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DISTRIBUTIVITY OF STRONG IMPLICATIONS OVER CONJUNCTIVE AND DISJUNCTIVE UNINORMS

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This paper deals with implications defined from disjunctive uninorms U by the expression $I(x, y) = U(N(x), y)$ where N is a strong negation. The main goal is to solve the functional equation derived from the distributivity condition of these implications over conjunctive and disjunctive uninorms. Special cases are considered when the conjunctive and disjunctive uninorm are a t-norm or a t-conorm respectively. The obtained results show a lot of new solutions generalyzing those obtained in previous works when the implications are derived from t-conorms.

Keywords: t-norm, t-conorm, uninorm, implication operator, S-implication, R-implication, distributivity

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1. INTRODUCTION

There are a lot of properties valid in classical logic that are generally not true when they are translated to fuzzy logic, using t-norms, t-conorms and strong negations to perform conjunctions, disjunctions and negations, respectively. A usual methodology in fuzzy logic research consists of study for which of these operators, a classical logic property remains true. This method usually derives in solving functional equations involving these kinds of operators. One particular example is given by the equivalence

$$(p \wedge q) \to r \equiv (p \to r) \vee (q \to r) \tag{1}$$

that was used in [2] to avoid combinatorial rule explosion in fuzzy systems, and that produced several discussions around it ([3], [4], [6], [13]). Translated to fuzzy logic, equation (1) becomes

$$I(T(x, y), z) = S(I(x, z), I(y, z)) \quad x, y, z \in [0, 1] \tag{2}$$

where I is an implication function, T is a t-norm and S is a t-conorm. This distributivity functional equation was solved in [18] for several kinds of implication functions derived from t-norms and t-conorms. Several related distributivities were also solved in [1] for the same kinds of implications.

On the other hand, uninorms ([7]) are a special kind of associative aggregation functions that has been extensively studied in the literature. They become specially interesting from the theoretical point of view because of their structure as a special combination of a t-norm and a t-conorm (see [7] or [9]). Moreover, they have proved to be useful for applications in many fields like expert systems, neural networks, aggregation, fuzzy system modelling, measure theory, etc. It is well known that a uninorm U can be conjunctive or disjunctive whenever $U(1,0) = 0$ or $U(1,0) = 1$, respectively. This fact allows to use them also as logical connectives. In this sense, fuzzy implications functions have been defined from uninorms in the following two ways:

- *Strong implications* (or *S-implications*) defined by $I(x,y) = U(N(x),y)$ for any disjunctive uninorm U and any strong negation N.

- *Residual implications* (or *R-implications*) defined by $I(x,y) = \sup\{z \in [0,1] \mid U(x,z) \leq y\}$ for any uninorm U such that $U(x,0) = 0$ for all $x < 1$.

An exhaustive study of both types of implications can be found in [5] and [15]. Uninorms and their derived implications are also used in aggregation applications like mathematical morphology (see [8]).

In this paper we want to solve equation (2) when conjunctions and disjunctions are performed by uninorms and the involved implications are strong implications also defined from uninorms. The case of residual implications derived from uninorms is studied in another paper of the authors (see [17]). A lot of new solutions of equation (1) appear apart from those already found in [18] and [1]. We solve equation (2) directly in the general case, but we derive some special solutions when someone of the considered uninorms is a t-norm or a t-conorm. Finally, we also solve three new equations related to (2). One is solved by duality and the others with quite similar reasoning as that used for the initial equation (2).

2. PRELIMINARIES

We suppose the reader to be familiar with basic results concerning t-norms and t-conorms that can be found in [9]. We recall only some notions on uninorms. For more details see [7, 10] and for implications derived from uninorms see [5] and [15].

Definition 1. A *uninorm* is a two-place function $U : [0,1] \times [0,1] \longrightarrow [0,1]$ which is associative, commutative, increasing in each place and such that there exists some element $e \in [0,1]$, called the *neutral element*, such that $U(e,x) = x$ for all $x \in [0,1]$.

It is clear that the function U becomes a t-norm when $e = 1$ and a t-conorm when $e = 0$. For any uninorm we have $U(0,1) \in \{0,1\}$, and a uninorm U is said conjunctive when $U(1,0) = 0$ and disjunctive when $U(1,0) = 1$.

Definition 2. A uninorm U with neutral element $e \in]0,1[$ is *representable* if and only if there is a strictly increasing, continuous function $h : [0,1] \to [-\infty, +\infty]$ with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that U is given by

$$U(a,b) = h^{-1}(h(a) + h(b))$$

for all $(a,b) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and $U(0,1) = U(1,0) \in \{0,1\}$. Function h is usually called an *additive generator* of U.

Note that any representable uninorm is continuous in $[0,1]^2 \setminus \{(0,1),(1,0)\}$ and all uninorms continuous in this set are in fact representable (see [16]).

