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ON ASYMPTOTIC BEHAVIOUR OF UNIVERSAL FUZZY MEASURES

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The asymptotic behaviour of universal fuzzy measures is investigated in the present paper. For each universal fuzzy measure a class of fuzzy measures preserving some natural properties is defined by means of convergence with respect to ultrafilters.

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1. INTRODUCTION

Universal fuzzy measures were introduced in [8] and further studied in [10]. Let us recall some notation and definitions. By $\mathbb{N} = \{1, 2, 3, \dots\}$ we denote the set of all positive integers. For a set A we denote by $|A|$ the cardinality of A and by 2^A the set of all subsets of A. For every $n \in \mathbb{N}$ and $A \subset \mathbb{N}$ let us denote $A_{(n)} = A \cap \{1, 2, \dots, n\}$, especially $\mathbb{N}_{(n)} = \{1, 2, \dots, n\}$.

Definition 1. A fuzzy measure on $\mathbb{N}_{(n)}$ is any mapping $\mu_n \colon 2^{\mathbb{N}_{(n)}} \to [0, 1]$ such that

(i) $\mu_n(\emptyset) = 0, \quad \mu_n(\mathbb{N}_{(n)}) = 1$,

(ii) $\mu_n(A) \le \mu_n(B)$ for all $A \subset B \subset \mathbb{N}_{(n)}$.

Definition 2. Universal fuzzy measure (shortly by UFM) is a system of fuzzy measures $\mu^* = (\mu_n)_{n\in\mathbb{N}}$ defined on measurable spaces $(\mathbb{N}_{(n)}, 2^{\mathbb{N}_{(n)}})_{n\in\mathbb{N}}$.

Definition 3. A UFM $(\mu_n)_{n\in\mathbb{N}}$ is called regular, if $\mu_n(A) \ge \mu_{n+k}(A)$ for all $A \subset \mathbb{N}_{(n)}$ and $k \in \mathbb{N}$.

Definition 4. Fuzzy measure (shortly by FM) is any monotone set function $\mu \colon 2^{\mathbb{N}} \to [0, \infty]$ with $\mu(\emptyset) = 0$.

Special interest is focused mainly on those UFM (FM) possessing some natural additional properties. The list of some follows, for more of them see for example [10].

A UFM $(\mu_n)_{n\in\mathbb{N}}$ possesses a particular property if and only if all μ_n; $n = 1, 2, \ldots$ have it. A FM possesses a particular property if and only if it holds in the range of *all* subsets of $\mathbb{N}$.

Definition 5. A fuzzy measure μ_n is called

(i) additive, if $\mu_n(A \cup B) = \mu_n(A) + \mu_n(B)$ for all pairs of disjoint $A, B \subset \mathbb{N}_{(n)}$,

(ii) sub-additive, if $\mu_n(A \cup B) \leq \mu_n(A) + \mu_n(B)$ for all $A, B \subset \mathbb{N}_{(n)}$,

(iii) super-additive, if $\mu_n(A \cup B) \geq \mu_n(A) + \mu_n(B)$ for all pairs of disjoint $A, B \subset \mathbb{N}_{(n)}$,

(iv) k-additive, if $\mu_n\left(\bigcup_{i=1}^{k+1} A_i\right) = \sum_{I \subset \neq \{1,2,\ldots,k+1\}} (-1)^{k-|I|} \mu_n\left(\bigcup_{i\in I} A_i\right)$ for arbitrary system of $k+1$ disjoint sets $A_1, A_2, \ldots A_{k+1} \subset \mathbb{N}_{(n)}$ and $k \in \mathbb{N}$,

(v) symmetric, if $\mu_n(A) = \mu_n(B)$ for all $A, B \subset \mathbb{N}_{(n)}$ such that $|A| = |B|$.

Notice that every additive fuzzy measure is also sub-additive and super-additive.

There are several means by which universal fuzzy measures can be defined (see e. g. [10]), one of them is that using weight functions. A weight function is any function $w\colon \mathbb{N} \to (0, \infty)$ and a UFM defined by a particular weight function w is given by $\mu_n(A) = \frac{\sum_{a\in A_{(n)}} w(a)}{\sum_{k\in\mathbb{N}_{(n)}} w(k)}$.

