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SHAPE FACTOR EXTREMES FOR PROLATE SPHEROIDS

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Microscopic prolate spheroids in a given volume of an opaque material are considered. The extremes of the shape factor of the spheroids are studied. The profiles of the spheroids are observed on a random planar section and based on these observations we want to estimate the distribution of the extremal shape factor of the spheroids. We show that under a tail uniformity condition the Maximum domain of attraction is stable. We discuss the normalising constants (n.c.) for the extremes of the spheroid and profile shape factor. Comparing the tail behaviour of the distribution of the profile and spheroid shape factor we show the relation between the n.c. of the profile shape factor (which can be estimated) and the n.c. of the spheroid shape factor (cannot be estimated directly) which are needed for the prediction of the tail behaviour of the shape factor. The paper completes the study [9] for prolate spheroids.

Keywords: stereology of extremes, shape factor, normalising constants, tail uniformity

Mathematics Subject Classification: 60G70, 62G32, 62P30

1. INTRODUCTION

In recent years the *stereology of extremes* is attracting more and more interest. The theoretical results and direct application to the material science can be seen e. g. in [3, 11, 12, 13], where the extreme of the size in the so called Wicksell's corpuscle problem is studied. In a series of papers [7, 8, 9, 10], e. g., we have discussed the extremes of the size or shape factor respectively for the *oblate spheroids* rather than for the balls. We continue in the paper [9] and in particular we complete the previous study for the *prolate spheroids*.

The main results of the paper are summarised in Theorems 1 and 4. The Maximum domain of attraction of the spheroid shape factor is shown to be 'translated' to the profiles under the tail uniformity condition. The tail uniformity condition is discussed briefly in Sections 3 and 5, for more detailed discussion see [10].

The possible statistical application of the main results is described in Section 4. We conclude the paper in Section 5 giving some examples of the tail behaviour of the spheroid shape factor and its relation to the tail behaviour of the profile shape factor. Comparison of these two tail behaviours is an essential step toward the adjustment of the normalising constants which are needed for the prediction of the distribution of the shape factor extremes of spheroids based on the observation of the profile shape factor extremes.

2. PROLATE SPHEROIDS

Random prolate spheroidal non-overlapping particles distributed in a given volume of an opaque material are considered in our study. Recall that prolate spheroids are ellipsoids with *two equal minor semi-axes* of length V, and *one major semi-axis* of length X. The random nature of the particles lies in the fact that (X, V) form a random vector independent on the position and orientation of the particle. The particles arrangement is, moreover, considered to be *isotropic*. The restriction to the prolate family rather than considering general ellipsoids is explained in [2].

The prolate spheroids are usually characterised by their *size* V and their *shape factor* S, where $S = X^2/V^2 - 1$. Evidently, (V, S) form a bivariate random vector. The joint distribution of (V, S) is further considered to be absolutely continuous w.r.t. bivariate Lebesgue measure. The joint probability density function (p.d.f.) of (V, S) is further denoted $g(v, s)$.

The spheroids cannot be observed directly. We can observe random profiles – result of a section of the given volume by a (random) plane. The profiles form a sample of random ellipses. The profiles are characterised, analogous to the original particles, by their size W (length of the minor semi-axis) and shape factor $T = Y^2/W^2 - 1$, where Y is the length of the major semi-axis. The joint distribution of (W, T) is again absolutely continuous and its joint p.d.f. is

$$f(w,t) = \frac{w}{2M(1+t)^2} \int_w^{X_f} \int_t^{S_f} \frac{(1+s)^{3/2} g(v,s)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-t}\sqrt{v^2-w^2}}, \tag{1}$$

where M is the population mean size of particles (half of the mean calliper diameter). ∞ is the upper endpoints of the distribution of the size, i.e., $X_f = \inf\{x : \mathsf{P}[X \le x] = 1\}$, and S_f is the upper endpoint of the shape factor distribution. In what follows we will always consider $X_f = \infty$.

3. STABILITY OF MAXIMUM DOMAIN OF ATTRACTION

Recall that if for a univariate cumulative distribution function (c.d.f.) H exist *normalising constants* (n.c.) a_n and b_n such that as $n \to \infty$ it holds

$$H^n(a_n x + b_n) \longrightarrow \begin{cases} \Lambda(x) = \exp(-\mathrm{e}^{-x}), & x \in \mathbb{R}, \text{ (Gumbel distr.), or} \\ \Phi_\alpha(x) = \exp(-x^{-\alpha}), & x \ge 0, \text{ (Fréchet distr.), or} \\ \Psi_\alpha(x) = \exp(-(-x)^\alpha), & x \le 0, \text{ (Weibull distr.)} \end{cases} \tag{2}$$

for some $\alpha > 0$, then H is said to belong into a *maximum domain of attraction* ($H \in \mathrm{MDA}(\cdot)$) of a c.d.f. Λ, Φ or Ψ respectively. There are no other possible limiting distributions.

