KYBERNETIKA — VOLUME 42 (2006), NUMBER 5, PAGES 617 – 628

SIMPLIFICATION OF THE GENERALIZED STATE EQUATIONS[1]

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The paper studies the problem of lowering the orders of input derivatives in nonlinear generalized state equations via generalized coordinate transformation. An alternative, computation-oriented proof is presented for the theorem, originally proved by Delaleau and Respondek, giving necessary and sufficient conditions for existence of such a transformation, in terms of commutativity of certain vector fields. Moreover, the dual conditions in terms of 1-forms have been derived, allowing to calculate the new generalized state coordinates in a simpler way. The result is illustrated with an example, originally given by Delaleau and Respondek (see [2]), but solved in an alternative way.

Keywords: generalized dynamics, generalized state transformations, input derivatives, classical state, prolonged vector fields

AMS Subject Classification: 93C10, 93B29, 93B17, 93B11

1. INTRODUCTION

The modern control theory concentrates on systems of the form

$$\dot{x} = f(x, u), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state variable, and $u \in \mathbb{R}^m$ is the control variable (or input). Various techniques from differential geometry yield a large variety of methods, allowing to control and analyze the behavior of the systems, described by equation (1). However, in several control problems one encounters more general dynamics than (1), containing in addition to inputs also a certain number of their time derivatives (see Fliess [4]):

$$\dot{x}_i = f_i\left(x_j, u_I, u_I^{(1)}, \ldots, u_I^{(\alpha_I)}\right). \tag{2}$$

Here and further, when not defined otherwise, the small Roman letters like i or j denote the indices of the state variables and have the values from 1 to n, the Roman capitals like I or J denote the indices of the inputs and have the values from 1 to m. The Greek indices with Roman capitals as subindices like α_I or κ_J denote the orders

[1] Preliminary version of this paper has been presented at the 4th IMACS/IFAC Symposium on Mathematical Modelling (MATHMOD), 2003.

of time derivatives of the corresponding input, and α_I labels always the highest time derivative order of u_I.

For example, one gets the dynamics (2) in system inversion or as an intermediate step in studying the problem of realization of a (single-input single-output) input-output equation

$$y^{(n)} = \varphi\left(y, y^{(1)}, \dots, y^{(n-1)}, u, u^{(1)}, \dots, u^{(n-1)}\right)$$

in the classical state space form (1). Namely, replacing the time derivatives of the output with the so-called generalized state variables, $x_i = y^{(i-1)}, \quad i = 1, \dots, n$, yields the generalized state equations (2) in the special form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= \varphi\left(x_1, x_2, \dots, x_n, u, u^{(1)}, \dots, u^{(n-1)}\right). \end{aligned} \tag{3}$$

Since most of the control theory is done for systems of the form (1), and not for the form (2), it is natural to study the problem of whether there exists a *generalized state transformation,* depending also on the inputs and their time derivatives

$$\tilde{x} = \Phi\left(x, u, \dots, u^{(\alpha)}\right) \tag{4}$$

transforming (2) into the classical state space form (1), i. e. removing the time derivatives of inputs from the generalized state space equations. Note that Φ is assumed to be invertible with respect to x. A natural framework to study this problem (see [2]) is to work in an *extended state space.*

Definition 1. (Delaleau and Respondek [2]) The direct product of the state space R^n and the input space R^K, where $K = \sum_{I=1}^{m}(\alpha_I + 1)$ is called the extended state space of a system with coordinates $x_i, u_I, \dot{u}_I, \dots, u_I^{(\alpha_I)}$.

Observe that (4) can be interpreted as a transformation of the extended state space having the special structure

$$\Psi(x, u, \dots, u^{(\alpha)}) = \left(\Phi(x, u, \dots, u^{(\alpha)}), u, \dots, u^{(\alpha)}\right).$$

Preserving the control-related coordinates means that we do not change u and its time-derivatives.

In many cases, especially when some input derivatives appear in equations (2) non-linearily, it is impossible to remove all of the time derivatives and then a more general problem can be studied – when it is possible to reduce, via a suitable co-ordinate transformation (4), the maximal orders of input time derivatives in the generalized state space equations, i. e. getting the new equations

$$\dot{\tilde{x}}_i = \tilde{f}_i\left(\tilde{x}_j, u_I, \dots, u_I^{(\beta_I)}\right) \tag{5}$$

where $\beta_I \le \alpha_I$ and at least for one input $\beta_I < \alpha_I$. The latter problem was studied by Delaleau and Respondek in [2], where the necessary and sufficient conditions (formulated as a Theorem) for existence of such a coordinate transformation were given in terms of the commutativity of certain vector fields, associated to system (2). Note that the paper [2] does not provide an algorithm for finding the generalized state coordinate transformation[2].