Notice that there is a significant difference between both, concept of fuzzy measures and that of universal fuzzy measures. While a UFM attaches to *every finite* subset of $\mathbb{N}$ an *infinite sequence of numbers*, a FM attaches to *every* subset of $\mathbb{N}$ a *unique number*. On the other hand, it is evident that both concepts FM and UFM are intimately related. A natural question arises to investigate these relations in more details.

In [10] it is shown that to any given locally finite[1] FM μ with $\mu(\{1\}) > 0$ one can attach in a natural way a regular UFM $(\mu_n)_{n\in\mathbb{N}}$ defined by $\mu_n(A) = \frac{\mu(A)}{\mu(\mathbb{N}_{(n)})}$ for $n \in \mathbb{N}$ and $A \subset \mathbb{N}_{(n)}$. Of course, there are also another possibilities how to produce a universal fuzzy measure from the given fuzzy measure. The purpose of the present paper is to discuss the reverse process, i. e. how fuzzy measures can be produced from a given universal fuzzy measure.

2. LOWER AND UPPER ASYMPTOTIC FUZZY MEASURES

In this section we will consider two natural fuzzy measures determined by a given universal fuzzy measure. The fact that both are fuzzy measures is immediate.

[1] i. e. $\mu(A)$ is finite for every finite $A \subset \mathbb{N}$

Definition 6. Let $\mu^* = (\mu_n)_{n\in\mathbb{N}}$ be a given UFM. Lower and upper asymptotic fuzzy measures determined by μ^* are given by

$$\underline{\mu^*}(A) = \liminf_{n\to\infty} \frac{\mu_n(A_{(n)})}{\mu_n(\mathbb{N}_{(n)})} \quad \text{and} \quad \overline{\mu^*}(A) = \limsup_{n\to\infty} \frac{\mu_n(A_{(n)})}{\mu_n(\mathbb{N}_{(n)})} \quad \text{for} \quad A \subset \mathbb{N},$$

respectively.

Lower and upper asymptotic fuzzy measures determined by universal fuzzy measures defined by means of weight functions were studied in number theory (see e. g. [3, 4]) and they are called lower and upper weighted densities. The most important among them are the lower and upper asymptotic densities which are defined by the weight function $w(n) = 1;\ n \in \mathbb{N}$ and the lower and upper logarithmic densities defined by the weight function $w(n) = \frac{1}{n};\ n \in \mathbb{N}$.

Unfortunately, the acceptance either of the lower or the upper asymptotic fuzzy measure as a natural extension of the original universal fuzzy measure can not be completely satisfactory. One reason is, as the following example shows, the fact that none of these FM shares most of natural properties of the original UFM.

Example 1. For $n \in \mathbb{N}$ and $A \subset \mathbb{N}_{(n)}$ denote $d_n(A) = \frac{|A|}{n}$. It can be easily seen that $d^* = (d_n)_{n\in\mathbb{N}}$ is a universal fuzzy measure possessing all properties (i) – (v). For $\underline{d^*}$ and $\overline{d^*}$ we have the following.

(i*) Define $A = \bigcup_{n\in\mathbb{N}}((2n)!, (2n+1)!] \cap \mathbb{N}$ and $B = \mathbb{N} - A$. Then we have

$$\underline{d^*}(A) = \underline{d^*}(B) = 0 \quad \text{and} \quad \overline{d^*}(A) = \overline{d^*}(B) = 1$$

and, as $\underline{d^*}(\mathbb{N}) = \overline{d^*}(\mathbb{N}) = 1$, none of $\underline{d^*}$ and $\overline{d^*}$ is additive.

(ii*) By elementary properties of lim sup, $\overline{d^*}$ is sub-additive. On the other hand, the example in part (i*) shows that $\underline{d^*}$ is not sub-additive.

(iii*) By elementary properties of lim inf, $\underline{d^*}$ is super-additive. On the other hand, the example in part (i*) shows that $\overline{d^*}$ is not super-additive.