There are well known sufficient conditions for $H \in \mathrm{MDA}(\cdot)$ if H is an absolutely continuous c.d.f. with a p.d.f. h. In particular if one of the following conditions

$$\lim_{s \nearrow S_f} \frac{h(s+ub(s))}{h(s)} = \mathrm{e}^{-u}, \quad u \in \mathbb{R}, \tag{3}$$

$$\lim_{s \to \infty} \frac{h(us)}{h(s)} = u^{-(\alpha+1)}, \quad u > 0,\ S_f = +\infty, \tag{4}$$

$$\lim_{s \searrow 0} \frac{h(S_f - us)}{h(S_f - s)} = u^{\alpha-1}, \quad u > 0,\ S_f < +\infty \tag{5}$$

hold for some auxiliary function $b(\cdot)$ or for some $\alpha > 0$ then

$$(3) \Rightarrow H \in \mathrm{MDA}(\Lambda), \quad (4) \Rightarrow H \in \mathrm{MDA}(\Phi_\alpha), \text{ and } \quad (5) \Rightarrow H \in \mathrm{MDA}(\Psi_\alpha).$$

Now we are ready to prove the stability of MDA.

Theorem 1. (Stability of MDA for the shape factor I) Suppose that the conditional density function $g_v(s)$ of the shape factor given the size $V = v$ satisfies one of the conditions (3), (4) or (5) uniformly in v. Then for the conditional d.f. $f_w(t)$ of the profile shape factor given the profile size $W = w$ the same condition holds for a parameter $\beta = \alpha + 1/2$ for the conditions (4) and (5).

Remark 2. In particular the Maximum domain of attraction is not changed up to the parameter of the limiting distribution. Since the MDA can be estimated from the observed data (profiles) and since for the Weibull and Fréchet distributions the parameter can be also estimated based on the profiles one can conclude that under the uniformity condition we know also the limiting distribution, and its parameter respectively, for the particle shape factor extremes.

Remark 3. The uniformity assumption of Theorem 1 in particular implies that the parameter α of (4) and (5) is the same for all values of v and also that the auxiliary function $b(\cdot)$ of (3) is the same for all values of v. It also means that the upper endpoint S_f does not depend on v.

P r o o f. We shall treat the three limiting distributions separately. More details can be found in the proof of similar theorem for the oblate spheroids, see [7, 8].

Fréchet limit: We need to study

$$\begin{aligned}\lim_{t\to\infty} \frac{f_w(at)}{f_w(t)} &= \lim_{t\to\infty} \frac{(1+t)^2}{(1+at)^2} \frac{\int_w^\infty \int_{at}^\infty \frac{(1+s)^{3/2} g(v,s)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-at}\sqrt{v^2-w^2}}}{\int_w^\infty \int_t^\infty \frac{(1+s)^{3/2} g(v,s)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-t}\sqrt{v^2-w^2}}} \\ &= \lim_{t\to\infty} \frac{a(1+t)^2}{(1+at)^2} \frac{\int_w^\infty \int_t^\infty \frac{(1+as)^{3/2} g_v(as) g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{as}\sqrt{as-at}\sqrt{v^2-w^2}}}{\int_w^\infty \int_t^\infty \frac{(1+s)^{3/2} g_v(s) g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-t}\sqrt{v^2-w^2}}}\end{aligned} \tag{6}$$

$$= \lim_{t\to\infty} a^{-1/2} \frac{(1+t)^2}{(a^{-1}+t)^2} \frac{\int_w^\infty \int_t^\infty \frac{(a^{-1}+s)^{3/2} g_v(as) g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-t}\sqrt{v^2-w^2}}}{\int_w^\infty \int_t^\infty \frac{(1+s)^{3/2} g_v(s) g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-t}\sqrt{v^2-w^2}}}$$

$$= a^{-\left((\alpha+1/2)+1\right)}.$$

The last equality follows from the fact that $(a^{-1}+t)/(1+t) \to 1$ as $t \to \infty$ and $g_v(as)/g_v(s) \to a^{-(\alpha+1)}$ as $s \geq t \to \infty$ uniformly in v and using Lemma 1.2.1 of [5] for the limit of the ratio of integrals.