This paper presents an alternative geometric proof for this theorem from [2] in terms of 1-forms which relies on invariants of vector fields, satisfying condition (7) (see below) and has the advantages that it

- leads directly to the algorithm for finding $\tilde{x}$ in (4), and
- allows to find the dual necessary and sufficient conditions in terms of 1-forms.

We suggest an explicit formula for finding these 1-forms, not necessary exact (see (30) below), and a system of first order partial differential equations for finding the integrating factors. Integration the 1-forms gives the new generalized state variables $\tilde{x}$. The state variables $\tilde{x}$ can be alternatively found as the solutions of a system of first order partial differential equations.

Note that necessary and sufficient conditions to lower the maximal order of the input derivatives in (2) for the single-input case in terms of 1-forms are also given in [6]. The relationship of these conditions to those given in this paper is clarified. Moreover, the problem of removing all the input derivatives in the special form of the generalized state equations (3) is studied in [1, 5, 8, 9, 10]. The relationship between the conditions is clarified in [7].

2. AN ALTERNATIVE PROOF OF DELALEAU–RESPONDEK THEOREM

Definition 2. (Delaleau and Respondek [2]) The prolongation F of a vector field $f = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i}$ into the extended state space is the vector field

$$F = \frac{\mathrm{d}}{\mathrm{d}t} = \sum_{i=1}^{n} f_i \frac{\partial}{\partial x_i} + \sum_{I=1}^{m} \left(\sum_{\kappa_I=0}^{\alpha_I} u_I^{(\kappa_I+1)} \frac{\partial}{\partial u_I^{(\kappa_I)}} \right) \tag{6}$$

as the operator of total time derivative.

Let $L_F \Psi$ denote the Lie derivative of an arbitrary tensor object Ψ, i. e. function, vector field or 1-form, with respect to the vector field F. Due to the fact that the vector field F is an operator of total time derivative, the Lie derivative $L_F \Psi$ and time derivative $\dot{\Psi}$ are identical.

[2]The generalized change of state coordinates Φ in [2] is defined as the restriction, to the original state space, of the (translated) inverse of the composition of flows of certain vector fields.

Theorem 1. (Delaleau and Respondek [2]) A generalized change of state coordinates Φ of the form (4), transforming the representation (2) into (5), exists if and only if for any $1 \leq I, J \leq m$ and $0 \leq P_I \leq \alpha_I - \beta_I$, $0 \leq P_J \leq \alpha_J - \beta_J$

$$\left[L_F^{P_I}\frac{\partial}{\partial u_I^{(\alpha_I)}}, L_F^{P_J}\frac{\partial}{\partial u_J^{(\alpha_J)}}\right] \equiv 0, \tag{7}$$

where

$$L_F^0\frac{\partial}{\partial u_I^{(\alpha_I)}} = \frac{\partial}{\partial u_I^{(\alpha_I)}}, \quad L_F^{P_I+1}\frac{\partial}{\partial u_I^{\alpha_I}} = \left[F, L_F^{P_I}\frac{\partial}{\partial u_I^{(\alpha_I)}}\right]. \tag{8}$$

According to [2], the validity of (7) allows to define the system of new coordinates $(\tilde{x}, u_I, \ldots, u_I^{(\alpha_I)})$ in the extended state space, in which the vector fields in (8) take the following form: $L_F^{P_I}\frac{\partial}{\partial u_I^{(\alpha_I)}} = (-1)^{P_I}\frac{\partial}{\partial u^{(\alpha_I - P_I)}}$. That is – they belong to the *natural basis*, consisting of partial derivative operators with respect to the new coordinates. In the new coordinates, the components $\tilde{f}_i$ in (5) no more depend on $u_I^{(\beta_I+1)}, \ldots, u_I^{(\alpha_I)}$. The validity of (7) allows to search the new coordinates $\tilde{x}$ as the invariants of the vector fields, satisfying (7).

This paper gives an alternative geometric proof for Theorem 1, based on the idea that the validity of (7) yields the existence of the involutive distribution

$$\Delta = \operatorname{span}\left\{(-1)^{P_I}L_F^{P_I}\frac{\partial}{\partial u_I^{(\alpha_I)}}\right\} \tag{9}$$

with dimension $K - \sum_{I=1}^m \beta_I$. The latter means, that the maximal annihillator of Δ, a codistribution $\Delta^\perp$ is integrable. The new generalized state variables $\tilde{x}$ will be defined so that their differentials belong to $\Delta^\perp$. Further always $0 \leq P_I \leq \alpha_I - \beta_I$.

