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# ON BAYESIAN ESTIMATION IN AN EXPONENTIAL DISTRIBUTION UNDER RANDOM CENSORSHIP

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The paper gives some basic ideas of both the construction and investigation of the properties of the Bayesian estimates of certain parametric functions of the parent exponential distribution under the model of random censorship assuming the Koziol–Green model. Various prior distributions are investigated and the corresponding estimates are derived. The stress is put on the asymptotic properties of the estimates with the particular stress on the Bayesian risk. Small sample properties are studied via simulations in the special case.

*Keywords:* exponential distribution, random censoring, survival data analysis, reliability, Koziol–Green model, Bayesian estimates, Bayesian risk, conjugate priors, asymptotic properties, small sample properties, simulation study

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## 1.. INTRODUCTION AND PRELIMINARIES

The exponential distribution is still one of the most popular distribution in survival data analysis and has been extensively studied by many authors. The basic ideas are given in [7]. The comparison of various reliability estimates from the confidential point of view has been given in [6]. A nice test of fit with the Koziol–Green model for random censorship has been suggested by Herbst [5]. More advanced models are treated in Franz [2]. A review of the topic can be found in [8]. Since the processes studied in reliability theory and survival data analysis are rather evolutionary than revolutionary, the prior information seems to be useful to improve the inference. The Bayesian approach is one possible way to implement a prior information into the model. In estimating reliability function and parameter of exponential distribution, Sarhan [13] exploits past experiments to approximate prior density. Liang [10] deals with random censorship with exponentially distributed censor, i. e. in the setting (1..3) but with known parameter of censoring distribution and in fact with restriction $\gamma < 1$ ($p > 1/2$) imposed by a prior. In [1] Jeffreys priors under several censoring mechanisms are derived, Bayesian estimates in the case of exponential distribution being treated in detail. Bayesian estimation for parameters of generalized exponential distribution under Type II censorship is dealt with in [12]. The present paper discusses Bayesian estimation in the exponential distribution under the Koziol–Green model of censorship. Several priors are proposed and corresponding estimators of characteristics of the distribution and the model are derived. Properties of the estimators are expressed in terms of almost sure convergence, asymptotic normality and Bayesian risks. Weak asymptotics of the Bayesian reliability estimator considered as a stochastic process is under the conjugate prior (2..3) studied in [4].

Let $X_1, \dots, X_n$ be independent identically distributed (i.i.d.) random variables (r.v.'s) with the distribution function $F$, the density function $f$ and let $T_1, \dots, T_n$ be i.i.d. r.v. which are independent of $X_j'$s and possess the distribution function $G$ and the density function $g$. In the model of random censorship we can only observe the i.i.d. pairs

$$(W_1, I_1), \dots, (W_n, I_n), \tag{1..1}$$

where $W_j = \min(X_j, T_j)$, $I_j = I\{X_j \le T_j\}$, $j = 1, \dots, n$. R.v.'s $X_j$ usually represent the lifetimes or times-to-failure while $T_j'$s represent time censors. The pair $(W_1, I_1)$ has the distribution with the density function

$$h(w, i) = \{f(w)[1 - G(w)]\}^i \{g(w)[1 - F(w)]\}^{1-i}, \qquad w \in \mathbb{R}, \quad i = 0, 1 \tag{1..2}$$

with respect to Lebesgue×counting product measure.

In the *Koziol–Green model* [9] it is supposed that the distributions of $X_j'$s and $T_j'$s are connected by

$$1 - G(t) = [1 - F(t)]^\gamma \tag{1..3}$$

for some $\gamma > 0$. In this case $W_1$ and $I_1$ are independent (see Herbst [5], e. g.). Instead of $\gamma$, we can consider the parameter

$$p = \Pr[X \le T], \tag{1..4}$$

$p \in (0, 1)$, since $p = 1/(1 + \gamma)$. The density (1..2) then becomes

$$f(w)[1 - F(w)]^\gamma \gamma^{1-i} \qquad w \in \mathbb{R}, \quad i = 0, 1. \tag{1..5}$$

In the present paper we suppose that $X_j'$s have an exponential distribution $\mathrm{Exp}(\theta)$, i. e., that they possess the density function

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta}, \qquad x > 0, \tag{1..6}$$

with the expectation $\mathrm{E}\,\mathrm{Exp}(\theta) = \theta$ and variance $\mathrm{var}\,\mathrm{Exp}(\theta) = \theta^2$, or after introducing a new parameter $\lambda = 1/\theta$

$$f(x; \lambda) = \lambda e^{-\lambda x}, \qquad x > 0, \tag{1..7}$$

where $\lambda$ represents the hazard rate of the distribution in question. Note that under the above assumptions $W_1$ possesses an $\mathrm{Exp}(p/\lambda)$ and $I_1$ is a zero-one r.v. with the parameter $p$. Also we will pay attention to the *reliability function*

$$R = e^{-\lambda} \tag{1..8}$$

at the mission time set to $t := 1$. Next we will mention further distributions used in this paper.

*Gamma distribution* $G(a, q)$**.** Gamma distribution with the parameters $a > 0$, $q > 0$ has the density function

$$\frac{a^q}{\Gamma(q)} x^{q-1} e^{-ax}, \qquad x > 0$$

with the expectation $\mathrm{E}\, G(a, q) = q/a$ and the variance $\mathrm{var}\, G(a, q) = q/a^2$.

*Logarithmic Gamma Distribution* $LG(a, q)$**.** If the distribution of the r.v. $X$ is $G(a, q)$ then the distribution of $Y = e^{-X}$ is $LG(a, q)$ with the density function

$$\frac{a^q}{\Gamma(q)} (-\ln y)^{q-1} y^a \frac{1}{y}, \qquad y \in (0, 1)$$

with the expectation

$$\mathrm{E}\, LG(a, q) = \left( \frac{a}{a+1} \right)^q$$

and the variance

$$\mathrm{var}\, LG(a, q) = \left( \frac{a}{a+2} \right)^q - \left( \frac{a}{a+1} \right)^{2q}.$$