Proposition 1. ([7]) A uninorm U with neutral element $e \in]0,1[$ is representable if and only if there exists a strictly increasing continuous function $h : [0,1] \to [0,+\infty]$ with $h(0) = 0$, $h(e) = 1$ and $h(1) = +\infty$ such that U is given by

$$U(x,y) = h^{-1}(h(x) \cdot h(y))$$

for all $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ and $U(0,1) = U(1,0) = 0$ or $U(0,1) = U(1,0) = 1$. Such a function h is called a *multiplicative generator* of U.

Remark 1. Note that representable uninorms are strictly increasing in the open square $]0,1[^2$ and consequently they satisfy $U(x,x) < x$ for all $x \in]0,e[$, and $U(x,x) > x$ for all $x \in]e,1[$.

Definition 3. A uninorm U with neutral element $e \in]0,1[$ is said to be in $\mathcal{U}_{\min}$ when it is given by

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2 \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2 \\ \min(x,y) & \text{otherwise} \end{cases} \tag{3}$$

and is said to be in $\mathcal{U}_{\max}$ when it is given by

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2 \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2 \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{4}$$

In both expressions T denotes a t-norm and S denotes a t-conorm.

Remark 2. In fact any uninorm U has the same structure that in (3) except for the values of $U(x,y)$ when $\min(x,y) < e < \max(x,y)$. In general, it is only known that these values are placed between the minimum and the maximum. Due to this general structure, any uninorm U is usually denoted by $U = (e,T,S)$. Note however that this notation is ambiguous because there are different uninorms with the same e, T and S.

Proposition 2. ([10]) U is an idempotent uninorm (that is, $U(x,x) = x$ for all $x \in [0,1]$) with neutral element $e \in [0,1]$ if and only if there exists a decreasing function $g : [0,1] \to [0,1]$ with $g(e) = e$, $g(x) = 0$ for all $x > g(0)$, $g(x) = 1$ for all $x < g(1)$, satisfying

$$\inf\{y \mid g(y) = g(x)\} \le g^2(x) \le \sup\{y \mid g(y) = g(x)\}$$

for all $x \in [0,1]$, such that

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } (y = g(x) \text{ and } x < g^2(x)) \\ \max(x,y) & \text{if } y > g(x) \text{ or } (y = g(x) \text{ and } x > g^2(x)) \\ \min(x,y) & \\ \quad\text{or} & \text{if } y = g(x) \text{ and } x = g^2(x) \\ \max(x,y) & \end{cases}$$

being commutative on the set of points (x,y) such that $y = g(x)$ with $x = g^2(x)$. Function g is usually called the *associated function* of U.

Idempotent uninorms will be denoted by $U = (e,g)$ although this notation is again ambiguous because depending on g it is possible to have many idempotent uninorms with the same e and g.

Definition 4. A binary operator $I : [0,1]\times[0,1] \to [0,1]$ is said to be an *implication operator*, or an *implication*, if it satisfies:

I1) I is nonincreasing in the first place and nondecreasing in the second one.

I2) $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not determined in general.

Strong implications from uninorms are defined in the following way:

$$I(x,y) = U(N(x),y) \quad \text{for all } x,y \in [0,1] \tag{5}$$

where N is a strong negation and U is a uninorm. Note that in order to be such I an implication, we need $U(1,0) = 1$, that is, U must be disjunctive. For this kind of implications we have the following properties.

Proposition 3. Let N be a strong negation, U a disjunctive uninorm with neutral element $e \in]0,1[$ and I the implication given by equation (5). Then,

i) $I(x,e) = N(x)$ for all $x \in [0,1]$.

ii) I satisfies contrapositive symmetry with respect to N, that is,

$$I(x,y) = I(N(y),N(x)) \qquad \text{for all} \quad x,y \in [0,1].$$

Finally, let us recall that the distributivity condition between uninorms has been studied for several classes. For instance, the solutions of such equation for idempotent uninorms can be found in [14] whereas for uninorms in $\mathcal{U}_{\max}$ can be found in [11] and [12].

3. DISTRIBUTIVE STRONG IMPLICATIONS

As we have commented, equation (1) was solved in [18] where the conjunction is performed by a continuous t-norm, the disjunction by a continuous t-conorm and the implication is a strong implication derived from another continuous t-conorm. Now, we are interested in solving the same equation in a more general setting: when the operators used to perform conjunctions and disjunctions are conjunctive and disjunctive uninorms, instead of t-norms and t-conorms, respectively, and the implications are strong implications derived from uninorms.

Thus, let $U = (e, T, S)$ be a disjunctive uninorm, N a strong negation, I the strong implication defined by $I(x, y) = U(N(x), y)$, $U_c = (e_c, T_c, S_c)$ a conjunctive uninorm and $U_d = (e_d, T_d, S_d)$ a disjunctive one. We want to solve the equation

$$I(U_c(x, y), z) = U_d(I(x, z), I(y, z)) \tag{6}$$

for all $x, y, z \in [0, 1]$ where the uninorm U is in one of the three classes stated in the preliminaries. Note however that, while the class of representable uninorms is disjoint with the other two classes, we have that $\mathcal{U}_{\min}$ (and $\mathcal{U}_{\max}$) and the class of idempotent uninorms have nonempty intersection. In fact, this intersection is given by those uninorms in $\mathcal{U}_{\min}$ (or $\mathcal{U}_{\max}$) with minimum and maximum as underlying t-norm and t-conorm.