Proof. Due to (8) one can write the vector fields $(-1)^{P_I}L_F^{P_I}\frac{\partial}{\partial u_I^{(\alpha_I)}}$ in original coordinates $(x_i, u_I, \ldots, u_I^{(\alpha_I)})$ as the sum of $\frac{\partial}{\partial u_I^{(\alpha_I - P_I)}}$ and a linear combination of vector fields $\frac{\partial}{\partial x_i}$ (see [10]):

$$(-1)^{P_I}L_F^{P_I}\frac{\partial}{\partial u_I^{(\alpha_I)}} = \frac{\partial}{\partial u_I^{(\alpha_I - P_I)}} + \sum_{i=1}^n \Xi_{P_I,i}(x_j, u_J, \ldots, u_J^{(\alpha_J)})\frac{\partial}{\partial x_i}. \tag{10}$$

Since the vector fields $\frac{\partial}{\partial u_I^{(\alpha_I - P_I)}}$ are linearly independent for different P_I values, the vector fields (10) are also linearly independent and define in the extended state space, instead of the *natural* basis, consisting of partial derivative operators with respect to the coordinates

$$\{\partial\} = \left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial u_I}, \ldots, \frac{\partial}{\partial u_I^{(\alpha_I)}}\right\}, \tag{11}$$

another basis, the so-called *adapted basis*, adapted to the distribution Δ, defined by (9). In the adapted basis

$$\{R\} = \left\{ \frac{\partial}{\partial x_i}, \frac{\partial}{\partial u_I}, \dots, \frac{\partial}{\partial u_I^{(\beta_I - 1)}}, (-1)^{\alpha_I - \beta_I} L_F^{\alpha_I - \beta_I} \frac{\partial}{\partial u_I^{(\alpha_I)}}, \dots, -L_F^1 \frac{\partial}{\partial u_I^{(\alpha_I)}}, \frac{\partial}{\partial u_I^{(\alpha_I)}} \right\}, \tag{12}$$

with dimension $n + K$ we replace the vector fields $\frac{\partial}{\partial u_I^{(\alpha_I - P_I)}}$ by the vector fields $(-1)^{P_I} L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}}$. The adapted basis $\{R\}$ and the natural basis $\{\partial\}$ are related by the following basis transformation (see [3], p. 28).

$$\{R\} = \{\partial\} \times M,$$

where

$$M = \begin{bmatrix} I_n & \Xi \\ 0 & I_K \end{bmatrix} \tag{13}$$

and the (P_I, i)th element of submatrix Ξ is defined by $\Xi_{P_I,i}$ in (10). Due to (10), the following scalar products for $0 \le P_I < \alpha_I$, $0 \le P_J < \alpha_J$ can be written as

$$(-1)^{P_J} \left\langle \mathrm{d}u_I^{P_I}, L_F^{P_J} \frac{\partial}{\partial u_J^{(\alpha_J)}} \right\rangle = \delta_{P_I, \alpha_J - P_J} \delta_{IJ}, \tag{14}$$

so an *adapted cobasis*, with dimension $n + K$, dual to $\{R\}$, is

$$\{\theta\} = \left\{ \omega_i, \mathrm{d}u_I, \mathrm{d}u_I^{(1)}, \dots, \mathrm{d}u_I^{(\alpha_I)} \right\}. \tag{15}$$

In the adapted cobasis the differentials $\mathrm{d}x_i$ in the *natural cobasis*

$$\{\mathrm{d}\} = \left\{ \mathrm{d}x_i, \mathrm{d}u_I, \mathrm{d}u_I^{(1)}, \dots, \mathrm{d}u_I^{(\alpha_I)} \right\} \tag{16}$$

are replaced by, usually not exact, 1-forms ω_i, which are orthogonal to $(-1)^{P_I} L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}}$. Due to the duality of $\{R\}$ and $\{\theta\}$, all scalar products

$$\left\langle \omega_i, L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}} \right\rangle = 0, \qquad \left\langle \mathrm{d}u_I^{(\eta_I)}, L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}} \right\rangle = 0 \tag{17}$$

where here and further $\eta_I = 0, \dots, \beta_I - 1$. Because of (17) one can define the codistribution with dimension $n + \sum_{I=1}^{m} \beta_I$,

$$\Delta^\perp = \operatorname{span} \left\{ \omega_i, \mathrm{d}u_I^{(\eta_I)} \right\}, \tag{18}$$

as the maximal annihillator of Δ. Since, due to (7), Δ is involutive, its maximal annihilator $\Delta^\perp$ is integrable and one can replace in (18) the non-exact 1-forms ω_i by differentials $\mathrm{d}\tilde{x}_i$:

$$\mathrm{d}\tilde{x}_i = \sum_{j=1}^{n} A_{ij} \omega_j + \sum_{I=1}^{m} \sum_{\eta_I = 0}^{\beta_I - 1} A_{iI(\eta_I)} \mathrm{d}u_I^{(\eta_I)}, \tag{19}$$

where the integrating factors A are the functions of u_I, their time derivatives up to $u_I^{(\beta_I-1)}$, and x.