Weibull limit: Using similar arguments as for (6) we obtain

$$\lim_{t\to 0} \frac{f_w(S_f - at)}{f_w(S_f - t)} = \lim_{t\to 0} \frac{(1+S_f-t)^2}{(1+S_f-at)^2} \frac{\int_w^\infty \int_{S_f-at}^{S_f} \frac{(1+s)^{3/2} g(v,s)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-(S_f-at)}\sqrt{v^2-w^2}}}{\int_w^\infty \int_{S_f-t}^{S_f} \frac{(1+s)^{3/2} g(v,s)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-(S_f-t)}\sqrt{v^2-w^2}}}$$

$$= \lim_{t\to 0} \frac{a(1+S_f-t)^2}{(1+S_f-at)^2} \frac{\int_w^\infty \int_0^t \frac{(1+S_f-as)^{3/2} g_v(S_f-as) g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{S_f-as}\sqrt{(S_f-as)-(S_f-at)}\sqrt{v^2-w^2}}}{\int_w^\infty \int_0^t \frac{(1+S_f-s)^{3/2} g_v(S_f-s) g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{S_f-s}\sqrt{(S_f-s)-(S_f-t)}\sqrt{v^2-w^2}}} \tag{7}$$

$$= \lim_{t\to 0} \frac{a^{1/2}(1+S_f-t)^2}{(1+S_f-at)^2} \frac{\int_w^\infty \int_0^t \frac{(1+S_f-as)^{3/2} g_v(S_f-as) g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{S_f-as}\sqrt{t-s}\sqrt{v^2-w^2}}}{\int_w^\infty \int_0^t \frac{(1+S_f-s)^{3/2} g_v(S_f-s) g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{S_f-s}\sqrt{t-s}\sqrt{v^2-w^2}}}$$

$$= a^{\alpha-1/2} = a^{(\alpha+1/2)-1}.$$

The second equality follows from the substitutions $s \leftrightarrow S_f - as$ and $s \leftrightarrow S_f - s$ in the numerator and denominator respectively. Lemma 1.2.1 of [5] and uniformity (in v) of the limit $g_v(S_f - as)/g_v(S_f - s) \to a^{\alpha-1}$ as $s \searrow 0$ finish the proof.

Gumbel limit: The situation is now complicated by the auxiliary function $b(\cdot)$. Recall (see e. g. Chapter 3 in [4]) that $b(s)$ can be chosen such that it is differentiable for $s < S_f$ and

$$\lim_{s\to\infty} b'(s) = 0 \quad \text{and} \quad \lim_{s\to\infty} s^{-1}b(s) = 0 \quad \text{if} \quad S_f = \infty, \quad \text{or}$$

$$\lim_{s\to S_f} b'(s) = 0 \quad \text{and} \quad \lim_{s\to S_f} (S_f - s)^{-1}b(s) = 0 \quad \text{if} \quad S_f < \infty.$$

Hence, using for the second equality below the substitution $s \leftrightarrow s + ab(s)$ and the limit

$$\lim_{t\to S_f} \frac{1+t+ab(t)}{1+t} = 1$$

following from the properties of $b(\cdot)$, we obtain

$$\begin{aligned}\lim_{t\to S_f}\frac{f_w(t+ab(t))}{f_w(t)} &= \lim_{t\to S_f}\frac{(1+t)^2}{(1+t+ab(t))^2}\frac{\int_w^\infty\int_{t+ab(t)}^{S_f}\frac{(1+s)^{3/2}g(v,s)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-(t+ab(t))}\sqrt{v^2-w^2}}}{\int_w^\infty\int_t^{S_f}\frac{(1+s)^{3/2}g(v,s)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-t}\sqrt{v^2-w^2}}}\\ &= \lim_{t\to S_f}\frac{\int_w^\infty\int_t^{S_f}\frac{(1+s+ab(s))^{3/2}(1+ab'(s))g_v(s+ab(s))g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s+ab(s)}\sqrt{(s+ab(s))-(t+ab(t))}\sqrt{v^2-w^2}}}{\int_w^\infty\int_t^{S_f}\frac{(1+s)^{3/2}g_v(s)g_V(v)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-t}\sqrt{v^2-w^2}}}\\ &= \exp\{-s\}\end{aligned} \tag{8}$$

by Lemma 1.2.1 of [5] again. We need in particular to check only that

$$\lim_{t\to S_f}\frac{(s+ab(s))-(t+ab(t))}{s-t}=1 \quad \text{where} \quad s>t.$$

