KYBERNETIKA — VOLUME *42* (2006), NUMBER 3, PAGES 367 – 378

S-MEASURES, T-MEASURES AND DISTINGUISHED CLASSES OF FUZZY MEASURES

Peter Struk and Andrea Stupňanová

S-measures are special fuzzy measures decomposable with respect to some fixed t-conorm S. We investigate the relationship of S-measures with some distinguished properties of fuzzy measures, such as subadditivity, submodularity, belief, etc. We show, for example, that each S_P-measure is a plausibility measure, and that each S-measure is submodular whenever S is 1-Lipschitz.

Keywords: fuzzy measure, T-norm, T-conorm, subadditivity, belief
AMS Subject Classification: 28E10

1. INTRODUCTION

Fuzzy measures may possess several properties, such as additivity, subadditivity, superadditivity, submodularity, supermodularity, k-additivity, S-decomposability, belief, plausibility, etc. [1, 3, 6, 7, 10, 11, 14, 15, 16].

Some relationships between these properties are immediate. For example, subadditivity and superadditivity hold simultaneously if and only if the discussed fuzzy measure is additive. Submodularity of a fuzzy measure ensures its subadditivity but not vice versa. Belief measures are supermodular while plausibility measures are submodular.

S-measures, i. e., S-decomposable fuzzy measures (where S is a triangular conorm [4]) were introduced and studied in [16] as a common generalization of probability and possibility (i. e., S_M-decomposable) measures [3, 6, 11, 15, 17]. Dual fuzzy measures to S-measures can be automatically introduced as T-measures, where T is a (dual to S) triangular norm. Though S-measures were studied from several aspects, especially in connection with integration [5, 6, 11, 15, 16], there is only little known about relationship of S-measures, i. e., S-decomposability of fuzzy measures, with other possible properties of fuzzy measures. This fact was observed, e. g., by D. Dubois, who asked during the 3rd Eusflat congress in Zittau, 2003, to clarify the relationship of S- (T-) measures and belief (plausibility) measures. Recall that only for the maximum t-conorm S_M it was known that each S_M-measure (i. e., each possibility measure) is necessarily also a plausibility measure [3, 6, 10, 11, 15]. By duality, each T_M-measure (i. e., each necessity measure) is also a belief measure.

The aim of this paper is to look closer at the relationship of S-(T-)decomposability of fuzzy measure and some of distinguished properties of fuzzy measures on finite universal spaces. The paper is organized as follows. In the next section we recall several distinguished properties of fuzzy measures we want to discuss. In Section 3, S- (T-)measures are recalled. Section 4 deals with S-measures with respect to a continuous Archimedean t-conorm S. Section 5 clarifies the relationship of S-measures with subadditivity, submodularity, superadditivity and supermodularity. Connections of S-measures with belief and plausibility measures are discussed in Section 6. Finally, in Conclusions, related results for T-measures are introduced, exploiting the duality of T-measures and S-measures.

2. PRELIMINARIES AND BASIC NOTIONS

Throughout this paper, X will denote a finite set and $\mathcal{P}(X)$ the corresponding power set. The following notions were introduced and discussed in [3, 6, 10, 13, 15].

Definition 1. A fuzzy measure m on X is a set function $m : \mathcal{P}(X) \to [0, 1]$, which satisfies the conditions:

1. $m(\emptyset) = 0$;
2. $m(X) = 1$;
3. $\forall E, F \in \mathcal{P}(X),\ E \subset F \Rightarrow m(E) \leq m(F)$.

Definition 2. A fuzzy measure p is an additive fuzzy measure (probability) if for arbitrary

$$A, B \in \mathcal{P}(X) : \ p(A \cup B) + p(A \cap B) = p(A) + p(B).$$

This property is equivalent with $p(A \cup B) = p(A) + p(B)$ whenever $A \cap B = \emptyset$.

Definition 3. Let m be a fuzzy measure on X. Its dual fuzzy measure m^d : $\mathcal{P}(X) \to [0, 1]$ is given by $m^d(A) := 1 - m(A^c)$ for $\forall A \in \mathcal{P}(X)$.

Definition 4. A set function $m_1 : \mathcal{P}(X) \to [0, 1]$ is called a basic probability assignment if it satisfies the conditions: $m_1(\emptyset) = 0$ and $\sum_{A \in \mathcal{P}(X)} m_1(A) = 1$.