Consequently, one can define the new generalized state variables

$$\tilde{x}_i = \Phi_i(x_j, u_I, \ldots, u_I^{(\beta_I-1)}), \quad i = 1, \ldots, n \tag{20}$$

as the integrals of codistribution $\Delta^\perp$, so that

$$\Delta^\perp = \operatorname{span}\{\mathrm{d}\tilde{x}_i, \mathrm{d}u_I^{(\eta_I)}\}. \tag{21}$$

Therefore, for $i = 1, \ldots, n$, $\langle \mathrm{d}\tilde{x}_i, \Delta \rangle = 0$, that, due to (9) yields,

$$\left\langle \mathrm{d}\tilde{x}_i, L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}} \right\rangle = 0. \tag{22}$$

Formulas (10) and (22) allow to replace the adapted cobasis $\{\theta\}$ by another natural cobasis

$$\{\tilde{\theta}\} = \{\mathrm{d}\tilde{x}_i, \mathrm{d}u_I, \mathrm{d}u_I^{(1)}, \ldots, \mathrm{d}u_I^{(\alpha_I)}\}. \tag{23}$$

The basis $\{\tilde{R}\}$, dual to $\{\tilde{\theta}\}$, must be also the natural basis:

$$\{\tilde{R}\} = \left\{ \frac{\partial}{\partial \tilde{x}_i}, \frac{\partial}{\partial u_I^{(\eta_I)}}, \ldots, \frac{\partial}{\partial u_I^{(\alpha_I - P_I)}} \right\} \tag{24}$$

i. e. the tangent map $T\Phi$ of the coordinate transformation (20) transforms the vector fields (10) into $(-1)^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I - P_I)}}$. Since the scalar product is invariant under arbitrary coordinate transformation, then due to (21),

$$\left\langle \mathrm{d}\tilde{x}_i, \frac{\partial}{\partial u_I^{(\alpha_I - P_I)}} \right\rangle = 0 \implies \frac{\partial \tilde{x}_i}{\partial u_I^{(\alpha_I - P_I)}} = 0. \tag{25}$$

Taking the time derivative of (25) gives

$$L_F \left\langle \mathrm{d}\tilde{x}_i, \frac{\partial}{\partial u_I^{(\alpha_I - P_I)}} \right\rangle = \left\langle \mathrm{d}\dot{\tilde{x}}_i, \frac{\partial}{\partial u_I^{(\alpha_I - P_I)}} \right\rangle - \left\langle \mathrm{d}\tilde{x}_i, \frac{\partial}{\partial u_I^{(\alpha_I - P_I - 1)}} \right\rangle = 0, \tag{26}$$

because in the new coordinates $\tilde{x}$,

$$L_F \frac{\partial}{\partial u_I^{(\alpha_I - P_I)}} := \left[F, \frac{\partial}{\partial u_I^{(\alpha_I - P_I)}} \right] = -\frac{\partial}{\partial u_I^{(\alpha_I - P_I - 1)}}. \tag{27}$$

Because of (25), the second term in the right hand side of (26) is zero for all $P_I = 0, \ldots, \alpha_I - \beta_I - 1$; therefore

$$\frac{\partial \dot{\tilde{x}}_i}{\partial u_I^{(\alpha_I - P_I)}} = 0$$

for $P_I = 0, \ldots, \alpha_I - \beta_I - 1$ and consequently for $\kappa_I > \beta_I$

$$\frac{\partial \dot{\tilde{x}}_i}{\partial u_I^{(\kappa_I)}} = 0,$$

which proves the theorem. □

3. THE DUAL CONDITION AND ALGORITHMS FOR FINDING THE NEW GENERALIZED STATE VARIABLES

The new proof of Theorem 1 allows to find the dual necessary and sufficient conditions in terms of 1-forms. When (7) holds, $\Delta^{\perp} = \mathrm{span}\{\omega_i, \mathrm{d}u_I^{\eta_I}\}$ as a maximal annihillator of involutive distribution Δ must be an integrable codistribution. Direct application of Frobenius' theorem shows that the exterior differentials of ω_i must belong to $\Delta^{\perp}$. The latter yields the equivalent dual condition for (7) in terms of 1-forms.