Remark 3. Note also that the other possible cases, taking in equation (6) both uninorms conjunctive or both disjunctive, give no solutions. It is enough to consider $x = 1$ and $y = z = 0$ to reach a contradiction.

A first characterization can be given in this general case.

Theorem 1. With the previous notations, I, U_c and U_d satisfy equation (6) if and only if U_c and U_d are N-dual and U is distributive over U_d.

Proof. Let us suppose first that I, U_c and U_d satisfy equation (6). Just taking $z = e$ in such equation we obtain

$$I(U_c(x, y), e) = U_d(I(x, e), I(y, e)) \Longrightarrow N(U_c(x, y)) = U_d(N(x), N(y))$$

that is, U_c and U_d are N-dual. Moreover, from this duality we have, on one hand, that

$$I(U_c(x, y), z) = U(N(U_c(x, y)), z) = U(U_d(N(x), N(y)), z)$$

and, on the other hand,

$$U_d(I(x, z), I(y, z)) = U_d(U(N(x), z), U(N(y), z)).$$

Since equation (6) holds we obtain the distributivity of U over U_d.

Conversely, if U_c and U_d are N-dual, by reversing the previous reasoning we can see that the distributivity of U over U_d implies equation (6). □

From this result the problem is reduced to find which uninorms U are distributive over U_d. We do this only when the uninorm U is in one of the three classes stated in the preliminaries, that are the most usual ones. Moreover, we distinguish three cases: when U is a t-conorm, when U_d is a t-conorm (and consequently U_c is a t-norm), and when U, U_d (and consequently U_c) are uninorms with neutral elements $e, e_d, (e_c)$ in the open interval $]0,1[$. In the following a subsection for each case is considered.

3.1. Case S, U_c and U_d

In this case, let us consider S a t-conorm, N a strong negation, I the strong implication generated by S and N, U_c a conjunctive uninorm and U_d a disjunctive one. To solve the equation

$$I(U_c(x,y),z) = U_d(I(x,z), I(y,z)) \tag{7}$$

for all $x, y, z \in [0,1]$, we only need to find which t-conorms S are distributive over the disjunctive uninorm U_d. This problem was already studied in [16] and from the results proved there we can derive the following theorem.

Theorem 2. Let S be a t-conorm, N a strong negation and I the strong implication given by $I(x,y) = S(N(x), y)$ for all $x, y \in [0,1]$. Let U_c be a conjunctive uninorm and U_d a disjunctive one with neutral elements e_c and e_d respectively. Then, I, U_c and U_d satisfy equation (7) if and only if U_d is an idempotent uninorm lying in $\mathcal{U}_{\max}$, U_c and U_d are N-dual and $S = (\langle 0, e_d, S_1\rangle, \langle e_d, 1, S_2\rangle)$ where S_1 and S_2 are t-conorms.

Remark 4. Figure 1 shows the structure of S, U_c and U_d satisfying equation (7).

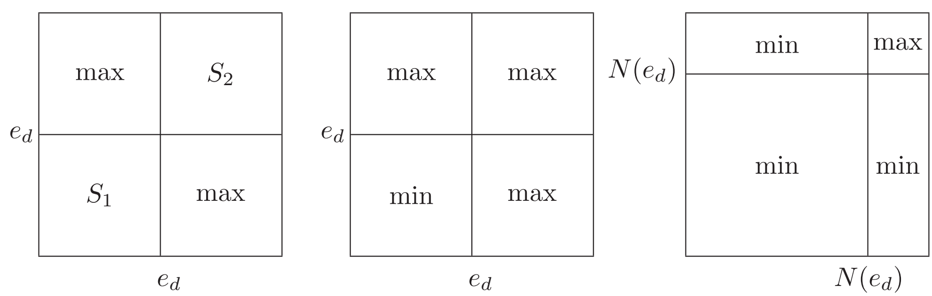

Fig. 1. General structure of a t-conorm S (left), a disjunctive uninorm U_d (middle) and a conjunctive uninorm U_c (right) satisfying equation (7), being $e_c = N(e_d)$.

3.2. Case U, T and S

In this case we will consider U a disjunctive uninorm with neutral element $e \in]0,1[$, N a strong negation, I the strong implication generated by U and N, T a t-norm, and S a t-conorm. Again, to solve the equation

$$I(T(x,y),z) = S(I(x,z),I(y,z)) \tag{8}$$

for all $x,y,z \in [0,1]$, we only need to find which disjunctive uninorms U are distributive over a t-conorm S. This problem was already solved in [16] for uninorms lying in one of the three mentioned classes and for continuous t-conorms. From the solutions obtained there we have the following theorem.

Theorem 3. Let U be a disjunctive uninorm with neutral element $e \in]0,1[$ which is idempotent, in $\mathcal{U}_{\max}$ or representable. Let N be a strong negation, I the strong implication generated by U and N, T a t-norm, and S a continuous t-conorm. Then, I, T and S satisfy equation (8) if and only if T and S are N-dual and we have one of the following two cases:

(a) $S = \max$, $T = \min$, or

(b) S is strict and U is representable and such that, if s is the additive generator of S with $s(e) = 1$, then s is also a multiplicative generator of U.