(iv*) It is a routine job to verify k-additivity of each d_n. On the other hand, we will show that neither $\underline{d^*}$ nor $\overline{d^*}$ is k-additive. Put

$$A_m = \bigcup_{n=0}^{\infty} ((n(k+1)+m)!, (n(k+1)+m+1)!] \cap \mathbb{N}; \quad m = 1, 2, \ldots k+1.$$

One can easily verify that equalities $\underline{d^*}\left(\bigcup_{i\in I} A_i\right) = 0, \quad \overline{d^*}\left(\bigcup_{i\in I} A_i\right) = 1$ hold for every choice of nonempty $I \subset\neq \{1, 2, \ldots, k+1\}$. Taking into account that $\bigcup_{i=1}^{k+1} A_i = \mathbb{N} - \{1\}$, we have

$$\underline{d^*}\left(\bigcup_{i=1}^{k+1} A_i\right) = 1 \neq 0 = \sum_{I\subset\neq\{1,2,\ldots,k+1\}} (-1)^{k-|I|}\, \underline{d^*}\left(\bigcup_{i\in I} A_i\right),$$

thus $\underline{d^*}$ is not k-additive. To see that $\overline{d^*}$ is not k-additive, notice that $\overline{d^*}\left(\bigcup_{i=1}^{k+1} A_i\right) = 1$ and, on the other hand,

$$
\begin{aligned}
&\sum_{I \subset\neq \{1,2,\dots,k+1\}} (-1)^{k-|I|}\, \overline{d^*}\left(\bigcup_{i\in I} A_i\right) = \sum_{j=1}^{k} \sum_{I\subset\{1,2,\dots,k+1\},|I|=j} (-1)^{k-j} \\
= \ & \sum_{j=1}^{k} (-1)^{k-j}\binom{k+1}{j} = -\sum_{j=0}^{k+1} (-1)^{k+1-j}\binom{k+1}{j} + 1 + (-1)^{k+1} \\
= \ & 1 + (-1)^{k+1} \neq 1.
\end{aligned}
$$

(v*) The set $B = \{2^n;\ n \in \mathbb{N}\}$ has the same cardinality as the set $\mathbb{N}$. On the other hand we have

$$\underline{d^*}(B) = \overline{d^*}(B) = 0 \quad \text{and} \quad \underline{d^*}(\mathbb{N}) = \overline{d^*}(\mathbb{N}) = 1,$$

thus none of $\underline{d^*}$ and $\overline{d^*}$ is symmetric.

The main point why, in general, both $\underline{\mu^*}$ and $\overline{\mu^*}$ fail to share nice properties of original UFM consists in the fact that, although both $\underline{\mu^*}(A)$ and $\overline{\mu^*}(A)$ are limit points of the sequence $(\mu_n(A))_{n\in\mathbb{N}}$ for every $A \subset \mathbb{N}$, the sequences of indices with respect to which these particular limit points are achieved, depend on A substantially. Consequently, one can not apply the standard rules for calculation of limits to derive required properties. Thus, to produce a FM preserving properties of the original UFM we need some common base of indices for which the corresponding subsequences of $(\mu_n(A))_{n\in\mathbb{N}}$ converge for all $A \subset \mathbb{N}$. The way how to do this provides a convergence with respect to ultrafilters.

3. ASYMPTOTIC FUZZY MEASURES WITH RESPECT TO ULTRAFILTERS

First, let us briefly recall some basic notions and properties of filters and filter convergence.

Definition 7. A nonempty class of subsets $\mathcal{F}$ of $\mathbb{N}$ is called filter on $\mathbb{N}$ if it does not contain the empty set and it is closed with respect to supersets and with respect to finite intersections, i. e.

$$A \in \mathcal{F} \quad \text{and} \quad A \subset B \quad \text{imply} \quad B \in \mathcal{F} \tag{1}$$

and

$$A, B \in \mathcal{F} \quad \text{implies} \quad A \cap B \in \mathcal{F}. \tag{2}$$

Notice that, by definition of filter, $\mathbb{N} \in \mathcal{F}$ for every filter $\mathcal{F}$. A filter is called ultrafilter if it is maximal with respect to $\subset$, i. e. if it is contained properly in no filter. An ultrafilter $\mathcal{U}$ is called fixed if $\bigcap_{A\in\mathcal{U}} A \neq \emptyset$, otherwise, it is called free or uniform.