Proposition 1. A generalized change of state coordinates Φ of the form (4), transforming the representation (2) into (5), exists if and only if for $i = 1, \dots, n$

$$D_{\wedge}\omega_i = \sum_{j=1}^{n} C_{ij} \wedge \omega_j + \sum_{I=1}^{m} \sum_{\eta_I=0}^{\beta_I - 1} C_{iI(\eta_I)} \wedge \mathrm{d}u_I^{(\eta_I)} \tag{28}$$

where ω_i are the 1-forms of adapted cobasis $\{\theta\}$ in (15), $D_{\wedge}$ denotes the exterior differentiation, and C_{ij} and $C_{i,I(\eta_I)}$ are the 1-forms obtained in computing the exterior differentials of ω_i.

To find ω_i's, the first n components of the adapted cobasis $\{\theta\}$ in (15), one can multiply the natural cobasis $\{\mathrm{d}\}$ in (16) with inverse of matrix M in (13) (see [3], p. 28):

$$\{\theta\} = M^{-1}\{\mathrm{d}\}. \tag{29}$$

If the dimension of the extended state space is large, calculation of M^{-1} may be complicated task and then it is easier to calculate ω_i's directly, by orthogonalizing the differentials $\mathrm{d}x_i$ with respect to all vector fields $(-1)^{P_I} L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}}$:

$$\omega_i = \mathrm{d}x_i - \sum_{I=1}^{m} \sum_{P_I=0}^{\alpha_I - \beta_I} (-1)^{P_I} \left\langle \mathrm{d}x_i, L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}} \right\rangle \mathrm{d}u_I^{(\alpha_I - P_I)}. \tag{30}$$

From (30), for $P_I = 0, \dots, \alpha_I - \beta_I$

$$\left\langle \omega_i, L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}} \right\rangle = 0, \tag{31}$$

which means that 1-forms ω_i annihilate the distribution Δ, and to find the new generalized state coordinates $\tilde{x}_i$, it remains to be replaced the nonexact one-forms ω_i by differentials $\mathrm{d}\tilde{x}_i$, as in (19). To find the integrating factors A in (19), one has to take the exterior differential of (19) giving

$$\sum_{j=1}^{n} (\mathrm{d}A_{ij} \wedge \omega_j + A_{ij} D_{\wedge}\omega_j) + \sum_{I=1}^{m} \sum_{\eta_I=0}^{\beta_I - 1} \mathrm{d}A_{iI(\eta_I)} \wedge \mathrm{d}u_I^{(\eta_I)} = 0. \tag{32}$$

That way we get a system of first order partial differential equations for finding the integrating factors A. Finally, integration of (19) gives the new state variables $\tilde{x}_i$.

In the extended state space the following codistributions can be defined

$$\begin{cases} \mathcal{H}_1^\beta &= \operatorname{span}\{\mathrm{d}x_i, \mathrm{d}u_I, \ldots, \mathrm{d}u_I^{(\alpha_I)}\} \\ \mathcal{H}_k^\beta &= \{\omega \in \mathcal{H}_{k-1}^\beta \mid \dot{\omega} \in (\mathcal{H}_{k-1}^\beta + \operatorname{span}_{\mathcal{K}}\{\mathrm{d}u_I, \ldots, \mathrm{d}u_I^{(\beta_I)}\})\} \end{cases} \tag{33}$$

for $k = 1, \ldots, \wedge + k$, where $\wedge = \max\{\alpha_i - \beta_i\}$. The 1-forms ω_i, or alternatively, $\Delta^\perp$, can be found as the limiting term $\mathcal{H}_{\wedge+2}^\beta$ of the decreasing sequence $\mathcal{H}_k^\beta$.

Theorem 2. The codistributions $\mathcal{H}_k^\beta$, $k = 1, \ldots, \max\{\alpha_I - \beta_I\}$ are integrable if and only if the Delaleau–Respondek condition (7) is satisfied.

P r o o f. $\mathcal{H}_k^\beta$ is integrable if and only if its maximal annihilator $S_k^\beta \triangleq (\mathcal{H}_k^\beta)^\perp$ is involutive. We will calculate the distributions S_k^β and show that they are involutive iff the condition (7) holds.