Definition 5. Let m_1 be a basic probability assignment on $\mathcal{P}(X)$. Then the belief measure $m : \mathcal{P}(X) \to [0, 1]$ induced by m_1 is defined by $m(A) = \sum_{B \subset A} m_1(B)$ for $\forall A \in \mathcal{P}(X)$.

Remark 1. Note that belief measure m can be defined as a fuzzy measure with the following property

$$\begin{aligned}&m(A_1 \cup A_2 \cup \cdots \cup A_n)\\ \geq\ &\sum_{i=1}^{n} m(A_i) - \sum_{i<j} m(A_i \cap A_j) + \cdots + (-1)^{n+1} m(A_1 \cap \cdots \cap A_n)\end{aligned}$$

for arbitrary $n \in N$ and $A_1, \ldots, A_n \in \mathcal{P}(X)$ (i. e., m is $\infty-$monotone [10]). Then the basic probability m_1 related to the belief measure m is the Möbius transform of m, see [6, 10, 15], given by

$$m_1(A) = \sum_{B \subset A} (-1)^{|A \setminus B|} m(B),$$

where $|A|$ is the cardinality of the set $A \in \mathcal{P}(X)$.

Recall that the dual fuzzy measure to a belief measure is called a plausibility measure.

Definition 6. The plausibility measure m is a fuzzy measure satisfying the condition

$$\begin{aligned}&m(A_1 \cap A_2 \cap \cdots \cap A_n)\\ \leq\ &\sum_{i=1}^{n} m(A_i) - \sum_{i<j} m(A_i \cup A_j) + \cdots + (-1)^{n+1} m(A_1 \cup \cdots \cup A_n)\end{aligned}$$

for all $n \in N$ and $A_1, \ldots A_n \in \mathcal{P}(X)$.

Remark 2. We can define the plausibility measure m by means of a basic probability assignment m_1 as follows

$$m(A) = \sum_{B \cap A \neq \emptyset} m_1(B) \text{ for } \forall\, A \in \mathcal{P}(X).$$

Definition 7. Let m be a fuzzy measure on X. Then m is called

- supermodular if $m(A \cup B) + m(A \cap B) \geq m(A) + m(B)$,
- submodular if $m(A \cup B) + m(A \cap B) \leq m(A) + m(B)$,
- superadditive if $m(A \cup B) \geq m(A) + m(B)$ for $A \cap B = \emptyset$,
- subadditive if $m(A \cup B) \leq m(A) + m(B)$,

for all $A, B \in \mathcal{P}(X)$.

Observe that we can equivalently introduce the subadditivity of m requiring $m(A \cup B) \leq m(A) + m(B)$ for disjoint A, B only.

Remark 3. We can easily show the dual relationship between supermodular and submodular fuzzy measure, but subadditive fuzzy measure need not be a dual measure of any superadditive fuzzy measure.

3. S-MEASURES AND T-MEASURES

As a genuine generalization of probability measures, S-measures and T-measures were introduced in [3, 16]. For more details about t-norms T and t-conorms S we recommend [4] and for the T-measures and S-measures we recommend [2, 3, 5, 10, 11, 16].

Definition 8. ([4, 16]) Let S be a t-conorm. A fuzzy measure $m : \mathcal{P}(X) \to [0,\, 1]$ is called an S-measure if for all $A, B \in \mathcal{P}(X)$ such that $A \cap B = \emptyset$

$$m(A \cup B) = S(m(A), m(B)).$$

Let T be a t-norm. A fuzzy measure $m^* : \mathcal{P}(X) \to [0,\, 1]$ is called a T-measure if for all $A, B \in \mathcal{P}(X)$ such that $A \cup B = X$

$$m^*(A \cap B) = T(m^*(A), m^*(B)).$$

If a t-norm T is dual to a t-conorm S, then each T-measure m^* is dual measure to some S-measure m, i. e, $m^* = m^d$. This fact allows us to rewrite results valid for S-measures directly for T-measures. Therefore, we will consider only S-measures since now.

A special type of S-measures when $S(x, y) = \max(x, y)$, i. e., when $S = S_M$, is *a possibility measure*. Its dual measure (T_M-measure with $T_M(x, y) = \min(x, y)$) is called *a necessity measure* [3].

Lemma 1. ([10, 15]) The class of all possibility measures form a special subclass of plausibility measures and the necessity measures are special type of belief measures.