Recall that ultrafilters are characterized as filters $\mathcal{F}$ possessing property

$$\text{if } \bigcup_{k=1}^{n} A_k \in \mathcal{F} \quad \text{then} \quad A_k \in \mathcal{F} \text{ for at least one } k = 1, 2, \ldots, n. \tag{3}$$

Notice that every fixed ultrafilter is of the form $\mathcal{U}_{(n)} = \{A \subset \mathbb{N};\ n \in A\}$ for some fixed $n \in \mathbb{N}$. Consequently, there are exactly countably many fixed ultrafilters. On the other hand an ultrafilter is free if and only if it contains no finite set. A nontrivial result by Pospíšil [9] says that there are exactly $2^{2^{\aleph_0}}$ (i. e. more than continuum) free ultrafilters (see also [2], Corollary 3.6.12). The set of all ultrafilters is denoted by $\beta\mathbb{N}$. Here we assume that $\mathbb{N}$ is naturally embedded in $\beta\mathbb{N}$ via mapping $n \to \mathcal{U}_{(n)}$. In order to understand better the concept of convergence with respect to a given filter, let us briefly add some more information on $\beta\mathbb{N}$. It can be endowed with topology whose base of open (and, at the same time closed) sets is the class $\{\overline{A} = \{\mathcal{U} \in \beta\mathbb{N};\ A \in \mathcal{U}\};\ A \subset \mathbb{N}\}$. The corresponding topological space is compact and it is called the *Stone-Čech compactification* of $\mathbb{N}$. Notice that $\mathbb{N}$ is an open dense subset of $\beta\mathbb{N}$, consequently $\beta\mathbb{N}^* = \beta\mathbb{N} - \mathbb{N}$, the set off all free ultrafilters is closed (consequently compact) subset of $\beta\mathbb{N}$. Also notice that given any filter $\mathcal{F}$, the set

$$\overline{\mathcal{F}} = \bigcap_{F\in\mathcal{F}} \overline{F} = \bigcap_{F\in\mathcal{F}} \{\mathcal{U} \in \beta\mathbb{N};\ F \in \mathcal{U}\} = \{\mathcal{U} \in \beta\mathbb{N};\ \mathcal{F} \subset \mathcal{U}\}$$

is closed in $\beta\mathbb{N}$.

The following proposition presents an important property of $\beta\mathbb{N}$ and it can be found in the most of standard textbooks in general topology, e. g. in [2].

Proposition 1. Every sequence of numbers $f\colon \mathbb{N} \to [0,1]$ can be uniquely extended to a continuous function $\tilde{f}\colon \beta\mathbb{N} \to [0,1]$.

Let us notice that the proposition holds also for every bounded sequence (see e. g. [2], Corollary 3.6.3).

Definition 8. We say that a sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ is convergent with respect to a filter $\mathcal{F}$ (briefly $\mathcal{F}$-convergent) to a number L if

$$\forall \varepsilon > 0 \quad A_\varepsilon = \{n \in \mathbb{N};\ |a_n - L| < \varepsilon\} \in \mathcal{F}.$$

In this case we write $\mathcal{F}\text{-}\lim a_n = L$.

This kind of convergence can be equivalently defined in terms of dual ideal $\mathcal{I}(\mathcal{F}) = \{\mathbb{N} - F;\ F \in \mathcal{F}\}$ and complementary sets $\mathbb{N} - A_\varepsilon$, it is called $\mathcal{I}$-convergence (see e. g. [6]). Notice that the usual convergence of sequences is equivalent to the convergence with respect to the filter $\mathcal{C} = \{A \subset \mathbb{N};\ \mathbb{N} - A \text{ is finite}\}$ of all cofinite sets.

In the case of a convergent sequence f the extension $\tilde{f}$, referred in Proposition 1, is constant on whole space $\beta\mathbb{N}^*$. In general, a sequence f converges to a number L with respect to a filter $\mathcal{F}$ if and only if $\overline{\mathcal{F}} \subset \tilde{f}^{-1}(L)$, i. e. $\tilde{f}$ is constantly equal to L on the whole set $\overline{\mathcal{F}}$. Thus we have:

a sequence f is $\mathcal{F}$-convergent if and only if $\tilde{f}$ is constant on $\overline{\mathcal{F}}$.