We represent the codistributions $\mathcal{H}_k^\beta$, for $k = 2, \ldots, \wedge + 2$, in a form that shows their explicit dependence on differentials $\mathrm{d}u_I, \ldots, \mathrm{d}u_I^{(\mu_I)}$:

$$\mathcal{H}_k^\beta = \operatorname{span}_{\mathcal{K}}\{\omega_i^{[k-1]}, \mathrm{d}u_I, \ldots, \mathrm{d}u_I^{(\mu_I)}\}, \tag{34}$$

where $\omega_i^{[0]} = \mathrm{d}x_i$, and $\mu_I = \max(\beta_I, \alpha_I - k + 1)$. From (33), the 1-forms $\omega_i^{[k-1]}$ in (34) are obtained by orthogonalizing the differentials $\mathrm{d}x_i$ step by step with respect to the vector fields

$$\left\{ L_F^{\rho_I} \frac{\partial}{\partial u_I^{(\alpha_I)}}, \ldots, \frac{\partial}{\partial u_I^{(\alpha_I)}} \right\} \tag{35}$$

where $\rho_I = \min(k-2, \alpha_I - \beta_I)$. That is

$$\begin{aligned} \omega_i^{[0]} &= \mathrm{d}x_i \\ \omega_i^{[k+1]} &= \mathrm{d}x_i - \sum_{i=1}^{m} \sum_{P_I=0}^{\min(k,\alpha_I-\beta_I)} (-1)^{(P_I)} \left\langle \mathrm{d}x_i, L_F^{P_I} \tfrac{\partial}{\partial u_I^{(\alpha_I)}} \right\rangle \mathrm{d}u_I^{(\alpha_I - P_I)}, \\ &\qquad\qquad k = 0, \ldots, \wedge. \end{aligned} \tag{36}$$

At the last step $k = \wedge$, one gets from (36), the 1-forms ω_i, defined by (30). Therefore, the maximal annihilator of codistribution $\mathcal{H}_k^\beta$ is a distribution S_k^β, spanned by the vector fields (35).

The necessary and sufficient involutivity condition of S_k^β is, that the Lie brackets of vector fields from (35) also belong to S_k^β, i. e. are the linear combinations of vector fields from (35). Due to formula (10), nonzero Lie barackets can not be expressed as the linear combinations of vector fields from (35). Consequently, the distributions S_k^β are involutive iff the vector fields (35) commute.

One can easily see, that

$$S_1^\beta \subseteq S_2^\beta \subseteq \ldots \subseteq S_{\wedge+2}^\beta = \Delta,$$

where Δ is spanned by the vectorfields specified in the Delaleau–Respondek condition (7). □

Remark. The conditions of Theorem 2 if applied to the special case of single-input systems reduce to the conditions, given in Theorem 1 in [6].

If the number of vectorfields, specified by the Delaleau–Respondek condition (7) is not much larger than the number of one-forms ω_i, then the new generalized state variables $\tilde{x}_i$ in (20) are easier to find as the solutions of the partial differential equations (22) rather than applying formula (30). According to (19), the new state variables $\tilde{x}$ myst be the integrals of the codistribution $\Delta^{\perp}$. Since $\Delta^{\perp}$ is the maximal annihilator of Δ, all one-forms, belonging to $\Delta^{\perp}$ must annihilate Δ. Consequently, the differentials $\mathrm{d}\tilde{x} \in \Delta^{\perp}$ must satisfy $\langle \mathrm{d}\tilde{x}_i, \Delta\rangle = 0$, $\forall i = 1, \ldots, n$. Then, due to (9), one can find the new generalized state variables as invariants of the vector fields $(-1)^{P_I} L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}}$, i.e. as functions Φ_i that remain constant along the trajectories of the vector fields:

$$\left\langle \mathrm{d}\tilde{x}, L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}} \right\rangle = 0. \tag{37}$$

Due to (10), all $u_I^{(\eta_I)}$, $\eta_I = 0, \ldots, \beta_I - 1$, are the solutions of (37), but additionally, the system of differential equations has n functionally independent solutions as functions of x, u and the time derivatives of u. These independent solutions can be defined as the new generalized state variables.

However, when the number of vector fields $L_F^{P_I} \frac{\partial}{\partial u_I^{(\alpha_I)}}$ compared to n, is very large, the system of differential equations (37) may be difficult to solve.

Remark. In some cases the codistribution $\Delta^{\perp}$ can be split into integrable and independent sub-codistributions $\Delta_1^{\perp}, \Delta_2^{\perp}, \ldots, \Delta_\rho^{\perp}$, $\Delta_1^{\perp} \cup \Delta_2^{\perp} \cup \ldots \cup \Delta_\rho^{\perp} = \Delta^{\perp}$ and $\Delta_i^{\perp} \cap \Delta_j^{\perp} = \emptyset$ for $\forall i, j = 1, \ldots, \rho$. Grouping the 1-forms ω_i as $\omega_{i_\zeta} \in \Delta_\zeta^{\perp}$ $\forall \zeta = 1, \ldots, \rho$, where $\sum_{\zeta=1}^{\rho} \operatorname{codim} \Delta_\zeta^{\perp} = n$, allows to simplify the determination of integrating factors A, since the condition (28) splits into ρ subsystems

$$D_\wedge \omega_{i_\zeta} \in \Delta_\zeta^{\perp}.$$

The interesting problem for the future research is to specify the restrictions for (2) when such splitting is possible.