Remark 5. Note that the only known solutions in [18] required T and S being the minimum and the maximum. On the contrary, here we have obtained strong implications that satisfy (8) with T and S strict.

3.3. Case U, U_c and U_d

From now on, U will denote a disjunctive uninorm with neutral element $e \in]0,1[$, N a strong negation, and I the strong implication generated by U and N. Also, in all this subsection, U_c will denote a conjunctive uninorm and U_d a disjunctive one with neutral elements e_c and e_d, respectively, such that $0 < e_c, e_d < 1$.

Again, to solve the equation

$$I(U_c(x,y),z) = U_d(I(x,z),I(y,z)) \tag{9}$$

for all $x,y,z \in [0,1]$ we need to find which disjunctive uninorms U are distributive over a disjunctive uninorm U_d. That is, we want to study the equation

$$U(x,U_d(y,z)) = U_d(U(x,y),U(x,z)) \tag{10}$$

for disjunctive uninorms U lying in one of the three classes stated in the preliminaries and for disjunctive uninorms U_d with underlying t-norm and t-conorm continuous.

Let us begin with several lemmata in order to establish the complete process that leads to the solutions. First, we deal with the case when U and U_d have the same neutral element.

Lemma 1. Let U and U_d be two uninorms with the same neutral element $e \in]0,1[$. If U is distributive over U_d, then U_d is idempotent and $U_d(x,y) = U(x,y)$ for all $(x,y) \in A(e) = [0,e] \times [e,1] \cup [e,1] \times [0,e]$.

Proof. Taking $y = z = e$ in (10) we directly obtain $U_d(x,x) = x$ and so U_d is idempotent. Moreover, taking $z = e$ we have:

$$U(x,y) = U(x, U_d(y,e)) = U_d(U(x,y), U(x,e)) = U_d(U(x,y), x) \tag{11}$$

for all $x, y \in [0,1]$. Now, let us prove that $U(x,y) \in \{x,y\}$ in $A(e)$. It is trivial when $x = e$ or $y = e$. On the other hand, if $x < e < y$ we can prove first that $U(x,y) \neq e$. If, on the contrary, we suppose $U(x,y) = e$ we obtain from equation (11) that $e = U(x,y) = U_d(U(x,y),x) = U_d(e,x) = x$, which is a contradiction. Thus we have the following two possibilities:

- If $x \le U(x,y) < e < y$. Since U_d is idempotent, equation (11) ensures that $U(x,y) = \min(U(x,y), x) = x$.
- If $x < e < U(x,y) \le y$. By commutativity, we also have from (11) that
$$U(x,y) = U(y,x) = U_d(U(y,x), y) \tag{12}$$
and so $U(x,y) = \max(U(y,x), y) = y$.

Thus, we have proved that $U(x,y) \in \{x,y\}$ in $A(e)$. Finally, we have again two possibilities in $A(e)$:

- If $U(x,y) = x$. By equation (12) we have $U(x,y) = U_d(U(y,x), y) = U_d(x,y)$.
- If $U(x,y) = y$. By equation (11) we have $U(x,y) = U_d(U(x,y), x) = U_d(y,x)$.

□

Next lemma proves that there are no solutions of equation (10) when U is representable.

Lemma 2. Let $U = (e, T, S)$ and $U_d = (e_d, T_d, S_d)$ be two uninorms. If U is distributive over U_d, then $U(e_d, e_d) = e_d$ and U can not be representable.

Proof. First note that from distributivity we have

$$U(e_d, U_d(e, e_d)) = U_d(U(e_d, e), U(e_d, e_d)), \quad \text{that is,} \quad e_d = U(e_d, e_d).$$

Now we distinguish two cases:

- If $e = e_d$, U_d is idempotent and U agrees with U_d in $A(e)$ by the previous lemma, and consequently U can not be representable.
- If $e \neq e_d$, then e_d is an idempotent element of U and U can not be representable because Remark 1. □

From the previous result and taking into account the only three considered classes of uninorms, we have that the uninorm U must be idempotent or in $\mathcal{U}_{\max}$. We will deal with both cases separately and we begin with the case when U is idempotent in the following lemma.

Lemma 3. Let $U = (e, g)$ and $U_d = (e_d, T_d, S_d)$ be uninorms such that U is idempotent and U_d has underlying t-norm T_d and t-conorm S_d continuous. If U is distributive over U_d, then U_d is idempotent.