*Inverse Gamma distribution* $IG(a, q)$**.** If the distribution of the r.v. $X$ is $G(a, q)$ then the distribution of $Z = 1/X$ is $IG(a, q)$ with the density function

$$\frac{a^q}{\Gamma(q)} \frac{1}{z^{q+1}} e^{-a/z}, \qquad z > 0.$$

If $q > 1$ then the expectation is $\mathrm{E}\, IG(a, q) = a/(q-1)$. If $q > 2$ then the variance is

$$\mathrm{var}\, IG(a, q) = \frac{a^2}{(q-1)^2(q-2)}.$$

*Beta distribution* $B(r, s)$**.** Beta distribution with the parameters $r > 0$, $s > 0$ has the density function

$$\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} x^{r-1} (1-x)^{s-1} \ , x \in (0, 1).$$

with the expectation $\mathrm{E}\, B(r, s) = r/(r+s)$ and the variance

$$\mathrm{var}\, B(r, s) = \frac{rs}{(r+s)^2(r+s+1)}.$$

*Beta distribution of the second order* $B2(r, s)$**.** If the distribution of the r.v. $X$ is $B(r, s)$, then the distribution of $V = 1/X - 1$ is $B2(r, s)$ with the density function

$$\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} \frac{v^{s-1}}{(1+v)^{r+s}}, \qquad v > 0.$$

If $r > 1$ then the expectation is $\mathrm{E}\, B2(r, s) = s/(r-1)$. If $r > 2$ then the variance is

$$\mathrm{var}\, B2(r, s) = \frac{(s+1)s}{(r-1)(r-2)} - \left( \frac{s}{r-1} \right)^2 = \frac{s(r+s-1)}{(r-1)^2(r-2)}.$$

**Likelihood.** In Bayesian inference we deal with the likelihood function which, under the above assumptions, reads

$$L(\lambda,\gamma;(W_1,I_1),\ldots,(W_n,I_n)) = \prod_{j=1}^{n}(\lambda e^{-\lambda W_j}(e^{-\lambda W_j})^{\gamma})\gamma^{1-I_j}.$$

If we denote

$$W = \sum_{j=1}^{n} W_j\,, \quad I = \sum_{j=1}^{n} I_j \tag{1..9}$$

then the likelihood becomes

$$L(\lambda,\gamma;W,I) = \lambda^n e^{-\lambda W} e^{-\lambda\gamma W}\gamma^{n-I}, \quad \lambda>0,\ \gamma>0. \tag{1..10}$$

For given $\lambda$, $p$, or $\gamma$ we obtain the expected values and variances of the conditional distributions of $I_1$ and $W_1$:

$$\mathrm{E}[I_1|\lambda,p] = p, \qquad \mathrm{var}[I_1|\lambda,p] = p(1-p)$$

and

$$\mathrm{E}[W_1|\lambda,p] = p/\lambda, \qquad \mathrm{var}[W_1|\lambda,p] = p^2/\lambda^2.$$

These facts will be utilized for calculating the unconditional expectations of $I$ and $W$ (which are the sums of i.i.d. with the same distributions as $I_1$ and $W_1$) utilizing the a priori knowledge. Also we will use them to establish the law of large numbers in particular cases.

**Bayesian estimation.** We restrict ourselves to the Bayesian estimates under the quadratic loss function, i.e. the estimator minimizing Bayesian risk function $\mathrm{E}(\tau-\widehat{\tau})^2$ if $\tau$ is the parameter in question. Thus the Bayesian estimate is simply the expected value of $\tau$ with respect to the posterior distribution, $\widehat{\tau} = \mathrm{E}(\tau|W,I)$, in our case. The Bayesian risk of this estimate can be expressed in different ways as

$$\begin{aligned}\varrho^\star_\tau &= \mathrm{E}(\mathrm{E}[\tau-\mathrm{E}(\tau|W,I)]^2|W,I) = \mathrm{E}[\tau-\mathrm{E}(\tau|W,I)]^2\\ &= \mathrm{E}\ \mathrm{var}(\tau|W,I) = \mathrm{var}\,\tau - \mathrm{var}[\mathrm{E}(\tau|W,I)] = \mathrm{var}\,\tau - \mathrm{var}\,\widehat{\tau}.\end{aligned} \tag{1..11}$$

As for computational aspects, the formula

$$\varrho^\star_\tau = \mathrm{E}\ \mathrm{var}(\tau|W,I) \tag{1..12}$$

is helpful if we know the posterior variance. In words, we just take the expectation of it.

## 2.. PRIORS AND BAYESIAN ESTIMATORS

### Conjugate prior

The natural conjugate prior for (1..10) is the system of densities

$$\left\{\frac{a^{c+1}}{\Gamma(c-b+1)\Gamma(b)}\lambda^c e^{-\lambda a}e^{-\lambda\gamma a}\gamma^{c-b}; a>0,\ b>0,\ c>b-1\right\} \tag{2..1}$$

or, after changing the parameters $a = a$, $b = r$, and $c = s + r - 1$

$$\boldsymbol{K}_{\lambda,\gamma} = \left\{ \frac{a^{r+s}}{\Gamma(r)\Gamma(s)} \lambda^{r+s-1} e^{-\lambda(1+\gamma)a} \gamma^{s-1}; a > 0,\, r > 0,\, s > 0 \right\}, \tag{2..2}$$

see also Franz [2].

Denote

$$K_{a,r,s}(\lambda,\gamma) = \frac{a^{r+s}}{\Gamma(r)\Gamma(s)} \lambda^{r+s-1} e^{-\lambda(1+\gamma)a} \gamma^{s-1} \tag{2..3}$$

the corresponding density function.

**Theorem 2.1.** The system $\boldsymbol{K}_{\lambda,\gamma}$ is conjugate with $L(\lambda,\gamma|W,I)$, the marginal distribution of $\lambda$ is $G(a,r)$ and that of $\gamma$ is $B2(r,s)$.

Proof. The proof is obvious. □

*Remark.* It follows from the above Theorem that $\mathrm{E}\,\lambda = r/a$ and for $r > 1$, $\mathrm{E}\,\gamma = s/(r-1)$. Moreover, the conditional distributions are

$$K_{a,r,s}(\lambda|\gamma) \sim \lambda^{r+s-1} e^{-\lambda(1+\gamma)a} \sim G((1+\gamma)a, r+s),$$
$$K_{a,r,s}(\gamma|\lambda) \sim \gamma^{s-1} e^{-\gamma\lambda a} \sim G(\lambda a, s).$$

For $r > 1$ we have cov $(\lambda,\gamma) = -\frac{s}{a}\frac{1}{r-1}$ and for $r > 2$ we have corr $(\lambda,\gamma) = -\sqrt{\frac{s(r-2)}{r(r+s-1)}}$.