4. EXAMPLE

We consider the same example of a crane as in Delaleau and Respondek [2]:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\dfrac{g \sin x_1}{u_1} - \dfrac{2x_2}{u_1}\dot{u}_1 - \dfrac{\cos x_1}{u_1}\ddot{u}_2, \end{cases} \tag{38}$$

with the inputs u_1 as the length of the rope and u_2 as the trolley position, and with the state variables x_1 as the angle between the rope and vertical axis and x_2 as its time derivative. In this example

$$F = x_2 \frac{\partial}{\partial x_2} - \left(\frac{g \sin x_1}{u_1} + \frac{2x_2}{u_1}\dot{u}_1 + \frac{\cos x_1}{u_1}\ddot{u}_2 \right) \frac{\partial}{\partial x_2} + \dot{u}_1 \frac{\partial}{\partial u_1} + \dot{u}_2 \frac{\partial}{\partial u_2} + \ddot{u}_2 \frac{\partial}{\partial \dot{u}_2},$$

$$L_F \frac{\partial}{\partial \dot{u}_1} = \frac{2x_2}{u_1} \cdot \frac{\partial}{\partial x_2} - \frac{\partial}{\partial u_1},$$

$$L_F \frac{\partial}{\partial \ddot{u}_2} = \frac{\cos x_1}{u_1} \cdot \frac{\partial}{\partial x_2} - \frac{\partial}{\partial \dot{u}_2}, \quad L_F^2 \frac{\partial}{\partial \ddot{u}_2} = \left(\frac{2 \cos x_1}{u_1^2} \dot{u}_1 - \frac{\sin x_1}{u_1} x_2 \right) \cdot \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u_2}$$

and the following Lie brackets equal to zero:

$$\left[\frac{\partial}{\partial \dot{u}_1}, L_F \frac{\partial}{\partial \dot{u}_1} \right] = \left[\frac{\partial}{\partial \dot{u}_1}, L_F \frac{\partial}{\partial \ddot{u}_2} \right] = 0,$$

$$\left[\frac{\partial}{\partial \ddot{u}_2}, L_F \frac{\partial}{\partial \dot{u}_1} \right] = \left[\frac{\partial}{\partial \ddot{u}_2}, L_F \frac{\partial}{\partial \ddot{u}_2} \right] = 0.$$

So, according to Theorem 1, one can remove from (38) either $\dot{u}_1$ or $\ddot{u}_2$. To remove $\dot{u}_1$, one has to replace the natural basis, consisting of partial derivatives of state variables, control variables and their time derivatives, by adapted basis (12) as follows:

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_1} - \frac{2x_2}{u_1} \cdot \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial \dot{u}_1}, \frac{\partial}{\partial \dot{u}_2}, \frac{\partial}{\partial \ddot{u}_2} \right\}.$$

The corresponding adapted co-basis (15) contains then the following 1-forms

$$\omega_1 = \mathrm{d}x_1, \quad \omega_2 = \mathrm{d}x_2 + \frac{2x_2}{u_1} \mathrm{d}u_1.$$

Note that $D_\wedge \omega_2 = -\frac{\omega_2}{x_2} \wedge \mathrm{d}x_2$. So, the 1-forms ω_1 and ω_2 satisfy the condition (28) and according to (19) we need to replace $\mathrm{d}x_2$ by $\mathrm{d}\tilde{x}_2 = A(x_2, u_1)\omega_2$. The exterior differentiation of the last formula gives us a differential equation with partial derivatives

$$\frac{\partial A}{\partial x_2} \cdot \frac{2x_2}{u_1} - \frac{\partial A}{\partial u_1} + \frac{2}{u_1} A = 0,$$

the solution of which is $A = u_1^2$ yielding

$$\mathrm{d}\tilde{x}_2 = u_1^2 \mathrm{d}x_2 + 2x_2 u_1 \mathrm{d}u_1.$$

Therefore, the new generalized state variables are

$$\tilde{x}_1 = x_1, \quad \tilde{x}_2 = u_1^2 x_2$$

yielding the same generalized state equations, as in [2]

$$\begin{cases} \dot{\tilde{x}}_1 &= \dfrac{\tilde{x}_2}{u_1^2} \\ \dot{\tilde{x}}_2 &= -u_1 \left(g \sin \tilde{x}_1 + \ddot{u}_2 \cos \tilde{x}_1 \right). \end{cases}$$

