization of geometric distribution is not linear in t for $S(t)/C(t)$. It is a periodic function with period 2π . The characterization of exponential and its discrete analog (geometric) are quite different.

The application of this result will be based on the empirical cf $\phi_n(t) = n^{-1} \sum_{j=1}^n \exp(itX_j)$, where X_1, \dots, X_n are random samples. A goodness-of-fit test of the empirical cf, as in the line of Henze and Meintanis [10] is under study. We hope to pursue this in a future communication.

ACKNOWLEDGEMENTS

The authors wish to thank two anonymous referees for their careful reading and valuable suggestions, which led some improvement over an earlier version of the paper. We also thank Professor S. C. Bagchi for some fruitful discussion during the preparation of the paper.

(Received September 14, 2005.)

REFERENCES

-
- [1] B. C. Arnold: Two characterizations of the geometric distribution. *J. Appl. Probab.* *17* (1980), 570–573.
 - [2] N. K. Bary: *A Treatise on Trigonometric Series*. Pergamon Press, Oxford 1964.
 - [3] G. B. Crawford: Characterization of geometric and exponential distributions. *Ann. Math. Statist.* *37* (1966), 1790–1795.
 - [4] E. El-Newehi and Z. Govindarajulu: Characterizations of geometric distributions and discrete IFR (DFR) distributions using order statistics. *J. Statist. Plann. Inf.* *3* (1979), 85–90.
 - [5] T. S. Ferguson: A characterization of the geometric distribution. *Amer. Math. Monthly* *71* (1965), 256–260.
 - [6] T. S. Ferguson: On characterizing distributions by properties of order statistics. *Sankhya Series A* *20* (1967), 265–278.
 - [7] T. S. Ferguson: On a Rao-Shanbhag characterization of the exponential/geometric distribution. *Sankhya Series A* *64* (2002), 247–255.
 - [8] E. B. Fosam and D. N. Shanbhag: Certain characterizations of exponential and geometric distributions. *J. Royal Statist. Soc. Series B* *56* (1994), 157–160.
 - [9] J. Galambos (1975): Characterizations of probability distributions by properties of order statistics. In: *Statistical Distributions in Scientific Work, Vol. 2: Characterizations and Applications* (G. P. Patil, S. Kotz, and J. K. Ord, eds.), D. Reidel, Boston 1975, II, pp. 89–101.
 - [10] N. Henze and S. G. Meintanis: Goodness-of-fit tests based on a new characterization of the exponential distribution. *Commun. Statist. – Theory and Methods* *31* (2002), 1479–1497.
 - [11] N. Hitha and U. N. Nair: Characterization of some discrete models by properties of residual life function. *Cal. Statist. Assoc. Bulletin* *38* (1989), 219–223.
 - [12] J. D. Kalbfleish and R. L. Prentice: *The Statistical Analysis of Failure Time Data*. Wiley, New York 1980.
 - [13] S. G. Meintanis and G. Iliopoulos: Characterizations of the exponential distribution based on certain properties of its characteristic function. *Kybernetika* *39* (2003), 295–298.

- [14] E. D. Rainville: Infinite Series. The Macmillan Company, New York 1967.
- [15] B. L. S. P. Rao and M. Sreehari: On some properties of geometric distribution. Sankhya Series A *42* (1980), 120–122.
- [16] G. S. Rogers: An alternate proof of the characterization of the density Ax^B . Amer. Math. Monthly *70* (1963), 857–858.
- [17] D. Roy and R. P. Gupta: Stochastic modeling through reliability measures in the discrete case. Statist. Probab. Letters *43* (1999), 197–206.
- [18] R. C. Srivastava: Two characterizations of the geometric distribution. J. Amer. Statist. Assoc. *69* (1974), 267–269.
- [19] E. Xekalaki: Hazard function and life distributions in discrete time. Commun. Statist. – Theory and Methods *12* (1983), 2503–2509.

Sudhansu S. Maiti, Department of Statistics, Visva-Bharati University, Santiniketan – 731 235. India.

e-mail: dssm1@rediffmail.com

Atanu Biswas, Applied Statistics Unit, Indian Statistical Institute, 203 B.T. Road, Kolkata – 700 108. India.

e-mail: atanu@isical.ac.in