**Theorem 2.2.** If we choose the prior density as $q(\lambda,\gamma) = K_{a,r,s}(\lambda,\gamma) \in \boldsymbol{K}_{\lambda,\gamma}$ then the a posteriori density is $q(\lambda,\gamma|W,I) = K_{a+W,r+I,s+n-I}(\lambda,\gamma) \in \boldsymbol{K}_{\lambda,\gamma}$. The corresponding Bayesian estimates under the quadratic loss function are

$$\widehat{\lambda} = \frac{I+r}{W+a}, \qquad \widehat{\gamma} = \frac{n-I+s}{I+r-1},$$
$$\widehat{R} = \left(\frac{W+a}{W+a+1}\right)^{I+r}, \quad \widehat{\theta} = \frac{W+a}{I+r-1} \text{ for } r > 1, \quad \widehat{p} = \frac{I+r}{n+r+s}.$$

Proof. The form of the a posteriori density follows from the construction of the conjugate priors. The first two estimates are simply the expected values with respect to the a posteriori densities. After substitution $R = e^{-\lambda}$, $\theta = \lambda^{-1}$, and $p = (1+\gamma)^{-1}$ the marginal priors are $K_{a,r,s}(R) \sim LG(a,r)$, $K_{a,r,s}(\theta) \sim IG(a,r)$, and $K_{a,r,s}(p) \sim B(r,s)$, respectively. Therefore, the respective posterior distributions are $K_{a+W,r+I,s+n-I}(R)$, $K_{a+W,r+I,s+n-I}(\theta)$, and $K_{a+W,r+I,s+n-I}(p)$. Taking expectations of these distributions we obtain the remaining estimates. □

*Remark.* The expectations of $I_1$ and $W_1$ are

$$\mathrm{E}\,I_1 = \mathrm{E}\,p = r/(r+s),$$
$$\mathrm{E}\,W_1 = \mathrm{E}(p/\lambda) = \mathrm{E}\,\frac{1}{\lambda(1+\gamma)} = \frac{a}{r+s-1}, \quad \text{if } r+s > 1. \tag{2..4}$$

**Independent priors**

A flexible system of independent priors is given by the density function

$$q(\lambda,\gamma)=\frac{a^r}{\Gamma(r)}\lambda^{r-1}e^{-a\lambda}\frac{b^s}{\Gamma(s)}\gamma^{s-1}e^{-b\gamma},\quad a>0,\ r>0,\ b>0,\ s>0, \tag{2..5}$$

i. e., $\lambda$ and $\gamma$ are independent with distributions $G(a,r)$ and $G(b,s)$, respectively. The corresponding posterior distribution possesses the density function

$$q(\lambda,\gamma|W,I)\sim\lambda^{n+r-1}e^{-\lambda(a+w)}e^{-\lambda\gamma w}e^{-b\gamma}\gamma^{n-i+s-1}. \tag{2..6}$$

In this case, it is not possible to express the usual Bayesian estimates in a close form with integrals evaluated. Still we are able to give so called *Bayesian estimates of the maximum likelihood type* or *generalized Bayesian estimates* which maximize the posterior density function.

**Theorem 2.3.** Suppose that $\lambda$ and $\gamma$ possess the density function (2..5), $n+r>1$, and $n-I+s-1>1$. Then

$$\widetilde{\lambda}=\frac{1}{2}\left[\frac{I+r-s}{a+W}-\frac{b}{W}+\sqrt{\frac{(I+r-s)^2}{(a+W)^2}+\frac{b^2}{W^2}+\frac{2b\,(2(n+r-1)-(I+r-s))}{W(a+W)}}\,\right]$$

and

$$\widetilde{\gamma}=\frac{1}{2}\left[-\frac{I+r-s}{b}-\frac{a+W}{W}+\sqrt{\frac{(I+r-s)^2}{b^2}+\frac{(a+W)^2}{W^2}+2\frac{I+r-s}{b}\frac{a+W}{W}+\frac{4(n-I+s-1)(a+W)}{Wb}}\right]$$

are the generalized Bayesian estimates.

P r o o f. The logarithm of the investigated posterior density function

$$\ell(\lambda,\gamma)=(n+r-1)\ln\lambda-\lambda(a+w)-\lambda\gamma W-b\gamma+(n-I+s-1)\ln\gamma.$$

takes its maximum at

$$\gamma(\lambda)=\frac{n-I+s-1}{b+W\lambda},$$

for fixed $\lambda>0$. Further,

$$\frac{\mathrm{d}\ell(\lambda,\frac{n-I+s-1}{b+W\lambda})}{\mathrm{d}\lambda}=\frac{n+r-1}{\lambda}-(a+W)-W\frac{n-I+s-1}{b+W\lambda}=\left.\frac{\partial\ell(\lambda,\gamma)}{\partial\lambda}\right|_{\gamma=\gamma(\lambda)}.$$

Multiplying the last expression by $\lambda(b+W\lambda)$ and setting it to zero we get the quadratic equation

$$-\lambda^2\left(W(a+W)\right)+\lambda\left(W(I+r-s)-b(a+W)\right)+b(n+r-1)=0$$

which has two real roots with the opposite signs so that only the positive one, i. e. $\widetilde{\lambda}$, is meaningful. Then $\widetilde{\gamma} = \gamma(\widetilde{\lambda})$ provides the desired estimate for $\gamma$. It is not difficult to show that the pair really maximizes the posterior density. □

Consider another system of independent priors for the pair $(\lambda, p)$:

$$q(\lambda, p) \sim \lambda^{q-1} e^{-a\lambda} p^{r-1} (1-p)^{s-1},\ \lambda > 0,\ p \in (0,1), \tag{2..7}$$

where $a > 0$ and $q$, $r$, $s$ are arbitrary prior parameters. In case $a > 0$, $q > 0$, $r > 0$, $s > 0$ we have $\lambda \sim G(a, q)$, $p \sim B(r, s)$ and $\lambda$ and $\gamma$ remain independent with distributions $\lambda \sim G(a, q)$ and $\gamma \sim B2(r, s)$, respectively. The density function of the corresponding posterior distribution is

$$q(\lambda, \gamma | W, I) \sim \lambda^{n+q-1} e^{-a\lambda} e^{-\lambda W} e^{-\lambda\gamma W} \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{r+s}},$$

with the marginal density of $\gamma$

$$q(\gamma | W, I) \sim \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{r+s} (\gamma W + W + a)^{n+q}},$$

and the conditional distribution of $\lambda$ given $(\gamma, W, I)$ is $q(\lambda | \gamma, W, I) = G(\gamma W + W + a, n+q)$.