But this follows from the properties of $b(\cdot)$. It holds

$$\frac{(s+ab(s))-(t+ab(t))}{s-t}=1+a\frac{b(s)-b(t)}{s-t}=1+ab'(\xi),$$

for some $t\le\xi\le s$ and since $b'(t)\to 0$ as $t\to S_f$ the proof is finished. □

Theorem 4. (Stability of MDA for the shape factor II) Suppose that the conditional density function $g_v(s)$ of the shape factor given the size $V=v$ satisfies one of the conditions (3), (4) or (5) uniformly in v and for some parameter $\alpha>0$ respectively. Then

1. the marginal d.f. $g(s)$ of the shape factor and the conditional d.f. $g_{>v}(s)$ of the shape factor given the size $V>v$ satisfy, with the parameter α respectively, the same of the conditions (3) – (5) as $g_v(s)$ does.

2. the marginal d.f. $f(t)$ of the profile shape factor and the conditional d.f. $f_{>w}(s)$ of the profile shape factor given the profile size $W>w$ satisfy, with the parameter $\beta=\alpha+1/2$ respectively, the same of the conditions (3) – (5) as $g_v(s)$ does.

P r o o f. The proof is an analogue of the proof of Theorem 1. Let us start with the marginal density functions.

To check that for the Fréchet limiting case it holds

$$\lim_{s\to\infty}\frac{g_S(as)}{g_S(s)}=\lim_{s\to\infty}\frac{\int_0^\infty g_v(as)g_V(v)\,\mathrm{d}v}{\int_0^\infty g_v(s)g_V(v)\,\mathrm{d}v}=a^{-(\alpha+1)}$$

is an easy consequence of the uniformity assumption. The other two limiting cases are completely similar.

The marginal density of the profile shape factor is

$$f_T(t) = \int_0^\infty f(w,t)\,\mathrm{d}w = \frac{1}{2M(1+t)^2}\int_0^\infty \int_t^{S_f} \frac{(1+s)^{3/2} v g(v,s)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-t}}. \qquad (9)$$

The arguments of (6) – (8) can be repeated again and the proof for the marginal density functions can be concluded.

For the conditional density functions let us first note that

$$g_{>v}(s) = \frac{\int_v^\infty g_u(s) g_V(u)\,\mathrm{d}u}{\int_v^\infty g_V(u)\,\mathrm{d}u}, \quad \text{and} \quad f_{>w}(t) = \frac{\int_w^\infty f_u(t) f_W(u)\,\mathrm{d}u}{\int_w^\infty f_W(u)\,\mathrm{d}u} \qquad (10)$$

and the denominators do not depend on the shape factor. Hence we must analyse for the Fréchet limiting case

$$\lim_{s\to\infty} \frac{g_{>v}(as)}{g_{>v}(s)} = \frac{\int_v^\infty g_u(as) g_V(u)\,\mathrm{d}u}{\int_v^\infty g_u(s) g_V(u)\,\mathrm{d}u} = a^{-(\alpha+1)}$$

again as a simple consequence of the uniformity assumption. The same argument may be used for the other two limiting cases.

For the last part of the proof it is sufficient to use arguments of (6) – (8) once again and the fact that

$$f_{>w}(t) = \frac{1}{2M(1+t)^2}\int_w^\infty \int_t^{S_f} \frac{(1+s)^{3/2}\sqrt{v^2-w^2}\, g(v,s)\,\mathrm{d}s\mathrm{d}v}{\sqrt{s}\sqrt{s-t}} \frac{1}{\int_w^\infty f_W(u)\,\mathrm{d}u}. \qquad \square$$

Remark 5. The reason for the uniformity assumption can be of course seen from the proofs of the Theorems 1 and 4. There is, however, also an intuitive reasoning for the uniformity assumption. Since it holds that $0 \le W \le V$ it is clear that in the sample of profiles with a given size, the spheroids of any greater size can belong. Hence the limiting behaviour of the profile shape factor extreme is influenced by the limiting behaviour of the shape factor extreme of any spheroid whose size is larger than the size of the observed profile.

If the uniformity assumption is not valid then we may expect that there can be some dominating extreme value behaviour for the shape factor extremes and this dominating behaviour is connected to some values of the size. But then it would be impossible to recover the non-dominating limiting behaviour of the shape factor for the other (conditioning) values of the size.