If $\mathcal{F}$ is not ultrafilter then $\overline{F}$ contains at least two points, by separation axioms and using density of $\mathbb{N}$ in $\beta\mathbb{N}$ one can find a sequence f whose extension $\tilde{f}$ is not constant on $\overline{\mathcal{F}}$ and, consequently, f is not $\mathcal{F}$-convergent. On the other hand, if $\mathcal{F}$ is ultrafilter, $\overline{\mathcal{F}} = \{\mathcal{F}\}$ and, as every function is constant on one point set, every sequence is $\mathcal{F}$-convergent. We obtain an important property of ultrafilters (see e. g. [1], Theorem 8.27).

Proposition 2. For every ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and every bounded sequence f of real numbers $\mathcal{U}$-lim f exists.

Definition 9. Let $\mu^* = (\mu_n)_{n\in\mathbb{N}}$ be a universal fuzzy measure and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. The fuzzy measure $\mu^*_{\mathcal{U}}$ defined by

$$\mu^*_{\mathcal{U}}(A) = \mathcal{U}\text{-lim}\, \mu_n(A_{(n)}); \quad A \subset \mathbb{N}$$

is called the asymptotic fuzzy measure (shortly by AFM) with respect to $\mathcal{U}$ determined by UFM μ^* or $\mathcal{U}$-AFM determined by UFM μ^*.

Now we are going to show that AFM with respect to an ultrafilter preserves some kind of properties possessing by UFM μ^*. First we define some classes of properties of UFM. Suppose that positive integers p and m and a continuous function $F\colon \mathbb{R}^p \to \mathbb{R}$ are given. Let

$$\mathcal{S}^n \subset \left(2^{\mathbb{N}_{(n)}}\right)^m; \; n = 1, 2, \dots \quad \text{and} \quad \mathcal{S} \subset \left(2^{\mathbb{N}}\right)^m$$

and

$$\varphi^n_k\colon \mathcal{S}^n \to 2^{\mathbb{N}_{(n)}}, \quad \varphi_k\colon \mathcal{S} \to 2^{\mathbb{N}}; \quad n = 1, 2, \dots\,, \quad k = 1, 2, \dots, p$$

be defined so that the conditions

$$\text{If } (A_1, A_2, \dots, A_m) \in \mathcal{S} \text{ then } (A_{1(n)}, A_{2(n)}, \dots, A_{m(n)}) \in \mathcal{S}^n \tag{4}$$

and

$$(\varphi_k(A_1, A_2, \dots, A_m))_{(n)} = \varphi^n_k(A_{1(n)}, A_{2(n)}, \dots, A_{m(n)}) \tag{5}$$

are satisfied for all $n \in \mathbb{N}$ and all $k = 1, 2, \dots, p$.

Definition 10. A universal fuzzy measure $\mu^* = (\mu_n)_{n\in\mathbb{N}}$ is called (E)-type UFM ((I)-type UFM) if

$$F(\mu_n(\varphi^n_1(\mathcal{Q}^n)), \mu_n(\varphi^n_2(\mathcal{Q}^n)), \dots, \mu_n(\varphi^n_p(\mathcal{Q}^n))) = 0 \quad (\geq 0) \tag{6}$$

holds for all choices of $\mathcal{Q}^n \in \mathcal{S}^n$ for all but finitely many $n \in \mathbb{N}$ for some collections of (φ^n_k); $k = 1, 2, \dots, p$, $n \in \mathbb{N}$, $(\mathcal{S}^n)_{n\in\mathbb{N}}$ and some F defined as above.