Under a more specific choice of prior parameters we can get explicit results.

**Theorem 2.4.** If $a = 0$, $q > -n$, $q > -r$, $r > 0$, $s > 0$, then

$$\widehat{\lambda} = \frac{I+r+q}{W} \frac{n+q}{n+q+r+s}, \quad \widehat{p} = \frac{I+r+q}{n+r+q+s}$$

are the Bayesian estimates under the quadratic loss function. If, moreover, $I+r+q>1$, then

$$\widehat{\theta} = \frac{W}{I+r+q-1} \frac{n+q+s+r-1}{n+q-1} \quad \text{and} \quad \widehat{\gamma} = \frac{n-I+s}{I+r+q-1}$$

are the Bayesian estimates under the quadratic loss function.

P r o o f. The posterior density function of $(\lambda, p)$ is

$$q(\lambda, p | W, I) \sim \lambda^{n+q-1} e^{-\lambda W/p} p^{-n+I+r-1} (1-p)^{n-I+s-1}$$

so that

$$q(p | W, I) = B(I+r+q, n-I+s), \qquad q(\lambda | p, W, I) = G(W/p, n+q).$$

The first two estimates are then calculated as

$$\widehat{p} = \mathrm{E}(p | W, I) = \frac{I+r+q}{n+r+q+s},$$

$$\widehat{\lambda} = \mathrm{E}(\lambda | W, I) = \mathrm{E}(\mathrm{E}[\lambda | p, W, I] | W, I) = \mathrm{E}(\frac{n+q}{W} p | W, I).$$

Further, $\gamma$ has the posterior distribution

$$q(\gamma|W,I) \sim \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{r+s}(\gamma W+W)^{n+q}} \sim \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{n+r+s+q}} \sim B2(I+r+q, n-I+s)$$

so that $\widehat{\gamma} = \mathrm{E}(\gamma|W,I)$. The conditional distribution of $\theta$ given $(p,W,I)$ is

$$q(\theta|p,W,I) \sim IG(W/p, n+q)$$

so that

$$\begin{aligned}\widehat{\theta} &= \mathrm{E}(\theta|W,I) = \mathrm{E}\left(\mathrm{E}[\theta|p,W,I]|W,I\right) = \mathrm{E}\left(\left.\frac{W}{(n+q-1)p}\right| W,I\right)\\ &= \frac{W}{n+q-1}\left(1+\mathrm{E}[(\frac{1}{p}-1)|W,I]\right) = \frac{W}{n+q-1}\left(1+\mathrm{E}[\gamma|W,I]\right)\\ &= \frac{W}{n+q-1}\left(1+\frac{n-I+s}{I+r+q-1}\right)\end{aligned}$$

and the result follows. □

With another restriction put on the prior parameters we can obtain the explicit form of the estimate of $R$.

**Theorem 2.5.** If $r=-s$, $s<q$, $s>0$, $a>0$, then

$$\widehat{\lambda} = \frac{I+q-s}{W+a}, \quad \widehat{R} = \left(\frac{W+a}{W+a+1}\right)^{I+q-s}$$

are the Bayesian estimates under the quadratic loss function. If, moreover $I+q-s>1$ then

$$\widehat{\theta} = \frac{W+a}{I+q-s-1}, \quad \widehat{\gamma} = \frac{W+a}{W}\frac{n-I+s}{I+q-s-1}$$

are the Bayesian estimates under the quadratic loss function.

Proof. The estimates are simply the posterior expectations again. The only explanation is needed for $\widehat{\gamma}$:

$$\begin{aligned}\widehat{\gamma} = \mathrm{E}(\gamma|W,I) &= \left(\int_0^\infty \frac{\gamma^{n-I+s}}{\left(1+\gamma\frac{W}{W+a}\right)^{n+q}}\,\mathrm{d}\gamma\right)\left(\int_0^\infty \frac{\gamma^{n-I+s-1}}{\left(1+\gamma\frac{W}{W+a}\right)^{n+q}}\,\mathrm{d}\gamma\right)^{-1}\\ &= \frac{W+a}{W}\frac{\int_0^\infty \frac{\gamma^{n-I+s}}{(1+\gamma)^{n+q}}\,\mathrm{d}\gamma}{\int_0^\infty \frac{\gamma^{n-I+s-1}}{(1+\gamma)^{n+q}}\,\mathrm{d}\gamma} = \frac{W+a}{W}\,\mathrm{E}\left[B2(I+q-s, n-I+s)\right],\end{aligned}$$

hence the result. □

### Jeffrey's prior

The Fisher information matrix is

$$\boldsymbol{J}(\lambda,\gamma) = \mathrm{E}\left[-\begin{pmatrix} -\frac{1}{\lambda^2} & -W_1 \\ -W_1 & -\frac{1}{\gamma^2}(1-I_1)\end{pmatrix}\right] = \begin{pmatrix} \frac{1}{\lambda^2} & \frac{1}{\lambda(1+\gamma)} \\ \frac{1}{\lambda(1+\gamma)} & \frac{1}{\gamma(1+\gamma)}\end{pmatrix}.$$

Hence the Jeffrey's prior density function of the pair $(\lambda, \gamma)$ may be expressed as

$$q(\lambda,\gamma) \sim |\det \boldsymbol{J}(\lambda,\gamma)|^{1/2} = \frac{1}{\lambda(1+\gamma)\sqrt{\gamma}} \qquad (2..8)$$

with marginal densities

$$q(\lambda) \sim \frac{1}{\lambda}\,, \quad q(\gamma) \sim \frac{1}{(1+\gamma)\sqrt{\gamma}}\,, \quad \lambda \text{ and } \gamma \text{ independent}\,.$$

**Theorem 2.6.** Suppose the Jeffrey's prior distribution (2..8). Then

$$\widehat{\lambda} = \frac{I+\frac{1}{2}}{W}\,\frac{n}{n+1}\,, \quad \widehat{\theta} = \frac{W}{I-\frac{1}{2}}\,\frac{n}{n-1}\,, \quad \widehat{p} = \frac{I+\frac{1}{2}}{n+1}\,, \quad \widehat{\gamma} = \frac{n-I+\frac{1}{2}}{I-\frac{1}{2}}$$

are the Bayesian estimates under the quadratic loss function.