Recall that for finite X these measures correspond to the nested structure of focal elements of the corresponding basic probability assignment.

4. S-MEASURES WITH RESPECT TO A CONTINUOUS ARCHIMEDEAN T-CONORMS

A continuous Archimedean t-conorm can be defined as follows [4].

Definition 9. A continuous t-conorm S is called Archimedean if and only if $(\forall\, x \in\,]0, 1[)\ (S(x, x) > x)$.

First we recall some basic properties of continuous Archimedean t-conorms.

Proposition 1. [4] A $[0,1]^2 \to [0,1]$ mapping S is a continuous Archimedean t-conorm if and only if there exists a continuous additive generator $s : [0,1] \nearrow [0,\infty]$ (i. e., increasing function, such that $s(0) = 0$) such that

$$S(x,y) = s^{-1} \left(\min(s(1), s(x) + s(y))\right).$$

Continuous Archimedean t-conorms can be divided into two classes: strict t-conorms and nilpotent t-conorms.

Definition 10. **(Strict t-conorms)** [4] A t-conorm S is called strict if and only if it is continuous and all partial mappings $S(x,\cdot)$, $x \in [0,1[$, are strictly increasing.

Definition 11. **(Nilpotent t-conorms)** [4] A t-conorm S is called nilpotent if and only if it is continuous, Archimedean and not strict.

Note that the probabilistic sum t-conorm $S_P(x,y) = x + y - xy$ is strict, while the Łukasiewicz t-conorm $S_L(x,y) = \min(1, x+y)$ is nilpotent.

We recall that an additive generator s of a strict t-conorm S is unbounded, i. e. $s(1) = \infty$. We can easily get the following lemmas, see also [2, 16].

Lemma 2. Let $s : [0,1] \nearrow [0,\infty]$ be some increasing bijection and let $m : \mathcal{P}(X) \to [0,\ 1]$ be a fuzzy measure. Then $S(x,y) = s^{-1}(s(x) + s(y))$ is a strict t-conorm and m is an S-measure if and only if $s \circ m$ is an additive measure ($s(m(X)) = \infty$).

For the nilpotent case we have the following result.

Lemma 3. Let $s : [0,1] \nearrow [0,1]$ be some increasing bijection and let $m : \mathcal{P}(X) \to [0,\ 1]$ be a fuzzy measure, such that $s \circ m$ is probability measure. Then $S(x,y) = s^{-1} \left(\min(1, s(x) + s(y))\right)$ is nilpotent t-conorm and m is an S-measure.

Note, however, that for a nilpotent t-conorm S, there are S-measures m such that $s \circ m$ is not a probability measure.

Example 1. Let $X = \{1, 2, \ldots, 10\}$ and $m : \mathcal{P}(X) \to [0,\ 1]$ be given by $m(A) = \min\left(1, \frac{\operatorname{card} A}{5}\right)$. Then m is an S_L-measure. However, $s_L \circ m = m$ is not a probability measure.

5. S-MEASURES, SUBADDITIVITY AND SUBMODULARITY

Now we describe the relationships between S-measures and subadditive (superadditive) measures.

Theorem 1. Let S be a t-conorm. Then the following are equivalent:

i) each S-measure m is subadditive

ii) $S \leq S_L$.

P r o o f. Let $S : [0,1]^2 \to [0,1]$ be a t-conorm such that $S \leq S_L$. Then for any S-measure m on X and any $A, B \in \mathcal{P}(X)$ we have

$$\begin{aligned} m(A \cup B) &= S(m(A), m(B)) \leq S_L(m(A) + m(B)) \\ &= \min(m(A) + m(B), 1) \leq m(A) + m(B) \end{aligned}$$

and thus m is subadditive. Conversely let for some $a, b \in [0,1]$ it holds $S(a,b) > S_L(a,b)$, i. e., $S(a,b) > a + b$. Put $X = \{x_1, x_2, x_3\}$ and define fuzzy measure $m : \mathcal{P}(X) \to [0,1]$ by $m(\{x_1\}) = a, m(\{x_2\}) = b, m(\{x_1, x_2\}) = S(a,b)$ and for all remaining $\emptyset \neq A \in \mathcal{P}(X)$ let $m(A) = 1$. Then m is an S-measure on X which is not subadditive. □

However, for superadditive measures we have only the following weaker result.

Theorem 2. Let S be a t-conorm such that each S-measure m is superadditive. Then $S \geq S_L$.