4. NORMALISING CONSTANTS AND STATISTICAL APPLICATION

In this section we shall discuss the possible statistical application of Theorems 1 and 4. In particular we shall describe a method how the extremes of the shape factor of spheroids can be predicted using the shape factor extremes of the profiles under the uniformity assumptions of Theorems 1 and 4.

Recall, that *normalising constants* form a sequence of pairs (a_n, b_n) such that (2) holds. In particular it follows by (2) that normalising constants are not defined *uniquely*. Actually if (a_n, b_n) are appropriate n.c. then consider any sequence (a'_n, b'_n) such that $a'_n/a_n \to 1$ and $(b_n - b'_n)/a_n \to 0$ as $n \to \infty$. It is easy to verify that a'_n and b'_n are also n.c. Thus one could consider a class of equivalent normalising constants. Below we shall refer to the normalising constants having the previous observation in mind. To determine the n.c. we are usually required to analyse the tail behaviour of the distribution function at hand.

The normalising constants for the observed profile shape factors can be estimated in many ways. The two most common include maximum likelihood estimator based on the k largest observations and the estimator based on the k block's maxima. The MDA of the observed sample can be also determined using the estimator of the power parameter α in (3)–(5). See [1], e. g., for more details.

Let us say that $(\widehat{a}_n, \widehat{b}_n)$ are the estimated normalising constants for the observed shape factor profiles (for the marginal distribution, e. g.) and that Υ_α is the limiting distribution, $\alpha = \infty$ for the Gumbel distribution. Hence we know that the shape factor of the spheroids belongs to the MDA$(\Upsilon_{\alpha-1/2})$. The approximated distribution of the extremal shape factor of the spheroids $S_{N:N}$ is therefore

$$\mathsf{P}\left[S_{N:N} < a_N x + b_N\right] \doteq \Upsilon_{\alpha-1/2}(x),$$

where N is the estimated number of particles in the given volume of material and (a_N, b_N) are the normalising constants which must be derived from the estimate $(\widehat{a}_n, \widehat{b}_n)$. This question will be discussed further.

The recalculation of the profile n.c. to the spheroid n.c. is a procedure requiring the determination of the normalising constants according to the parametric tail assumption on the corresponding distribution function. In this connection the following lemma is often useful.

Lemma 6. (Normalising constants) Suppose that a distribution function K has an upper endpoint M_f. Then the following statements are valid.

1. If $M_f = \infty$ the d.f. K belongs to the Gumbel domain of attraction and if there exist constants $\alpha > 0, \beta, \gamma > 0, \delta > 0$ such that

$$\lim_{v\to\infty} \frac{1 - K(v)}{\alpha v^\beta \mathrm{e}^{-\gamma v^\delta}} = 1,$$

then the normalising constants can be chosen as

$$a_n = \left(\frac{\log n}{\gamma}\right)^{1/\delta - 1} \frac{1}{\gamma\delta},$$
$$b_n = \left(\frac{\log n}{\gamma}\right)^{1/\delta} + \frac{\frac{\beta}{\delta}(\log\log n - \log\gamma) + \log\alpha}{\left(\frac{\log n}{\gamma}\right)^{1-1/\delta}\gamma\delta}.$$

2. If the distribution function K belongs to the Fréchet domain of attraction and if there exist constants $\alpha > 0, \beta, \gamma > 0$ such that

$$\lim_{v\to\infty} \frac{1-K(v)}{\alpha v^{-\gamma}} = 1,$$

then the normalising constants can be chosen as

$$a_n = (n\alpha)^{1/\gamma},\ b_n = 0.$$

3. If the distribution function K belongs to the Weibull domain of attraction and if there exist constants $\beta > 0$ and $\gamma > 0$ such that

$$\lim_{v\to M_f} \frac{1-K(v)}{\gamma(v/M_f)^{\beta}(M_f-v)^{\alpha}} = 1,$$

then the normalising constants can be chosen as

$$a_n = (n\gamma)^{-1/\alpha},\ b_n = M_f.$$

Hence our plan is the following. For some parametric model describing the asymptotic tail behaviour of the spheroid shape factor we shall derive the normalising constants (a_n^s, b_n^s) using Lemma 6, where n is considered to be a variable rather than the constant. The same Lemma 6 and the relation (1) can be used to derive the normalising constants (a_m^p, b_m^p) for the profile shape factor. Comparison of these two pairs of the normalising constants will suggest the way how to estimate the n.c. of the spheroid shape factor using the estimated n.c. of the profile shape factor.