Definition 11. Suppose that a finite set $\{\mathcal{E}_1, \mathcal{E}_2, \ \ldots, \ \mathcal{E}_s\}$ of conditions like (6) are given. A universal fuzzy measure $\mu^* = (\mu_n)_{n\in\mathbb{N}}$ is called:
— $(\wedge)$-type if for every $\mu_n;\ n \in \mathbb{N}$ each $\mathcal{E}_j;\ j = 1, 2, \ldots, s$ is satisfied,
— $(\vee)$-type if for every $\mu_n;\ n \in \mathbb{N}$ at least one $\mathcal{E}_j;\ j = 1, 2, \ldots, s$ is satisfied.

Remark 1. A fuzzy measure μ is called (E)-, (I)-, $(\wedge)$- or $(\vee)$-type if it satisfies the corresponding conditions with corresponding inputs.

Theorem 1. Let $\mu^* = (\mu_n)_{n\in\mathbb{N}}$ be a $(\wedge)$- or $(\vee)$-type UFM and let $\mathcal{U} \in \beta\mathbb{N}$ be an arbitrary ultrafilter. Then $\mu^*_{\mathcal{U}}$ possesses the same property as μ^*.

P r o o f. We will prove theorem for $(\vee)$-type UFM, the proof for $(\wedge)$-type UFM is similar, even easier. Suppose that a UFM $\mu^* = (\mu_n)_{n\in\mathbb{N}}$ satisfies for every $n \in \mathbb{N}$ at least one of the following conditions

$$F_j(\mu_n(\varphi^n_{1,j}(\mathcal{Q}^n_j)), \mu_n(\varphi^n_{2,j}(\mathcal{Q}^n_j)), \ldots, \mu_n(\varphi^n_{p_j,j}(\mathcal{Q}^n_j))) = 0;\ j = 1, \ldots, s. \tag{7}$$

for for all possible choices of $\mathcal{Q}^n_j \in \mathcal{S}^n_j$. Here all $F_j, \varphi^n_{k,j}$ and $\mathcal{S}^n_j$ are as described above. We will suppose only equations in (7), if inequalities occur, the proof is similar. Choose $\mathcal{U} \in \beta\mathbb{N}$ arbitrarily. If $\mathcal{U}$ is fixed, say $\mathcal{U} = \mathcal{U}_n$ then $\mu^*_{\mathcal{U}}(A) = \mu_n(A_{(n)})$ and the statement of theorem follows. Thus suppose that $\mathcal{U}$ is free. By (3) there is a number $j \in \{1, 2, \ldots, s\}$ and a set $I \in \mathcal{U}$ such that (7) holds for j and every $n \in I$. To prove theorem for $(\vee)$-type UFM it is sufficient to show that for every choice of

$$\mathcal{Q} = (A_1, A_2, \ldots, A_{m_j}) \in \mathcal{S}_j$$

we have

$$F_j\left(\mu^*_{\mathcal{U}}(\varphi_{1,j}(\mathcal{Q})), \mu^*_{\mathcal{U}}(\varphi_{2,j}(\mathcal{Q})), \ldots, \mu^*_{\mathcal{U}}(\varphi_{p_j,j}(\mathcal{Q}))\right) = 0. \tag{8}$$

Let $\varepsilon > 0$. By continuity of F_j there is a $\delta > 0$ such that for every $(x_1, x_2, \ldots, x_{p_j})$ such that $|x_k - \mu^*_{\mathcal{U}}(\varphi_{k,j}(\mathcal{Q}))| < \delta;\ k = 1, 2, \ldots, p_j$ we have

$$\left|F_j\left(\mu^*_{\mathcal{U}}(\varphi_{1,j}(\mathcal{Q})), \mu^*_{\mathcal{U}}(\varphi_{2,j}(\mathcal{Q})), \ldots, \mu^*_{\mathcal{U}}(\varphi_{p_j,j}(\mathcal{Q}))\right) - F_j(x_1, x_2, \ldots, x_{p_j})\right| < \varepsilon. \tag{9}$$