P r o o f. For any $a, b \in]0,1[$ define a fuzzy measure m on $X = \{x_1, x_2, x_3\}$ in the same way as in the proof of the previous theorem. Then m is an S-measure and from its superadditivity it follows

$$S(a,b) = S(m(\{x_1\}), m(\{x_2\})) = m(\{x_1, x_2\}) \geq m(\{x_1\}) + m(\{x_2\}) = a + b,$$

i. e., $S(a,b) \geq S_L(a,b)$. Thus $S \geq S_L$.

Observe that we cannot reverse the implication in the above theorem. Indeed, fuzzy measure m from Example 1 is an S_L-measure. Put $A = \{1,2,3,4,5\}$ and $B = \{6,7,8,9,10\}$. Then $A \cap B = \emptyset$ but $m(A \cup B) = 1 \not\geq 2 = m(A) + m(B)$, i. e., m is not superadditive. □

For submodular fuzzy measures we have the next result.

Theorem 3. Let S be a continuous t-conorm. Then the following are equivalent:

i) each S-measure m is submodular

ii) S is 1-Lipschitz t-conorm.

P r o o f. Let S be a 1-Lipschitz t-conorm and let m be an S-measure on X. For arbitrary chosen subsets $A, B \in \mathcal{P}(X)$, denote $a = m(A \backslash B), b = m(B \backslash A), c = m(A \cap B)$. Then

$$S(a, S(b,c)) - S(a,c) \leq S(b,c) - c$$

due to the 1-Lipschitz property of S, i. e.,

$$m(A \cup B) + m(A \cap B) = S(a, S(b, c)) + c \leq S(a, c) + S(b, c) = m(A) + m(B),$$

proving the submodularity of m. Conversely suppose that S is not 1-Lipschitz, i. e., there are $a, c, d \in [0, 1]$ such that $c < d$ and $S(a, d) - S(a, c) > d - c$. As $S(\cdot, c)$ is continuous then there is b such that $d = S(b, c)$. Now, it is enough to introduce an S-measure on $X = \{x_1, x_2, x_3, x_4\}$ determined by $m(\{x_1\}) = a, m(\{x_2\}) = b, m(\{x_3\}) = c$ and $m(\{x_4\}) = 1$. For $A = \{x_1, x_3\}$ and $B = \{x_2, x_3\}$ we have

$$m(A \cup B) + m(A \cap B) > m(A) + m(B),$$

i. e., m is not submodular. □

Observe that 1-Lipschitz t-conorms are exactly associative dual copulas [9] and if such S is Archimedean then its additive generator is concave.

Note that there are non-continuous t-conorms S such that each S-measure m is necessarily submodular.

Example 2.

i) Let $S^\diamond$ be a t-conorm dual to the t-norm $T^\diamond$ introduced by [8] and studied by [12] compare also [4], given by

$$T^\diamond(a, b) = \sum_{n=1}^{\infty} \frac{1}{2^{a_n + b_n - n}}$$

whenever $a, b \in]0, 1]$, where $a = \sum_{n=1}^{\infty} \frac{1}{2^{a_n}}$ and $b = \sum_{n=1}^{\infty} \frac{1}{2^{b_n}}$ are the (unique) infinite dyadic expansions of a and b. Evidently, for a given t-conorm S, each S-measure m is submodular whenever

$$S(a, S(b, c)) + c \leq S(a, c) + S(b, c) \tag{1}$$

for all $a, b, c \in [0, 1]$. For the dual t-norm T, (1) is equivalent to

$$T(a, c) + T(b, c) \leq T(a, T(b, c)) + c \tag{2}$$

for all $a, b, c \in [0, 1]$.

Observe that for any integers $a_n, b_n, c_n \in \{n, n + 1, \dots\}$ we have

$$\frac{1}{2^{a_n + c_n - n}} + \frac{1}{2^{b_n + c_n - n}} \leq \frac{1}{2^{a_n + b_n + c_n - 2n}} + \frac{1}{2^{c_n}}$$

and thus the inequality (2) is valid for $T^\diamond$ and any $a, b, c \in]0, 1]$. If one of the arguments a, b, c becomes 0, then for each T the inequality (2) is trivially fulfilled. Summarizing, $T^\diamond$ satisfies (2) for all possible arguments and thus $S^\diamond$ satisfies (1), i. e., each $S^\diamond$-measure is submodular though $S^\diamond$ is a non-continuous t-conorm (observe that $S^\diamond \leq S_L$ is fulfilled as a consequence of $T^\diamond \geq T_L$, compare Theorem 1).