P r o o f. First, taking $y = z = e$ in (10) we obtain

$$U_d(x, x) = U(x, U_d(e, e)) \quad \text{for all} \quad x \in [0, 1]. \tag{13}$$

Thus, we have three possibilities:

- If $U_d(e, e) = e$, then clearly U_d is idempotent from the equation above.
- If $U_d(e, e) > e$. Let us now prove that in this case we reach a contradiction. Really, if $U_d(e, e) > e$ it necessarily holds $g(U_d(e, e)) \leq e$.
  - On one hand, for all $x \in [0, 1]$ such that $g(U_d(e, e)) < x < U_d(e, e)$ we have by Proposition 2

    $$U(x, U_d(e, e)) = \max(x, U_d(e, e)) = U_d(e, e)$$

    that reduces to $U_d(x, x) = U_d(e, e)$ by equation (13). This implies in particular that $x > e_d$ for all these values, and consequently, $e_d \leq g(U_d(e, e))$.
  - On the other hand, for all $x < g(U_d(e, e))$ we similarly have by Proposition 2 that

    $$U(x, U_d(e, e)) = \min(x, U_d(e, e)) = x$$

    which implies $U_d(x, x) = x$ for all $x < g(U_d(e, e))$.

  Now, the continuity of S_d gives a contradiction.
- If $U_d(e, e) < e$. Then we similarly obtain a contradiction using that T_d is continuous.

Thus, we have proved that U_d must be idempotent. □

Now, we deal with the case when U is a uninorm in $\mathcal{U}_{\max}$.

Lemma 4. Let $U = (e, T, S)$ and $U_d = (e_d, T_d, S_d)$ be uninorms with $U \in \mathcal{U}_{\max}$ and with T_d and S_d continuous. If U is distributive over U_d, then U_d is idempotent.

P r o o f. Since $U \in \mathcal{U}_{\max}$, for all $x > e$ we have

$$U(x, U_d(e, 0)) = U_d(U(x, e), U(x, 0)) = U_d(x, x)$$

Now we distinguish several cases:

- If $e < e_d$, then we have $U_d(x, x) = U(x, U_d(e, 0)) = U(x, 0) = x$.
- If $e \geq e_d$ then we have $U_d(e, 0) \in [0, e]$. Since $U(x, y) = x$ for all (x, y) such that $y \leq e < x$ it is clear that $U(x, U_d(e, 0)) = x$ and consequently $U_d(x, x) = x$.

Then $U_d(x, x) = x$ for all $x > e$ and by continuity of S_d and T_d we have that $U_d(e, e) = e$. But then, taking $y = z = e$ in the distributivity equation, we obtain $U_d(x, x) = x$ for all $x \in [0, 1]$, that is, U_d is idempotent. □

Lemma 5. Let $U = (e, T, S)$ and $U_d = (e_d, T_d, S_d)$ be uninorms, with T_d and S_d continuous, and $U \in \mathcal{U}_{\max}$. If U is distributive over U_d and $e \leq e_d$, then $U_d \in \mathcal{U}_{\max}$.

Proof. First of all, we know that U_d is idempotent by the previous lemma. Thus, let g_d be the associated function of U_d. The case $e = e_d$ follows from Lemma 1. On the other hand, if $e < e_d$ then $g_d(e) \geq e_d$ and let us divide our proof in several steps.

i) First we prove that $g_d(e) = e_d$. Suppose on the contrary that $g_d(e) > e_d$, then there exist x and z such that

$$e < e_d < x < z < g_d(e)$$

and for these values we have

$$U(x, U_d(e, z)) = U(x, \min(e, z)) = U(x, e) = x$$

but, as $e_d < x < z \leq U(x, z)$,

$$U_d(U(x, e), U(x, z)) = U_d(x, U(x, z)) = \max(x, U(x, z)) = U(x, z) > x,$$

that gives a contradiction from equation (10). Then $g_d(e) = e_d$.

ii) By decreasingness of g_d and the previous step, $g_d(x) = e_d$ for all $x \in [e, e_d]$. Therefore, for all $x > e_d = g_d(e)$, we have $U_d(e, x) = \max(x, e) = x$, that is, $g_d(x) \leq e$ for all $x > e_d$.

iii) Now, we prove that $g_d(x) = 0$ for all $x > e_d$. Suppose that $0 < g_d(x)$ for some $x > e_d$. There exists z such that

$$0 < z < g_d(x) \leq e < e_d = g_d(e) < x$$

and then, from one side

$$U(z, U_d(e, x)) = U(z, x) = x$$

and from the other

$$U_d(U(z, e), U(z, x)) = U_d(z, x) = z$$

that is a contradiction. Then, $g_d(x) = 0$ for all $x > e_d$.

iv) Next we prove that $g_d(x) = e_d$ for all $x \in]0, e_d]$. The previous step shows that

$$U_d(a, y) = \max(a, y) \qquad \text{for all} \qquad a \geq e_d \quad \text{and} \quad y > 0.$$

Thus, when $y > 0$, we have by commutativity that $U_d(y, a) = \max(y, a)$, proving that $g_d(y) \leq a$, for all $a > e_d$. Consequently, $g_d(y) \leq e_d$ for all $y > 0$ and also $g_d(y) = e_d$ for all $0 < y \leq e_d$ using the decreasingness of g_d and step (ii) above.

v) Finally, we prove that $g_d(0) = e_d$. If we suppose $g_d(0) > e_d$ we can take an element x such that $0 < e < e_d < x < g_d(0)$ and for this value we have

$$U(0, U_d(e, x)) = U(0, x) = x$$

whereas

$$U_d(U(0, e), U(0, x)) = U_d(0, x) = 0$$

obtaining a contradiction.