P r o o f. Jeffrey's prior is the special case of Theorem 2.4 where we put $q = 0$, $r = s = \frac{1}{2}$. □

## 3.. ASYMPTOTIC RESULTS

In this Section we present some asymptotic results for the estimates given above. For the purpose of this Section denote $\overline{I}_n = I/n$ and $\overline{W}_n = W/n$.

### Bayesian risks

**Theorem 3.1.** If $r > 8$ then for the natural conjugate prior (2..2)

$$\lim_{n\longrightarrow\infty} n\varrho^\star_\lambda = \frac{r(r+s+1)}{a^2}$$

$$\lim_{n\longrightarrow\infty} n\varrho^\star_\theta = \frac{a^2(r+s-3)}{(r-1)(r-2)(r-3)}$$

$$\lim_{n\longrightarrow\infty} n\varrho^\star_\gamma = \frac{(r+s-1)(r+s-2)s}{(r-1)(r-2)(r-3)}$$

$$\lim_{n\longrightarrow\infty} n\varrho^\star_p = \frac{rs}{(r+s)(r+s-1)}$$

$$\lim_{n\longrightarrow\infty} n\varrho^\star_R = \frac{ra^{r-1}}{(a+2)^{r+1}}\left(s + \frac{a}{a+2}(r+1)\right)$$

are the Bayesian risks of the corresponding estimates.

P r o o f . Since the posterior variances are known, we can compute the Bayesian risk using (1..12). Derivation of the first four formulæ is analogous so that we present the details for $\varrho^\star_\lambda$ only. We make use of Theorem 5, b1) in Hurt [7] to obtain the asymptotic expansion for

$$\varrho^\star_\lambda = \mathrm{E}\ \frac{I+r}{(W+a)^2} = \frac{1}{n}\ \mathrm{E}\ \frac{\overline{I}_n + r/n}{(\overline{W}_n + a/n)^2}.$$

Put $g(w,i,n) = (i+r/n)/(w+r/n)^2$, $q=1$, $u=0$, $p=6$ in the mentioned theorem. We have $|g(w,i,n)| \le C_1 + C_2 n^2$, $\max[\mathrm{E}_*(\overline{I}_n - \mathrm{E}\, I_1)^2, \mathrm{E}_*(\overline{W}_n - \mathrm{E}_*\, W_1)^2] = C_3/n$, $\max[\mathrm{E}_*(\overline{I}_n - \mathrm{E}\, I_1)^6, \mathrm{E}_*(\overline{W}_n - \mathrm{E}_*\, W_1)^6] = C_4/n^3 + O(1/n^4)$, where we denote $\mathrm{E}_*(\cdot) = \mathrm{E}[\cdot|\lambda,\gamma]$, so that the assumptions of the Theorem 5, loc. cit., are satisfied. If we moreover add the existence of $\mathrm{E}(\mathrm{E}_*\, W_1)^6$ $(r+s>6)$ we can even conclude (following the proof of the mentioned theorem) $\mathrm{E}\, g(\overline{W}_n, \overline{I}_n, n) = \mathrm{E}\, g(\mathrm{E}_*\, \overline{W}_n, \mathrm{E}_*\, \overline{I}_n, n) + O(1/n)$ and the result follows.

The calculation of $\varrho^\star_R$ is a bit more complicated. If we denote (and apply expansion of $\ln(1+x)$)

$$A_1 = (I+r)\ln\left(1 + \frac{2}{W+a}\right) = 2\frac{I+r}{W+a} - \frac{2^2}{2}\frac{I+r}{(W+a)^2} + R_1,$$
$$A_2 = 2(I+r)\ln\left(1 + \frac{1}{W+a}\right) = 2\frac{I+r}{W+a} - \frac{2}{2}\frac{I+r}{(W+a)^2} + R_2,$$

where $0 \le R_1 \le \frac{2^3}{3}\frac{I+r}{(W+a)^3}$ and $0 \le R_2 \le \frac{2}{3}\frac{I+r}{(W+a)^3}$, the Bayesian risk is

$$n\varrho^\star_R = \mathrm{E}\ n(\exp(-A_1) - \exp(-A_2)).$$

Since

$$\exp\left(2\frac{I+r}{(W+a)^2}\right) = 1 + 2\frac{I+r}{(W+a)^2} + \frac{e^{S_1}}{2!}\left(2\frac{I+r}{(W+a)^2}\right)^2,$$
$$\exp(-R_1) = 1 - e^{-T_1}R_1,$$

where $0 \le S_1 \le 2\frac{I+r}{(W+a)^2}$ and $0 \le T_1 \le R_1$, we have

$$\begin{aligned}\exp(-A_1) &= e^{-2\frac{I+r}{W+a}}(1 - e^{-T_1}R_1)\left(1 + 2\frac{I+r}{(W+a)^2}\right)\\ &\quad + e^{-2\frac{I+r}{W+a}}e^{-R_1}\frac{e^{S_1}}{2!}\left(2\frac{I+r}{(W+a)^2}\right)^2\\ &= e^{-2\frac{I+r}{W+a}}\left(1 + 2\frac{I+r}{(W+a)^2}\right) + Q_1,\end{aligned}$$

where

$$\begin{aligned}|Q_1| &\le e^{-2\frac{I+r}{W+a}}e^{-T_1}R_1\left(1 + 2\frac{I+r}{(W+a)^2}\right) + \frac{1}{2!}\left(2\frac{I+r}{(W+a)^2}\right)^2 e^{-2\frac{I+r}{W+a} - R_1 + S_1}\\ &\le 1\cdot 1\cdot \frac{8}{3}\frac{I+r}{(W+a)^3}\left(1 + 2\frac{I+r}{(W+a)^2}\right) + 2\left(\frac{I+r}{(W+a)^2}\right)^2 e^{-A_1}\\ &\le K_1\frac{n}{W^3} + K_2\frac{n^2}{W^5} + K_3\frac{n^2}{W^4}\cdot 1.\end{aligned}$$