Remark 6. Note that the adjustment of the estimated n.c. of the profiles to the n.c. of the spheroids results in a need of a parametric tail assumption. If we can observe the spheroids directly there is no need to assume parametric model since both the n.c. and the MDA can be estimated directly from the observations nonparametrically.

It is, however, sufficient to consider only an asymptotic class of the parameter model specifying the dominating terms only. If we consider, for example, the *Hall class* (see [6]) the tail of the distribution function K is specified simply by

$$1 - K(u) = Cu^{-\alpha} + o(u^{-\alpha}), \quad \text{as} \quad u \to \infty. \tag{11}$$

Lemma 6 can be applied directly to such a class of distributions.

Note that on the other hand the convergence of the term $o(u^{-\alpha})$ to zero in (11) can be extremely slow. In this case the generalisation of Lemma 6 [10, Lemma 3.8] may be of particular interest. For the case when $1 - K(u) = C(\log u)^{\beta}u^{-\alpha} + o\big((\log u)^{\beta}u^{-\alpha}\big)$ see Lemma 2 in [8].

5. APPROXIMATION OF THE SHAPE FACTOR TAILS

Let us introduce a relation $\approx$ for the distribution functions and probability densities (p.d.f.). Let $K(\cdot)$ be a distribution function and $k(\cdot)$ its p.d.f. and let $\Xi(\cdot)$ and $\xi(\cdot)$ be some suitable functions. We write for M_f being the upper endpoint of the d.f. K

$$\begin{aligned} 1-K(x) \approx \Xi(x) \quad &\Longleftrightarrow \quad \lim_{x\to M_f} \frac{1-K(x)}{\Xi(x)} = 1 \\ k(x) \approx \xi(x) \quad &\Longleftrightarrow \quad \lim_{x\to M_f} \frac{\int_x^{M_f} k(u)\,\mathrm{d}u}{\int_x^{M_f} \xi(u)\,\mathrm{d}u} = 1 \end{aligned} \tag{12}$$

respectively.

In order to apply Lemma 6 we need to study the behaviour of

$$1-F_T(t) = \frac{1}{2M}\int_0^\infty v g_V(v) I(v,t)\,\mathrm{d}v, \tag{13}$$

$$1-F_w(t) = \frac{1}{2M f_W(w)}\int_w^\infty \frac{w g_V(v)}{\sqrt{v^2-w^2}} I(v,t)\,\mathrm{d}v, \tag{14}$$

$$1-F_{>w}(t) = \frac{1}{2M\big(1-F_W(w)\big)}\int_w^\infty \sqrt{v^2-w^2}\, g_V(v) I(v,t)\,\mathrm{d}v, \tag{15}$$

where

$$I(v,t) = \int_t^{S_f} \frac{1}{2\sqrt{s}}\left[\frac{2\sqrt{(s-t)(1+s)}}{1+t} + \log\frac{1+\sqrt{\frac{s-t}{1+s}}}{1-\sqrt{\frac{s-t}{1+s}}}\right] g_v(s)\,\mathrm{d}s,$$

and to compare it with

$$1-G_S(s) = \int_0^\infty g_V(u)\int_s^{S_f} g_u(r)\,\mathrm{d}r\mathrm{d}u, \tag{16}$$

$$1-G_v(s) = \int_s^{S_f} g_v(r)\,\mathrm{d}r, \tag{17}$$

$$1-G_{>v}(s) = \frac{1}{1-G_V(v)}\int_v^\infty g_V(u)\int_s^{S_f} g_u(r)\,\mathrm{d}r\mathrm{d}u, \tag{18}$$

respectively. It is obvious that it is in particular the behaviour of $g_v(s)$ for s close to the upper endpoint S_f (recall that it is assumed to be uniform in v) which needs to be analysed.

The question of uniformity assumption of Theorems 1 and 4 is broadly discussed in [10] and [9]. Let us recall that the generalised Farlie–Gumbel–Morgenstern (FGM) family of bivariate distributions is considered as an appropriate model for the bivariate distribution of size and shape factor when the tail uniformity is assumed.

If we use the generalised FGM family, then the conditional p.d.f. $g_v(s)$ can be for s close to S_f approximated by

$$g_v(s) \approx \zeta(s)L(s,v)\psi\big(G_V(v)\big) + o\big(\zeta(s)\big),$$

where the function $L(\cdot, v)$ is slowly varying at S_f (uniformly in v) and the function $\zeta(s)$ is either regularly varying (therefore the limiting distribution is Fréchet or Weibull) or rapidly varying in S_f (and hence the limiting distribution is Gumbel). For more facts about the regular variation and its application to the theory of sample extremes see e. g. [5].