Joining all the previous steps we have

$$g_d(x) = \begin{cases} e_d & \text{if } \ x \leq e_d \\ 0 & \text{otherwise} \end{cases}$$

that is, $U_d \in \mathcal{U}_{\max}$. □

Lemma 6. Let $U = (e, T, S)$ and $U_d = (e_d, T_d, S_d)$ be uninorms, with T_d and S_d continuous, and $U \in \mathcal{U}_{\max}$. If U is distributive over U_d and $e_d < e$, then U_d is the only possible idempotent uninorm with associated function g_d given by

$$g_d(x) = \begin{cases} e & \text{if } x < e_d \\ e_d & \text{if } e_d \leq x \leq e \\ 0 & \text{if } x > e. \end{cases} \tag{14}$$

That is, U_d is given by

$$U_d(x, y) = \begin{cases} \min(x, y) & \text{if } \min(x, y) < e_d \text{ and } \max(x, y) \leq e \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Moreover, the uninorm U must be given by the minimum in all points (x, y) such that $\min(x, y) \leq e_d \leq \max(x, y) \leq e$.

Proof. We already know that U_d must be idempotent, say $U_d = (e_d, g_d)$ and let us prove that g_d must be given by equation (14) in several steps.

i) We prove first that $g_d(x) = e_d$ for all $x \in [e_d, e]$. Since $e_d < e$ we have $g_d(e) \leq e_d$. Suppose that $g_d(e) < e_d$ and take x, z such that $g_d(e) < z < x < e_d < e$. On one hand,

$$U(x, U_d(e, z)) = U(x, \max(e, z)) = U(x, e) = x$$

and on the other, since $U(x, z) \leq \min(x, z) = z < x < e_d$,

$$U_d(U(x, e), U(x, z)) = U_d(x, U(x, z)) = \min(x, U(x, z)) = U(x, z) < x$$

obtaining contradiction. Thus, $g_d(e) = e_d$ and by decreasingness we have $g_d(x) = e_d$ for all $x \in [e_d, e]$.

ii) We prove now that $g_d(x) = e$ for all $x < e_d$. By the previous step and using commutativity, it must be $g_d(x) \geq e$ for all these values. Suppose then that there is $x < e_d$ such that $g_d(x) > e$ and take z such that $x < e_d = g_d(e) < e < z < g_d(x)$. Since $U \in \mathcal{U}_{\max}$, we have $U(x, z) = z$ and then

$$U(x, U_d(e, z)) = U(x, z) = z$$

and also

$$U_d(U(x, e), U(x, z)) = U_d(x, z) = x,$$

obtaining again contradiction.

iii) The step above also proves, by commutativity of U_d, that $g_d(x) = 0$ for all $x > e$.

Thus, g_d must be given by equation (14) and now it is easy to see that there is only one possible idempotent uninorm with this associated function, which is the one given in the lemma.

Finally, it only remains to prove that $U(x, y) = \min(x, y)$ when $\min(x, y) \leq e_d \leq \max(x, y) \leq e$. To see this, take $x < e_d < y < e$. We have $U(x, U_d(y, e)) = U(x, e) = x$ whereas, since $U(x, y) \leq x$,

$$U_d(U(x, y), U(x, e)) = U_d(U(x, y), x) = \min(U(x, y), x) = U(x, y)$$

that is $U(x, y) = x$ as we wanted to prove. □

In view of all the previous lemmata and taking into account the results on distributivity involving uninorms in $\mathcal{U}_{\max}$ proved in [11], and those involving idempotent uninorms proved in [14], we can give now the following theorem that shows all solutions of equation (10).

Theorem 4. Let $U = (e, T, S)$ be a uninorm which is idempotent, in $\mathcal{U}_{\max}$ or representable. Let $U_d = (e_d, T_d, S_d)$ be a uninorm with T_d and S_d continuous. Then U is distributive over U_d if and only if U_d is idempotent, say $U_d = (e_d, g_d)$, and one of the following cases holds:

i) U is idempotent, say $U = (e, g)$, and

  a) If $e < e_d$, then
    1. $g(x) \leq g_d(x)$ for all $x \in [0, 1]$.
    2. If $g(x) < g_d(x)$ then $g_d(x) = e_d$ and $x \leq e_d$.
    3. If there are x and z satisfying $g(z) = g_d(z) = x$ and $g(x) = g_d(x) = z$, then $U(x, z) = U_d(x, z)$.

  b) If $e = e_d$, then $U = U_d$.

  c) If $e > e_d$, then
    1. $g_d(x) \leq g(x)$ for all $x \in [0, 1]$.
    2. If $g_d(x) < g(x)$ then $g_d(x) = e_d$ and $x \geq e_d$.