Similarly we could get

$$\exp(-A_2) = e^{-2\frac{I+r}{W+a}} \left(1 + \frac{I+r}{(W+a)^2}\right) + Q_2.$$

Together

$$n\varrho_R^\star = \mathrm{E}\ n\mathrm{e}^{-2(I+r)/(W+a)} \left(2\frac{I+r}{(W+a)^2} - \frac{I+r}{(W+a)^2}\right) + \mathrm{E}\ n(Q_1 - Q_2).$$

Applying the cited theorem to the first term finishes the proof, since the second term is of the order $O(1/n)$: $\mathrm{E}(1/W)^k$ is $O(1/n^k)$ (given $\lambda, \gamma$, $1/W \sim IG(\lambda(1+\gamma), n)$).□

**Almost sure convergence**

It has been shown that given $\lambda$, $\gamma$, $p = 1/(1+\gamma)$, $R = e^{-\lambda}$, $\theta = 1/\lambda$

$$\mathrm{E}\, I_1 = p, \quad \mathrm{var}\, I_1 = p(1-p), \quad \mathrm{E}\, W_1 = p/\lambda, \quad \mathrm{var}\, W_1 = p^2/\lambda^2$$

hold. Since the derived estimates are of the similar form we can state the general theorem concerning the almost sure convergence.

**Theorem 3.2.** If $a$, $c_1$, $c_2$, $c$ are arbitrary constants then

$$\lim_{n\longrightarrow\infty} \frac{n - I + c_1}{I + c_2} = \gamma \ \text{ a. s.}, \qquad \lim_{n\longrightarrow\infty} \frac{I + c_1}{n + c_2} = p \ \text{ a. s.},$$
$$\lim_{n\longrightarrow\infty} \frac{I + c_1}{W + c_2} = \lambda \ \text{ a. s.}, \qquad \lim_{n\longrightarrow\infty} \frac{W + c_2}{I + c_1} = \theta \ \text{ a. s.},$$
$$\lim_{n\longrightarrow\infty} \left(\frac{W + a}{W + a + 1}\right)^{I+c} = R \ \text{ a. s.}$$

holds.

P r o o f. The assertion is a simple consequence of the fact that $\overline{I}_n \longrightarrow p$ and $\overline{W}_n \longrightarrow p/\lambda$ almost surely for $n \longrightarrow \infty$. □

**Asymptotic normality**

**Theorem 3.3.** If $a$, $c_1$, $c_2$, $c$ are arbitrary constants then

$$\lim_{n\longrightarrow\infty} \pounds \left(\sqrt{n}(\frac{I + c_1}{n + c_2} - p)\right) = N\left(0, p(1-p)\right),$$
$$\lim_{n\longrightarrow\infty} \pounds \left(\sqrt{n}(\frac{n - I + c_1}{I + c_2} - \gamma)\right) = N\left(0, \gamma(1+\gamma)^2\right),$$
$$\lim_{n\longrightarrow\infty} \pounds \left(\sqrt{n}(\frac{I + c_1}{W + c_2} - \lambda)\right) = N\left(0, \frac{\lambda^2}{p}\right),$$
$$\lim_{n\longrightarrow\infty} \pounds \left(\sqrt{n}(\frac{W + c_1}{I + c_2} - \theta)\right) = N\left(0, \frac{\theta^2}{p}\right),$$

$$\lim_{n\longrightarrow\infty} £\left(\sqrt{n}(\left(\frac{W+a}{W+a+1}\right)^{I+c} - R)\right) = N\left(0, \frac{R^2(\ln R)^2}{p}\right).$$

P r o o f . It follows from the central limit theorem for i.i.d. with the finite nonzero variance that

$$\lim_{n\longrightarrow\infty} £\left(\sqrt{n}(\frac{I}{n} - p)\right) = N\left(0, p(1-p)\right),$$
$$\lim_{n\longrightarrow\infty} £\left(\sqrt{n}(\frac{W}{n} - \frac{p}{\lambda})\right) = N\left(0, p^2/\lambda^2\right).$$

Using Cramér–Slutsky and Sverdrup theorem we get

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} £\left(\sqrt{n}(\frac{I+c_1}{n+c_2} - p)\right) \\
&\quad = \lim_{n\longrightarrow\infty} £\left(\sqrt{n}\frac{n}{n+c_2}(\frac{I}{n} - p) + \sqrt{n}\frac{c_1}{n+c_2} + \sqrt{n}(\frac{n}{n+c_2} - 1)p\right) \\
&\quad = \lim_{n\longrightarrow\infty} £\left(\sqrt{n}(\frac{I}{n} - p)\right) = N\left(0, p(1-p)\right),
\end{aligned}$$

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} £\left(\sqrt{n}(\frac{n-I+c_1}{I+c_2} - \gamma)\right) \\
&\quad = \lim_{n\longrightarrow\infty} £\left(\sqrt{n}\frac{I}{I+c_2}(\frac{n-I}{I} - \gamma) + \sqrt{n}\frac{c_1}{I+c_2} + \sqrt{n}(\frac{I}{I+c_2} - 1)\gamma\right) \\
&\quad = \lim_{n\longrightarrow\infty} £\left(\sqrt{n}(\frac{n-I}{I} - \gamma)\right) = \lim_{n\longrightarrow\infty} £\left(\frac{\sqrt{n}(1-\overline{I}_n - \gamma\overline{I}_n)}{\overline{I}_n}\right) \\
&\quad = \lim_{n\longrightarrow\infty} £\left(\frac{\sqrt{n}(p-\overline{I}_n)}{pp}\right) = N\left(0, \frac{p(1-p)}{p^4}\right) \\
&\quad = N\left(0, \frac{(1-p)}{p^3}\right) = N\left(0, \gamma(1+\gamma)^2\right).
\end{aligned}$$