Hence all we need is to analyse the integral

$$J(t) = \int_t^{S_f} \frac{1}{2\sqrt{s}} \left[\frac{2\sqrt{(s-t)(1+s)}}{1+t} + \log \frac{1+\sqrt{\frac{s-t}{1+s}}}{1-\sqrt{\frac{s-t}{1+s}}} \right] \zeta(s)\,\mathrm{d}s \tag{19}$$

when $t \to S_f$ for different choices of the limiting behaviour of $\zeta(s)$. The further analysis of the normalising constants is then analogous to the discussion of Sections 4 and 5 in [9]. We give here just two examples of exponential and polynomial tail. We shall further assume that the right endpoint $S_f = \infty$ hence we do not need to study the Weibull limiting case.

5.1. Exponential tails with infinite right endpoint

Let us consider that $S_f = \infty$, and the p.d.f. $g_v(s) \approx \zeta(s)\psi\big(G_V(v)\big)$, where the function $\zeta(s) = \alpha s^\beta \exp\{-\gamma s\}$ as $s \to \infty$ (see (12) for the $\approx$ notation). Hence (19) becomes

$$\begin{aligned}
J(t) &= \int_t^{\infty} \frac{1}{2\sqrt{s}} \left[\frac{2\sqrt{(s-t)(1+s)}}{1+t} + \log \frac{1+\sqrt{\frac{s-t}{1+s}}}{1-\sqrt{\frac{s-t}{1+s}}} \right] \alpha s^\beta \exp\{-\gamma s\}\,\mathrm{d}s \\
&= \int_0^{\infty} \frac{1}{2\sqrt{s+t}} \left[\frac{2\sqrt{s(1+s+t)}}{1+t} + \log \frac{1+\sqrt{\frac{s}{1+s+t}}}{1-\sqrt{\frac{s}{1+s+t}}} \right] \alpha (s+t)^\beta \mathrm{e}^{-\gamma(s+t)}\,\mathrm{d}s \\
&= \frac{\alpha t^\beta \mathrm{e}^{-\gamma t}}{t+1} \int_0^\infty \sqrt{\frac{1+s+t}{s+t}}\sqrt{s}\left(1+\frac{s}{t}\right)^\beta \mathrm{e}^{-\gamma s}\,\mathrm{d}s \\
&\quad + \frac{\alpha t^\beta \mathrm{e}^{-\gamma t}}{t^{1/2}} \int_0^\infty \frac{1}{2}\log \frac{1+\sqrt{\frac{s}{1+s+t}}}{1-\sqrt{\frac{s}{1+s+t}}} \left(1+\frac{s}{t}\right)^{\beta-1/2} \mathrm{e}^{-\gamma s}\,\mathrm{d}s \\
&\approx \alpha t^{\beta-1}\mathrm{e}^{-\gamma t} \int_0^\infty \sqrt{s}\mathrm{e}^{-\gamma s}\,\mathrm{d}s + \frac{1}{2}\alpha t^{\beta-1/2}\mathrm{e}^{-\gamma t} \int_0^\infty \log \frac{1+\sqrt{\frac{s}{1+s+t}}}{1-\sqrt{\frac{s}{1+s+t}}} \mathrm{e}^{-\gamma s}\,\mathrm{d}s \\
&\approx 2\alpha t^{\beta-1}\mathrm{e}^{-\gamma t} \int_0^\infty \sqrt{s}\mathrm{e}^{-\gamma s}\,\mathrm{d}s = \alpha\gamma^{-3/2} t^{\beta-1}\mathrm{e}^{-\gamma t}\Gamma\left(\frac{1}{2}\right),
\end{aligned}$$

where $\Gamma(\cdot)$ is the Euler gamma function. Lemma 6 can be now applied directly to this approximation as, for example, (13) and (16) become

$$1 - F_T(t) \approx \frac{1}{2M} \int_0^\infty v g_V(v) \psi\big(G_V(v)\big) \, \mathrm{d}v \, \alpha \gamma^{-3/2} t^{\beta-1} \mathrm{e}^{-\gamma t} \Gamma\left(\frac{1}{2}\right), \tag{20}$$

$$1 - G_S(s) \approx \int_0^\infty g_V(u) \psi\big(G_V(u)\big) \, \mathrm{d}u \, \alpha \gamma^{-1} s^{\beta} \mathrm{e}^{-\gamma s}. \tag{21}$$