By (4), for every $n \in \mathbb{N}$ we have $\mathcal{Q}^n = (A_{1(n)}, A_{2(n)}, \ldots, A_{m_j\,(n)}) \in \mathcal{S}^n_j$ and by (5) the equations

$$(\varphi_{k,j}(\mathcal{Q}))_{(n)} = \varphi^n_{k,j}(\mathcal{Q}^n) \tag{10}$$

hold for every $k = 1, 2, \ldots, p_j$. Definition of the measure $\mu^*_{\mathcal{U}}$ guarantees the existence of sets $J_k \in \mathcal{U};\ k = 1, 2, \ldots, p_j$ such that, taking into account also (10), for every $n \in J_k$ we have

$$\left|\mu^*_{\mathcal{U}}(\varphi_{k,j}(\mathcal{Q})) - \mu_n(\varphi^n_{k,j}(\mathcal{Q}^n))\right| = \left|\mu^*_{\mathcal{U}}(\varphi_{k,j}(\mathcal{Q})) - \mu_n((\varphi_{k,j}(\mathcal{Q}))_{(n)})\right| < \delta \tag{11}$$

By (2) the set $K = I \cap \left(\bigcap_{k=1}^{p_j} J_k\right) \in \mathcal{U}$ is nonempty. Take $n \in K$, put $x_k = \mu_n(\varphi^n_{k,j}(\mathcal{Q}^n))$ for $k = 1, 2, \ldots, p_j$, use definition of the set I and (7), (9), (11) to obtain

$$\begin{aligned} & |F_j\left(\mu^*_{\mathcal{U}}(\varphi_{1,j}(\mathcal{Q})), \mu^*_{\mathcal{U}}(\varphi_{2,j}(\mathcal{Q})), \ldots, \mu^*_{\mathcal{U}}(\varphi_{p_j,j}(\mathcal{Q}))\right)| \\ = \ & |F_j\left(\mu^*_{\mathcal{U}}(\varphi_{1,j}(\mathcal{Q})), \mu^*_{\mathcal{U}}(\varphi_{2,j}(\mathcal{Q})), \ldots, \mu^*_{\mathcal{U}}(\varphi_{p_j,j}(\mathcal{Q}))\right) - F_j(x_1, x_2, \ldots, x_{p_j})| < \varepsilon. \end{aligned}$$

As $\varepsilon > 0$ was arbitrary, (8) follows and theorem is proved. □

Corollary 1. Let $\mu^* = (\mu_n)_{n\in\mathbb{N}}$ fulfils any of properties (i) to (iv) in Definition 5. Then also $\mu^*_{\mathcal{U}}$ does so for every $\mathcal{U} \in \beta\mathbb{N}$, in particular, any $\mathcal{U}$-limit of additive UFM is additive FM.

Proof. We will sketch the proof for additivity, the proofs for any other of mentioned properties are similar. To prove the corollary for additivity, put $p = 3,\ m = 2$, define $(A, B) \in \mathcal{S}$ if and only if A, B are disjoint, set $\varphi_1(A, B) = A \cup B,\ \varphi_2(A, B) = A, \varphi_3(A, B) = B$ and $F(x_1, x_2, x_3) = x_1 - x_2 - x_3$ and just add index n to define $\mathcal{S}^n,\ \varphi_j^n;\ j = 1, 2, 3.$ □

The following example shows that the statement of Theorem 1 does not hold for every kind of property of UFM.

Example 2. Consider the same UFM $(d_n)_{n\in\mathbb{N}}$ as in Example 1. It is easy to see that every d_n is symmetric. On the other hand, use the same argument as in Example 1 to show that $\mu^*_{\mathcal{U}}$ is not symmetric.

4. FURTHER RESULTS, COMMENTS, QUESTIONS

FM which are $\mathcal{U}$-limits of UFM

Theorem 1 provides a support for the construction of fuzzy measures derived from universal fuzzy measures in a natural way and preserving some kind of their properties. As every such fuzzy measure is necessary normalized (i.e measure of $\mathbb{N}$ is one), there are fuzzy measures which can not be obtained as limits of UFM with respect to some ultrafilter. One can ask if there are some normalized fuzzy measures which are not $\mathcal{U}$-limits of UFM. Thus natural problems arise.

Open problem 1. Characterize these normalized fuzzy measures which can be obtained as $\mathcal{U}$-limits of universal fuzzy measures.

By Pospíšil's theorem, by $\mathcal{U}$-limits can be produced from every UFM theoretically $2^{2^{\aleph_0}}$ different fuzzy measures (one FM for each of $2^{2^{\aleph_0}}$ ultrafilters). In practice, many of them can coincide.