Also

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} £\left(\sqrt{n}(\frac{I+c_1}{W+c_2} - \lambda)\right) \\
&\quad = \lim_{n\longrightarrow\infty} £\left(\sqrt{n}\frac{W}{W+c_2}(\frac{I}{W} - \lambda) + \sqrt{n}\frac{c_1}{W+c_2} + \sqrt{n}(\frac{W}{W+c_2} - 1)\lambda\right) \\
&\quad = \lim_{n\longrightarrow\infty} £\left(\sqrt{n}(\frac{I}{W} - \lambda)\right) = \lim_{n\longrightarrow\infty} £\left(\frac{\sqrt{n}(\overline{I}_n - \lambda\overline{W}_n)}{\overline{W}_n}\right) \\
&\quad = \lim_{n\longrightarrow\infty} £\left(\frac{\overline{I}_n - \lambda\overline{W}_n}{p/\lambda}\right) = N\left(0, \frac{1}{p^2/\lambda^2}\left(p(1-p) + \lambda^2\frac{p^2}{\lambda^2}\right)\right) \\
&\quad = N\left(0, \frac{\lambda^2}{p}\right),
\end{aligned}$$

since $W_1$ and $I_1$ are independent and thus

$$\begin{pmatrix} \overline{I}_n - p \\ \overline{W}_n - p/\lambda \end{pmatrix} \xrightarrow[n\to\infty]{\mathrm{D}} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} p(1-p) & 0 \\ 0 & p^2/\lambda^2 \end{pmatrix}\right)$$

and the limiting distribution $\lim_{n\longrightarrow\infty} \pounds(\overline{I}_n - \lambda\overline{W}_n)$ may be obtained as the limit of the linear combination of the independent normal distributions. Similarly

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}(\frac{W+c_1}{I+c_2} - \theta)\right) = \\
&\quad = \lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}(\frac{W}{I} - \theta) + \sqrt{n}(\frac{W}{I+c_2} - \frac{W}{I}) + \sqrt{n}\frac{c_1}{I+c_2}\right) \\
&\quad = \lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}\frac{1}{\overline{I}_n}(\overline{W}_n - \theta\overline{I}_n)\right) \\
&\quad = N\left(0, \frac{1}{p^2}\left(\frac{p^2}{\lambda^2} + \theta^2 p(1-p)\right)\right) = N\left(0, \theta^2(1 + \frac{1-p}{p})\right) \\
&\quad = N\left(0, \frac{\theta^2}{p}\right).
\end{aligned}$$

To justify the next relationship let us note that

$$n^{2/3}\frac{I}{(W+a)^2} = n^{-1/3}\frac{\overline{I}_n}{\left(\overline{W}_n + \frac{a}{n}\right)^2} \xrightarrow[n\to\infty]{} 0 \text{ a. s.},$$

so that $I/(W+a)^2 = o(n^{-2/3})$ for $n \to \infty$ with probability 1 which together with

$$Q_1 = \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m(W+a)^m} = O\left(\frac{1}{(W+a)^2}\right) \quad \text{for } n \to \infty \text{ a. s.},$$

(following from the expansion $\ln(1+x)$ for $x \to 0$) gives

$$\sqrt{n} I Q_1 \xrightarrow[n\to\infty]{} 0 \quad \text{a. s.}$$

and

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}\left(I\ln(1+\frac{1}{W+a}) - \lambda\right)\right) = \lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}\left(\frac{I}{W+a} + IQ_1\right)\right) \\
&\quad = \lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}\frac{I}{W+a}\right) = N\left(0, \frac{\lambda^2}{p}\right).
\end{aligned}$$

It now follows that

$$n^{1/3}\left(I\ln(1+\frac{1}{W+a}) - \lambda\right) \xrightarrow[n\to\infty]{\mathrm{P}} 0$$

and

$$\sqrt{n}\left(I\ln(1+\frac{1}{W+a}) - \lambda\right)^2 \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

We use Taylor expansion of the exponential function to derive the last assertion for $c = 0$. Let us denote

$$Q_2 = \sum_{m=2}^{\infty} \frac{I\ln(1+\frac{1}{W+a}) - \lambda}{m!}.$$

There exist $K > 0$ and $\delta > 0$ such that for $|I\ln(1+1/(W+a)) - \lambda| < \delta$ we have $|Q_2| < K(I\ln(1+1/(W+a)) - \lambda)^2$ and

$$\Pr\left[|\sqrt{n}e^{-\lambda}Q_2| > \varepsilon\right] \le \Pr\left[\sqrt{n}e^{-\lambda}K(I\ln(1+1/(W+a)) - \lambda)^2 > \varepsilon\right] \underset{n\to\infty}{\longrightarrow} 0$$

for every $\varepsilon > 0$ or

$$\sqrt{n}e^{-\lambda}Q_2 \xrightarrow[n\to\infty]{\mathrm{P}} 0.$$

Hence

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}(\left(\frac{W+a}{W+a+1}\right)^I - R)\right)\\
&\quad= \lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}e^{-\lambda}(e^{-I\ln\frac{W+a+1}{W+a}+\lambda} - 1)\right)\\
&\quad= \lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}e^{-\lambda}(1 - \frac{I\ln(1+\frac{1}{W+a}) - \lambda}{1!} + Q_2 - 1)\right)\\
&\quad= \lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}e^{-\lambda}(I\ln(1+\frac{1}{W+a}) - \lambda)\right)\\
&\quad= N\left(0, e^{-2\lambda}\frac{\lambda^2}{p}\right) = N\left(0, \frac{R^2(\ln R)^2}{p}\right).
\end{aligned}$$

For arbitrary $c$

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}(\left(\frac{W+a}{W+a+1}\right)^{I+c} - R)\right)\\
&\quad= \lim_{n\longrightarrow\infty} \pounds\left(\sqrt{n}\left(\frac{W+a}{W+a+1}\right)^c\right.\\
&\qquad\times\left.\left(\left(\frac{W+a}{W+a+1}\right)^I - R\right) - \sqrt{n}R\left(1 - \left(\frac{W+a}{W+a+1}\right)^c\right)\right)\\
&\quad= N\left(0, \frac{R^2(\ln R)^2}{p}\right),
\end{aligned}$$

since

$$\lim_{n\longrightarrow\infty}\left(\frac{W+a}{W+a+1}\right)^c = \left(\frac{\overline{W}_n + \frac{a}{n}}{\overline{W}_n + \frac{a+1}{n}}\right)^c = 1 \text{ a.s.},$$

$$\sqrt{n}\left(\left(\frac{W+a}{W+a+1}\right)^I - R\right) \xrightarrow[n\to\infty]{\mathrm{D}} N\left(0, \frac{R^2(\ln R)^2}{p}\right) \quad \text{and}$$

$$\lim_{n\longrightarrow\infty}\sqrt{n}R\left(1 - \left(\frac{W+a}{W+a+1}\right)^c\right) = \lim_{n\longrightarrow\infty}\sqrt{n}R\left(1 - \left(1 - \frac{1}{W+a+1}\right)^c\right)$$

$$= \lim_{n\longrightarrow\infty} \sqrt{n}R\left(1 - \left(1 + O\left(\frac{1}{W+a+1}\right)\right)\right) = 0 \text{ a. s.} \qquad \square$$