5.2. Polynomial tails with infinite right endpoint

Consider the function $\zeta(s) = \alpha s^{-\beta-1}$ now and let us approximate (19) again. It is now little bit more difficult. We get

$$\begin{aligned}
J(t) &= \int_t^\infty \frac{1}{2\sqrt{s}} \left[\frac{2\sqrt{(s-t)(1+s)}}{1+t} + \log \frac{1+\sqrt{\frac{s-t}{1+s}}}{1-\sqrt{\frac{s-t}{1+s}}} \right] \alpha s^{-\beta-1} \, \mathrm{d}s \\
&= \int_0^\infty \frac{1}{2\sqrt{s+t}} \left[\frac{2\sqrt{s(1+s+t)}}{1+t} + \log \frac{1+\sqrt{\frac{s}{1+s+t}}}{1-\sqrt{\frac{s}{1+s+t}}} \right] \alpha (s+t)^{-\beta-1} \, \mathrm{d}s \\
&\approx \alpha \int_0^\infty \sqrt{s}(t+s)^{-\beta-2} \frac{2t+s+1}{t+1} \, \mathrm{d}s \\
&= \alpha(t+1)^{-1} t^{-\beta-1} \int_0^\infty \sqrt{s} \left(1 + \frac{s}{t}\right)^{-\beta-2} \left(2 + \frac{s+1}{t}\right) \mathrm{d}s \\
&\approx \alpha t^{-\beta-1/2} \int_0^\infty \sqrt{s}(1+s)^{-\beta-2}(2+s) \, \mathrm{d}s = t^{-\beta-1/2} \int_1^\infty \sqrt{s-1} s^{-\beta-2} (1+s) \, \mathrm{d}s \\
&= \alpha t^{-\beta-1/2} \int_0^1 \sqrt{(1-s)} s^{\beta-3/2} (1+s) \, \mathrm{d}s \\
&= \alpha t^{-\beta-1/2} \left[\mathrm{B}\left(\frac{3}{2}, \beta - \frac{1}{2}\right) + \mathrm{B}\left(\frac{3}{2}, \beta + \frac{1}{2}\right) \right],
\end{aligned}$$

where $\mathrm{B}(\cdot,\cdot)$ is the beta function. Lemma 6 can be again applied directly to this approximation.

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REFERENCES

[1] S. Coles: An Introduction to Statistical Modeling of Extreme Values. Springer–Verlag, Berlin 2001.

[2] L.-M. Cruz-Orive: Particle size-shape distributions; the general spheroid problem. J. Microsc. *107* (1976), 235–253.

[3] H. Drees and R.-D. Reiss: Tail behavior in Wicksell's corpuscle problem. In: Probability Theory and Applications (J. Galambos and J. Kátai, eds.), Kluwer, Dordrecht 1992, pp. 205–220.

[4] P. Embrechts, C. Klüppelberg, and T. Mikosch: Modelling Extremal Events for Insurance and Finance. Springer–Verlag, Berlin 1997.

[5] L. de Haan: On Regular Variation and Its Application to the Weak Convergence of Sample Extremes. (Mathematical Centre Tract 32.) Mathematisch Centrum, Amsterdam 1975.

[6] P. Hall: On some simple estimates of an exponent of regular variation. J. Roy. Statist. Soc. Ser. B *44* (1982), 37–42.

[7] D. Hlubinka: Stereology of extremes; shape factor of spheroids. Extremes *5* (2003), 5–24.

[8] D. Hlubinka: Stereology of extremes; size of spheroids. Mathematica Bohemica *128* (2003), 419–438.

[9] D. Hlubinka: Extremes of spheroid shape factor based on two dimensional profiles. Kybernetika *42* (2006), 77–94.

[10] D. Hlubinka and S. Kotz: The generalized FGM distribution and its application to stereology of extremes. Far East J. Theoret. Statist. (to appear).

[11] S. Kötzer and I. Molchanov: On the domain of attraction for the lower tail in Wicksell's corpuscle problem. In: Proc. S^4G (V. Beneš, R. Lechnerová, and I. Saxl, eds.), Prague 2006, pp. 91–96.

[12] R. Takahashi and M. Sibuya: The maximum size of the planar sections of random spheres and its application to metalurgy. Ann. Inst. Statist. Math. *48* (1996), 127–144.

[13] R. Takahashi and M. Sibuya: Maximum size prediction in Wicksell's corpuscle problem for the exponential tail data. Extremes *5* (2002), 55–70.

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