Open problem 2. Let μ^* be UFM and let $\mathcal{U}, \mathcal{V}$ be ultrafilters. Find necessary and sufficient condition for equality $\mu^*_{\mathcal{U}} = \mu^*_{\mathcal{V}}$.

Open problem 3. For each cardinal number λ less than or equal to $2^{2^{\aleph_0}}$, characterize these universal fuzzy measures for which there are exactly λ many $\mathcal{U}$-asymptotic fuzzy measures determined by them.

Values of asymptotic fuzzy measures

It is easy to see that for every UFM μ^* and every ultrafilter $\mathcal{U}$ the inequalities

$$0 \le \underline{\mu^*}(A) \le \mu^*_{\mathcal{U}}(A) \le \overline{\mu^*}(A) \le 1 \tag{12}$$

hold for every $A \subset \mathbb{N}$. Consider the same UFM $(d_n)_{n\in\mathbb{N}}$ as in Example 1. In [7] it is shown that for every given pair $0 \le \alpha \le \beta \le 1$ there is a set $A \subset \mathbb{N}$ such that $\underline{d}(A) = \alpha$ and $\overline{d}(A) = \beta$. More generally, for a given $A \subset \mathbb{N}$ let us call the set $\{(\overline{d}(X), \underline{d}(X));\ X \subset A\}$ the density set of A. These sets were studied and characterized in papers [3] and [5].

Open problem 4. For a given UFM μ^* characterize the set

$$\Lambda_{\mu^*} = \{(\underline{\mu^*}(A), \overline{\mu^*}(A));\ A \subset \mathbb{N}\}.$$

For a given UFM μ^* and given $A \subset \mathbb{N}$ characterize the set

$$\Lambda_{\mu^*}(A) = \{(\underline{\mu^*}(X), \overline{\mu^*}(X));\ X \subset A\}.$$

It is clear that for a given UFM μ^* and given $A \subset \mathbb{N}$ the set $\{\mu^*_{\mathcal{U}}(A);\ \mathcal{U} \in \beta\mathbb{N}\}$ is the set of all limit points of the sequence $(\mu_n(A))_{n\in\mathbb{N}}$. It can be also interesting to characterize the following sets.

Open problem 5. For a given UFM μ^* and given $\mathcal{U} \in \beta\mathbb{N}$ characterize the set $\{\mu^*_{\mathcal{U}}(A);\ A \subset \mathbb{N}\}$. For a given UFM μ^*, given $\mathcal{U} \in \beta\mathbb{N}$ and given $A \subset \mathbb{N}$ characterize the set $\{\mu^*_{\mathcal{U}}(X);\ X \subset A\}$.

Continuity of fuzzy measures with respect to a parameter

For every $\alpha \in [-1, \infty)$ let us define the UFM $d^*_\alpha = (d_{\alpha,n})_{n\in\mathbb{N}}$ by $d_{n,\alpha}(A) = \frac{\sum_{a\in A} a^\alpha}{\sum_{k=1}^n k^\alpha}$ for every $n \in \mathbb{N},\ A \subset \mathbb{N}_{(n)}$. In [4] a continuity of $\underline{d^*_\alpha}$ and $\overline{d^*_\alpha}$ with respect to parameter $\alpha \in (-1, \infty)$ for every $A \subset \mathbb{N}$ was proved and there were found examples of sets for which discontinuity at $\alpha_0 = -1$ can occur. Let us consider more general problem.

Open problem 6. Let X be a topological space and suppose that UFM $\mu^*_\alpha;\ \alpha \in X$ are defined so that for every $n \in \mathbb{N}$ and every $A \subset \mathbb{N}_{(n)}$ the mapping $\alpha \to \mu_{\alpha,n}(A)$ is continuous on X. For $A \subset \mathbb{N}$ and ultrafilter $\mathcal{U}$, what can be said about the continuity of $\underline{\mu^*_\alpha}(A), \overline{\mu^*_\alpha}(A)$ and $\mu^*_{\alpha,\mathcal{U}}(A)$ with respect to $\alpha \in X$?

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