## 4.. SIMULATION

In case of the independent gamma priors (2..5) we were not able to express the Bayesian risks in a close form. To get an idea about the behaviour of the Bayesian risks we performed a limited simulation study. The prior parameters were chosen as to achieve the expected value of $\lambda$ equal to 1 and the expected value of $\gamma$ corresponding to the portion of uncensored observations $p = 1/(1+\gamma)$ equal to 0.5, 0.8, and 0.9. The accuracy of the prior knowledge is expressed by the coefficients of variation (the standard deviation divided by the mean) $V_\lambda$ and $V_\gamma$ equal to 0.5, 0.3, and 0.1. The estimated $n$-multiples of the Bayesian risks for 5000 realizations of the samples of size $n$ = 20, 50, and 100 are summarized in the following tables.

| $n\cdot\varrho^\star_\lambda$ | $V_\gamma$ | = | 0.5 | $n$ | $V_\gamma$ | = | 0.3 | $n$ | $V_\gamma$ | = | 0.1 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $V_\lambda$ | 1.33 | 1.12 | 1.01 | 20 | 1.22 | 1.04 | 1.05 | 20 | 1.06 | 1.01 | 0.95 | $V_\lambda$ |
| = | 1.72 | 1.29 | 1.16 | 50 | 1.60 | 1.23 | 1.21 | 50 | 1.23 | 1.12 | 1.12 | = |
| 0.5 | 2.02 | 1.39 | 1.24 | 100 | 1.88 | 1.31 | 1.21 | 100 | 1.42 | 1.19 | 1.16 | 0.5 |
| $p$ | 0.5 | 0.8 | 0.9 | | 0.5 | 0.8 | 0.9 | | 0.5 | 0.8 | 0.9 | $p$ |
| $V_\lambda$ | 0.86 | 0.70 | 0.69 | 20 | 0.80 | 0.71 | 0.70 | 20 | 0.70 | 0.69 | 0.69 | $V_\lambda$ |
| = | 1.30 | 1.01 | 0.96 | 50 | 1.21 | 0.94 | 0.90 | 50 | 0.93 | 0.87 | 0.90 | = |
| 0.3 | 1.62 | 1.12 | 1.04 | 100 | 1.47 | 1.09 | 0.99 | 100 | 1.12 | 1.04 | 0.96 | 0.3 |
| $p$ | 0.5 | 0.8 | 0.9 | | 0.5 | 0.8 | 0.9 | | 0.5 | 0.8 | 0.9 | $p$ |
| $V_\lambda$ | 0.18 | 0.16 | 0.16 | 20 | 0.17 | 0.16 | 0.17 | 20 | 0.16 | 0.16 | 0.17 | $V_\lambda$ |
| = | 0.39 | 0.34 | 0.33 | 50 | 0.37 | 0.33 | 0.34 | 50 | 0.34 | 0.33 | 0.34 | = |
| 0.1 | 0.63 | 0.55 | 0.51 | 100 | 0.63 | 0.53 | 0.50 | 100 | 0.53 | 0.51 | 0.49 | 0.1 |
| | $V_\gamma$ | = | 0.5 | n | $V_\gamma$ | = | 0.3 | n | $V_\gamma$ | = | 0.1 | |

| $n\cdot\varrho^\star_\gamma$ | $V_\gamma$ | = | 0.5 | n | $V_\gamma$ | = | 0.3 | n | $V_\gamma$ | = | 0.1 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $V_\lambda$ | 2.154 | 0.193 | 0.047 | 20 | 1.219 | 0.089 | 0.020 | 20 | 0.184 | 0.012 | 0.002 | $V_\lambda$ |
| = | 3.240 | 0.297 | 0.085 | 50 | 2.154 | 0.164 | 0.042 | 50 | 0.434 | 0.028 | 0.006 | = |
| 0.5 | 4.082 | 0.346 | 0.109 | 100 | 2.865 | 0.234 | 0.062 | 100 | 0.783 | 0.051 | 0.011 | 0.5 |
| p | 0.5 | 0.8 | 0.9 | | 0.5 | 0.8 | 0.9 | | 0.5 | 0.8 | 0.9 | p |
| $V_\lambda$ | 2.114 | 0.191 | 0.048 | 20 | 1.134 | 0.091 | 0.020 | 20 | 0.181 | 0.011 | 0.002 | $V_\lambda$ |
| = | 3.227 | 0.286 | 0.082 | 50 | 2.029 | 0.172 | 0.042 | 50 | 0.426 | 0.028 | 0.006 | = |
| 0.3 | 3.677 | 0.340 | 0.109 | 100 | 2.755 | 0.233 | 0.062 | 100 | 0.768 | 0.052 | 0.011 | 0.3 |
| p | 0.5 | 0.8 | 0.9 | | 0.5 | 0.8 | 0.9 | | 0.5 | 0.8 | 0.9 | p |
| $V_\lambda$ | 1.712 | 0.184 | 0.050 | 20 | 1.047 | 0.087 | 0.020 | 20 | 0.183 | 0.012 | 0.002 | $V_\lambda$ |
| = | 2.293 | 0.259 | 0.079 | 50 | 1.639 | 0.155 | 0.041 | 50 | 0.389 | 0.028 | 0.006 | = |
| 0.1 | 2.812 | 0.311 | 0.101 | 100 | 2.152 | 0.237 | 0.063 | 100 | 0.738 | 0.052 | 0.011 | 0.1 |
| | $V_\gamma$ | = | 0.5 | n | $V_\gamma$ | = | 0.3 | n | $V_\gamma$ | = | 0.1 | |

We can see that under the accurate knowledge of $\lambda$ the Bayesian risk is not too much influenced either by the accuracy of the knowledge of $p$ or the value $p$ itself. Similarly, under the accurate knowledge of $\gamma$ the risk is not influenced by the accuracy of $\lambda$. The risk $\varrho^\star_\gamma$ substantially depends on the portion of uncensored observations $p$, however. Obviously, with the increasing portion of the uncensored observations and increasing sample sizes the risks decrease.

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