ii) The non-continuous t-conorm $S^{nM} : [0,1]^2 \to [0,1]$ (nilpotent maximum) given by
$$S^{nM}(a,b) = \begin{cases} \max(a,b) & a+b<1, \\ 1 & \text{else}, \end{cases}$$
fulfils $S^{nM} \le S_L$ and thus each S^{nM}-measure is subadditive. However, define an S^{nM}-measure on $X = \{x_1, x_2, x_3\}$ putting $m(\{x_1\}) = m(\{x_2\}) = 0.6$ and $m(\{x_3\}) = 0.3$. For $A = \{x_1, x_2\}$ and $B = \{x_2, x_3\}$ we have
$$m(A \cup B) + m(A \cap B) = 1 + 0.6 > 0.6 + 0.6 = m(A) + m(B),$$
i. e., this S^{nM}-measure is not submodular.

However, there is no t-conorm S such that each S-measure is supermodular. Indeed, it is sufficient to put $X = \{1,2,3\}$, $m(A) = [\frac{\text{card} A}{2}]$. For each $S \ge S_L$, m is S-measure which is not supermodular, because of for $A = \{1,2\}$, $B = \{2,3\}$ we have $m(A \cup B) + m(A \cap B) = 1 + 0 \not\ge 2 = m(A) + m(B)$. Now it is sufficient to recall Theorem 2 and the fact that supermodularity of m ensures its superadditivity.

6. S-MEASURES, BELIEF AND PLAUSIBILITY MEASURES

Each belief measure is supermodular and thus there is no t-conorm S which will ensure that each S-measure is belief. Moreover, for strict t-conorm S we know that it cannot be stronger than S_L and thus for the corresponding S-measures we cannot ensure neither the superadditivity. However, for nilpotent t-conorms we have the following result based on results from [14] and [1].

Theorem 4. Let S be a nilpotent t-conorm with a normed additive generator s and let p be a probability measure. Then the S-measure $m = s^{-1} \circ p$ is supermodular if s^{-1} is convex, i. e., s is concave function. Moreover it is always belief whenever s^{-1} has all derivatives non-negative, i. e., if s^{-1} is an absolutely monotonic function.

P r o o f. Observe first that for all $a, b, c \ge 0$ and any convex function f, it holds $f(a+c) - f(c) \le f(a+c+b) - f(c+b)$, i. e., $f(a+b+c) + f(c) \ge f(a+c) + f(b+c)$. Thus for any probability measure p on X, subsets $A, B \in \mathcal{P}(X)$ and a convex s^{-1} we have $m(A \cup B) + m(A \cap B) = s^{-1}(a+b+c) + s^{-1}(c) \ge s^{-1}(a+c) + s^{-1}(b+c) = m(A) + m(B)$, where $a = p(A \backslash B)$, $b = p(B \backslash C)$ and $c = p(A \cap B)$, i. e., m is supermodular.

The second part of theorem follows from results from [14], compare also [1], where we have shown that absolutely monotonic distortions preserve the belief property of fuzzy measures (evidently, each probability measure is also a belief measure). □

There are some typical examples of nilpotent t-conorms S for which s^{-1} has all derivatives non-negative:

- $S = S^Y_{\frac{1}{n}} = \min(1, (x^{\frac{1}{n}} + y^{\frac{1}{n}})^n)$, $n \ge 1$ is Yager t-conorm with additive generator $s(x) = x^{\frac{1}{n}}$, $s^{-1}(x) = x^n$. Note that for any probability p, $s^{-1} \circ p$ is an n-additive belief measure, compare [7].

- $S(x, y) = \min(1, x + y + xy)$ with additive generator $s(x) = \log_2(1 + x)$, $s^{-1}(x) = 2^x - 1$.

For subadditive (plausibility) measures we have the next result (its first part follows directly from Theorem 3, the second part follows from [14], compare also [1]).

Theorem 5. Let S be a nilpotent t-conorm with a normed additive generator s and p be a probability measure. Then the S-measure $m = s^{-1} \circ p$ is submodular if s^{-1} is concave function. Moreover it is always plausibility whenever derivatives of s^{-1} change the sign, i. e., if s^{-1} is a completely monotonic function.