3. If there are x and z satisfying $g(z) = g_d(z) = x$ and $g(x) = g_d(x) = z$, then $U(x,z) = U_d(x,z)$.

ii) $U \in \mathcal{U}_{\max}$ and

a) If $e \leq e_d$, then U_d is in $\mathcal{U}_{\max}$ and U is given by

$$U(x,y) = \begin{cases} eT\left(\frac{x}{e}, \frac{y}{e}\right) & \text{if } x, y \leq e \\ e + (e_d - e)S_1\left(\frac{x-e}{e_d-e}, \frac{y-e}{e_d-e}\right) & \text{if } e \leq x, y \leq e_d \\ e_d + (1-e_d)S_2\left(\frac{x-e_d}{1-e_d}, \frac{y-e_d}{1-e_d}\right) & \text{if } e_d \leq x, y \\ \max(x,y) & \text{otherwise} \end{cases}$$

b) If $e > e_d$, then U is given by

$$U(x,y) = \begin{cases} e_d T_1\left(\frac{x}{e_d}, \frac{y}{e_d}\right) & \text{if } x, y \leq e_d \\ e_d + (e - e_d)T_2\left(\frac{x-e_d}{e-e_d}, \frac{y-e_d}{e-e_d}\right) & \text{if } e_d \leq x, y \leq e \\ \min(x,y) & \text{if } \min(x,y) \leq e_d \\ & \leq \max(x,y) \leq e \\ e + (1-e)S\left(\frac{x-e}{1-e}, \frac{y-e}{1-e}\right) & \text{if } e \leq x, y \\ \max(x,y) & \text{otherwise} \end{cases}$$

and U_d and g_d are as in Lemma 6.

Proof. From left to right is a direct consequence of the previous lemmata and the results on distributive uninorms given in [11] and in [14]. Conversely, cases i)–a), i)–b) and i)–c) are proved to verify distributivity in [14] whereas case ii)–a) is proved in [11]. Thus, it only remains to prove the distributivity of case ii)–b) that is a tedious but straightforward computation. □

Remark 6. Solutions in ii)–a) and ii)–b) of the previous theorem can be viewed in Figures 2 and 3, respectively.

And consequently we have the following theorem showing all solutions of equation (6).

Theorem 5. Let $U = (e, T, S)$ be a uninorm which is idempotent, in $\mathcal{U}_{\max}$ or representable. Let N be a strong negation and I the strong implication generated by U and N. Let $U_c = (e_c, T_c, S_c)$ and $U_d = (e_d, T_d, S_d)$ be uninorms such that T_d and S_d are continuous. Then I, U_c and U_d satisfy equation (6) if and only if U_d is idempotent, U_c and U_d are N-dual and one of the cases in Theorem 4 holds.

Proof. Straightforward from Theorems 1 and 4. □

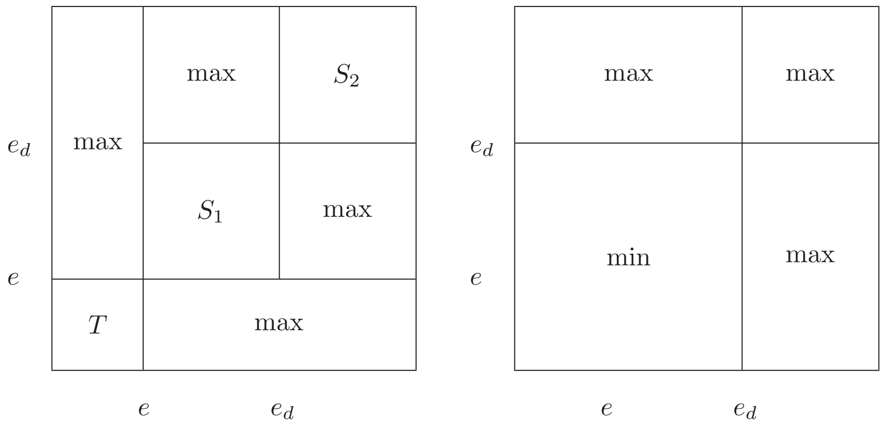

Fig. 2. General structure of a uninorm U in $\mathcal{U}_{\max}$ (left), that is distributive over a disjunctive uninorm U_d (right), with neutral elements $e \le e_d$.

4. OTHER RELATED EQUATIONS

Finally, let us deal in this section with several related equations on distributivity, also studied in [1] for the case of t-norms and t-conorms. Namely, those derived from the following equivalences in classical logic:

$$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r), \tag{15}$$

$$p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r), \tag{16}$$

$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r). \tag{17}$$

In our context, they become respectively

$$I(U_d(x,y),z) = U_c(I(x,z), I(y,z)), \tag{18}$$

$$I(x, U_c(y,z)) = U_c'(I(x,y), I(x,z)), \tag{19}$$

$$I(x, U_d(y,z)) = U_d'(I(x,y), I(x,z)) \tag{20}$$

for all $x, y, z \in [0,1]$. Note that equation (20) is dual of the equation solved in the section above and then it can be easily solved using this duality. Namely, let I be a strong implication defined by $I(x,y) = U(N(x), y)$ and U_d, U_d' disjunctive uninorms. Then I, U_d and U_d' satisfy equation (20) if and only if