It is more complicated to remove $\ddot{u}_2$ from the generalized state equations. The adapted basis (12) in this case is

$$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \frac{\partial}{\partial \dot{u}_1}, \frac{\partial}{\partial \dot{u}_2} - \frac{\cos x_1}{u_1} \cdot \frac{\partial}{\partial x_2}, \frac{\partial}{\partial \ddot{u}_2} \right\}$$

and the corresponding co-basis contains the 1-forms

$$\omega_1 = \mathrm{d}x_1, \quad \omega_2 = \mathrm{d}x_2 + \frac{\cos x_1}{u_1}\mathrm{d}\dot{u}_2.$$

Since

$$D_\wedge \omega_2 = -\frac{\sin x_1}{u_1}\omega_1 \wedge \mathrm{d}\dot{u}_2 + \frac{\omega_2}{u_1} \wedge \mathrm{d}u_1,$$

according to (28) there must exist an involutive distribution, annihillating the 1-forms $\{\omega_1, \omega_2, \mathrm{d}u_1\}$. The new state variables must be defined according to (19) as follows

$$\mathrm{d}\tilde{x}_1 = \mathrm{d}x_1, \quad \mathrm{d}\tilde{x}_2 = A\omega_1 + B\omega_2 + C\mathrm{d}u_1 = A\mathrm{d}x_1 + B\mathrm{d}x_2 + B\frac{\cos x_1}{u_1}\mathrm{d}\dot{u}_2 + C\mathrm{d}u_1.$$

The exterior differentiation of $\mathrm{d}\tilde{x}_2$ yields the following system of differential equations with partial derivatives:

$$\begin{cases} \dfrac{\partial A}{\partial x_2} = \dfrac{\partial B}{\partial x_1}, \quad \dfrac{\partial A}{\partial u_1} = \dfrac{\partial C}{\partial x_1} \\[2ex] \dfrac{\partial A}{\partial \dot{u}_2} - \dfrac{\partial B}{\partial x_1} \cdot \dfrac{\cos x_1}{u_1} + B\dfrac{\sin x_1}{u_1} = 0 \\[2ex] \dfrac{\partial B}{\partial u_1} = \dfrac{\partial C}{\partial x_2}, \quad \dfrac{\partial B}{\partial \dot{u}_2} - \dfrac{\partial B}{\partial x_2} \cdot \dfrac{\cos x_1}{u_1} = 0 \\[2ex] \dfrac{\partial B}{\partial u_1} \cdot \dfrac{\cos x_1}{u_1} - B\dfrac{\cos x_1}{u_1^2} - \dfrac{\partial C}{\partial \dot{u}_2} = 0. \end{cases}$$

By solving this system of equations we get the following integration factors:

$$A = -\frac{\dot{u}_2}{u_1}\sin x_1, \quad B = 1, \quad C = -\frac{\dot{u}_2}{u_1^2}\cos x_1.$$

The trivial solution $A = B = C = 0$ does not fit since it yields $\mathrm{d}\tilde{x}_2 = 0$ which does not define the coordinate transformations. Therefore, the new state variables are

$$\tilde{x}_1 = x_1, \quad \tilde{x}_2 = x_2 + \frac{\dot{u}_2}{u_1}\cos x_1,$$

yielding the new system of state equations

$$\begin{cases} \dot{\tilde{x}}_1 & = \tilde{x}_2 - \frac{\dot{u}_2}{u_1}\cos\tilde{x}_1 \\ \dot{\tilde{x}}_2 & = -\frac{1}{u_1}\left(g\sin\tilde{x}_1 + 2\dot{u}_1\tilde{x}_2 + \dot{u}_2\tilde{x}_2\sin\tilde{x}_1\right) + \frac{1}{u_1^2}\left(\dot{u}_1\dot{u}_2\cos\tilde{x}_1 + \dot{u}_2^2\cos\tilde{x}_1\sin\tilde{x}_1\right) \end{cases}$$

as in [2].

5. CONCLUSIONS

In this paper, the problem of lowering the maximal orders of time derivatives of inputs in generalized state equations via generalized state coordinate transformation is studied. The alternative, computation oriented proof is presented for theorem proved in [2], which gives necessary and sufficient conditions for existence of such a transformation in terms of commutativity of certain vector fields. The dual conditions in terms of 1-forms are given on the basis of the new proof. The new state variables can be defined either as the invariants of the vector fields, defined by the necessary and sufficient conditions, or as the functions, whose differentials are the linear combinations of certain, not necessarily exact 1-forms. The formula to find these 1-forms is presented.

ACKNOWLEDGEMENT

This work is partially supported by the Estonian Science Foundation under Grant 5405.

(Received January 11, 2005.))

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