Observe that the Yager t-conorm S_2^Y, see below, is a nilpotent t-conorm with a convex additive generator s, $s(x) = x^2$, such that $s^{-1}(x) = \sqrt{x}$ changes the sign of its derivatives. This fact together with Lemma 3 imply the next result.

Corollary 1. Let m be an S_2^Y-measure. Then m is a plausibility measure such that m^2 is a probability measure.

There are some typical examples of nilpotent t-conorms S such that s^{-1} changes the sign of its derivatives:

- $S = S_n^Y = \min(1, (x^n + y^n)^{\frac{1}{n}})$, $n \in N$ is Yager t-conorm with additive generator $s(x) = x^n$, $s^{-1}(x) = x^{\frac{1}{n}}$,
- $S(x, y) = \min(1, \log_2(2^x + 2^y - 1))$ with additive generator $s(x) = 2^x - 1$, $s^{-1}(x) = \log_2(1 + x)$.

Note that a similar claim holds also for strict t-norms. For example, let $X = \{x_1, \ldots, x_n\}$ and let m be an S_P-measure on X. Denote $w_i = m(\{x_i\})$, $i = 1, \ldots, n$. Observe that then $w_j = 1$ for some $j \in \{1, \ldots, n\}$. Then for each $A \in P(X)$, $m(A) = 1 - \prod_{x_i \in A}(1 - w_i)$. For the dual fuzzy measure $m^d(A) = \prod_{x_i \notin A}(1 - w_i)$. Its Möbius transform [6] is given by

$$
\begin{aligned}
M_{m^d}(A) &= \sum_{B \subset A} (-1)^{\mathrm{card}(A \setminus B)} m^d(B) = \sum_{B \subset A} \left((-1)^{\mathrm{card}(A \setminus B)} \prod_{x_i \notin B} (1 - w_i) \right) \\
&= \left(\prod_{x_i \notin A} (1 - w_i) \right) \left(\sum_{B \subset A} (-1)^{\mathrm{card}(A \setminus B)} \prod_{x_i \in A \setminus B} (1 - w_i) \right) \\
&= \left(\prod_{x_i \notin A} (1 - w_i) \right) \left(\prod_{x_i \in A} w_i \right) \geq 0
\end{aligned}
$$

and thus M_{m^d} is a basic probability assignment, m^d is a belief measure and therefore m is a plausibility measure. We summarize the above results for S_P-measures in the next theorem.

Theorem 6. Let m be an S_P-measure. Then m is a plausibility measure.

For S_L-measures, we have the next result.

Theorem 7. Let m be an S_L-measure.Then m is an upper probability measure, i. e., $m = \sup(p|p \le m)$. However, m need not be a plausibility measure.

Proof. Consider the finite set $X = \{x_1, \dots, x_n\}$, then any S_L-measure m can be represented as follows:

$$w_i := m(\{x_i\}) \text{ for all } i = 1, \dots, n \text{ and for all } A \subset X$$

$$m(A) = \min\left(\sum_{x_i \in A} w_i, 1\right),$$

where $w_i \ge 0$ and $\sum_{i=1}^{n} w_i = 1$. For any fixed $A \in \mathcal{P}(X)$, such that $m(A) = 1$, define a probability measure $p_A : X \to [0,1]$ by $p_A(B) = \sum_{i \in B} u_i$, where

$$u_i = \frac{w_i}{\sum_{j \in A} w_j} \text{ if } i \in A \text{ and } u_i = 0 \text{ if } i \notin A.$$

Then $p \le m$ and $p_A(A) = m(A)$. Similarly, if $m(A) < 1$, a probability measure $p_A : X \to [0,1]$ can be introduced putting $p_A(B) = \sum_{i \in B} u_i$, where

$$u_i = w_i \text{ if } i \in A \text{ and } u_i = \frac{w_i}{\sum_{j \notin A} w_j} \text{ if } i \notin A.$$

Again we have $p \le m$ and $p_A(A) = m(A)$. Then $\sup(p_A|A \in \mathcal{P}(X)) \le \sup(p|p \le m) \le m = \sup(p_A|A \in \mathcal{P}(X))$, i. e., m is an upper probability measure. □

Next fuzzy measure is another example of S_L-measure, however it is not a plausibility measure.

Example 3. Consider the finite set $X = \{1,2,3\}$, $A = \{1\}$, $B = \{2\}$ and $C = \{3\}$. Let

	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
m	0	0.4	0.7	0.1	1	0.5	0.8	1

Then $m(A \cap B \cap C) = 0 \not\le -0.1 = m(A) + m(B) + m(C) - m(A \cup B) - m(B \cup C) - m(A \cup C) + m(A \cup B \cup C)$, i. e., m is not a plausibility measure. However $m = \max(p_1, p_2, p_3)$ where

	$\{1\}$	$\{2\}$	$\{3\}$
p_1	0.3	0.7	0
p_2	0.4	0.5	0.1
p_3	0.2	0.7	0.1.

7. CONCLUSIONS

We have discussed some properties of fuzzy measures and their relationship with S-measures. A special attention was paid to S-measures related to probability measures. Due to the duality of S-measures and T-measures, we can conclude:

i) each T-measure m is superadditive if and only if $T \geq T_L$

ii) for a continuous t-norm T, each T-measure m is supermodular if and only if T is 1-Lipschitz

iii) for a nilpotent t-norm T with an additive generator t, for any probability measure p, $m = 1 - t \circ p$ is a belief measure if and only if the derivatives of t^{-1} change the sign (i. e., T is n-copula for each $n \in N$ [9]).

ACKNOWLEDGEMENT

This work was supported by Science and Technology Assistance Agency under the contract No. APVT-20-046402 and APVT-20-003204. A partial support of the grant VEGA 1/2032/05 is also kindly announced.

(Received October 16, 2005.)

REFERENCES

[1] A. G. Bronevich: On the closure of families of fuzzy measures under eventwise aggregations. Fuzzy Sets and Systems *153* (2005), 1, 45–70.
[2] A. Chateauneuf: Decomposable capacities, distorted probabilities and concave capacities. Math. Soc. Sci. *31* (1996), 19–37.
[3] D. Dubois and H. Prade: Fuzzy Sets and Systems: Theory and Applications. Academic Press, New York 1980.
[4] E. P. Klement, R. Mesiar and E. Pap: Triangular Norms. Kluwer Academic Publishers, Dortrecht 2000.
[5] E. P. Klement, R. Mesiar and E. Pap: Integration with respect to decomposable measures, based on a conditionally distributive semiring on the unit interval. Internat. J. Uncertain. Fuzziness Knowledge-Based Systems *8* (2000), 701–717.
[6] G. J. Klir and T. A. Folger: Fuzzy Sets, Uncertainty and Information. Prentice Hall, Englewood Cliffs, New Jersey 1988.
[7] R. Mesiar: Generalizations of k-order additive discrete fuzzy measures. Fuzzy Sets and Systems *102* (1999), 423–428.
[8] R. Mesiar: Triangular norms – an overview. In: Computational Inteligence in Theory and Practice (B. Reusch, K.-H. Temme, eds.), Physica–Verlag, Heidelberg 2001, pp. 35–54.
[9] R. B. Nelsen: An Introduction to Copulas. (Lecture Notes in Statistics 139.), Springer, New York 1999.
[10] E. Pap: Null-additive Set Functions. Kluwer Academic Publishers, Dordrecht 1995.
[11] E. Pap: Pseudo-additive measures and their applications. In: Handbook of Measure Theory, Volume II (E. Pap, ed.), Elsevier, North–Holland, Amsterdam 2002, pp. 1403–1465.
[12] D. Smutná: On a peculiar t-norm. Busefal *75* (1998), 60–67.

[13] M. Sugeno: Theory of Fuzzy Integrals and Applications. Ph.D. Thesis, Tokyo Institute of Technology, Tokyo 1974.

[14] Ľ. Valášková and P. Struk: Preservation of Distinguished Fuzzy Measure Classes by Distortion. In: MDAI 2004, Barcelona (V. Torra, Y. Narukawa, eds., Lecture Notes in Artificial Intelligence 3131), Springer–Verlag, Berlin 2004, pp. 175–182.

[15] Z. Wang and G. J. Klir: Fuzzy Measures Theory. Plenum Press, New York 1992.

[16] S. Weber: $\perp$-decomposable measures and integrals for Archimedean t-conorms. J. Math. Anal. Appl. *101* (1984), 114–138.

[17] L. Zadeh: Fuzzy sets as a basis for a theory of possibility. Fuzzy Sets and Systems *1* (1978), 3–28.

Peter Struk and Andrea Stupňanová, Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68. Bratislava. Slovak Republic.

e-mails: struk@math.sk, andy@math.sk