$$I(N(x), U_d(N(y), N(z))) = U_d'(I(N(x), N(y)), I(N(x), N(z)))$$

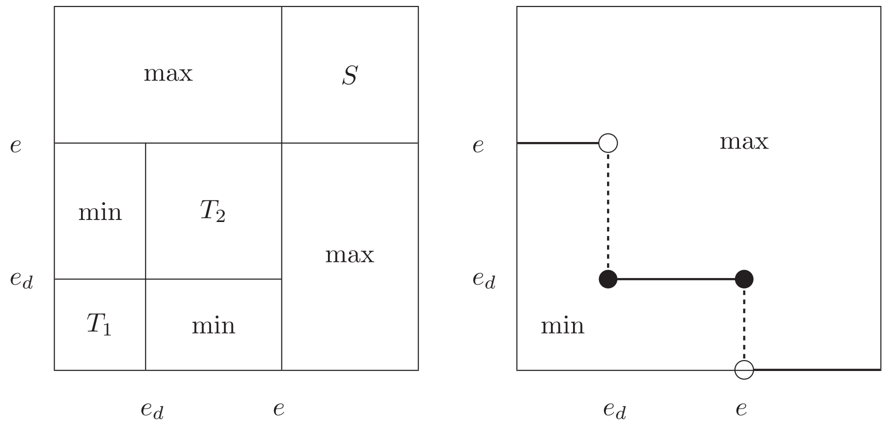

Fig. 3. General structure of a uninorm U in $\mathcal{U}_{\max}$ (left), that is distributive over a disjunctive uninorm U_d (right), with neutral elements $e_d < e$.

for all $x, y, z \in [0, 1]$ which is equivalent to

$$I(N(x), N(U_c(y, z))) = U'_d(I(N(x), N(y)), I(N(x), N(z)))$$

where U_c is the N-dual of U_d. Now, since any strong implication satisfies contrapositive symmetry with respect to N (see Proposition 3 *ii*)), this is equivalent to

$$I(U_c(y, z), x) = U'_d(I(y, x), I(z, x))$$

which is exactly equation (6). Thus, solutions of equation (20) directly derive from Theorems 4 and 5 as follows.

Theorem 6. Let $U = (e, T, S)$ be a uninorm which is idempotent, in $\mathcal{U}_{\max}$ or representable. Let N be a strong negation and I the strong implication generated by U and N. Let $U_d = (e_d, T_d, S_d)$ and $U'_d = (e'_d, T'_d, S'_d)$ be two disjunctive uninorms with T'_d and S'_d continuous. Then I, U_d and U'_d satisfy equation (20) if and only if $U_d = U'_d$ and one of the following cases holds:

- $U = S$ is a t-conorm and U_d and S are as in Theorem 2.
- $U_d = S$ is a t-conorm and $S = \max$ or S and U are as in Theorem 3 (b).
- $0 < e, e_d < 1$, U_d is idempotent, say $U_d = (e_d, g_d)$, and U and U_d are as in one of the cases of Theorem 4.

With respect to equations (18) and (19), they are also dual one of each other and so we need to solve only one of them since the other will follow by duality. Let us deal for instance with equation (18). Similarly to the case studied in the previous section we obtain the following characterization.

Theorem 7. Let $U = (e, T, S)$ be a disjunctive uninorm, N a strong negation, I the strong implication defined by $I(x, y) = U(N(x), y)$, $U_d = (e_d, T_d, S_d)$ a disjunctive uninorm and $U_c = (e_c, T_c, S_c)$ a conjunctive one. Then, I, U_d and U_c satisfy equation (18) if and only if U_c and U_d are N-dual and U is distributive over U_c.

Proof. It is similar to the one given in Theorem 1. □

Now, the problem is to solve the distributivity of a disjunctive uninorm over a conjunctive one, that is

$$U(x, U_c(y, z)) = U_c(U(x, y), U(x, z)) \quad \text{for all} \quad x, y, z \in [0, 1] \tag{21}$$

where U is a disjunctive uninorm and U_c is a conjunctive one. We will prove that there are less solutions than in the case previously studied. To do this, we again distinguish three cases according to whether the involved uninorms is a t-norm or a t-conorm:

i) Case when $U = S$ is a t-conorm. In this case, there are no solutions of equation (18) since there are no t-conorms distributive over conjunctive uninorms (see [16]).

ii) Case when $U_c = T$ is a t-norm. In this case, again from results in [16] we have the following solutions.

Theorem 8. Let U be a disjunctive uninorm with neutral element $e \in]0, 1[$ which is idempotent, in $\mathcal{U}_{\max}$ or representable. Let N be a strong negation, I the strong implication generated by U and N, S a t-conorm, and T a continuous t-norm. Then, I, S and T satisfy equation (18) if and only if S and T are N-dual and we have one of the following two cases:

(a) $T = \min$, or

(b) T is strict and U is representable and such that, if t is the additive generator of T such that $t(e) = 1$, then $\frac{1}{t}$ is also a multiplicative generator of U.

iii) General case when the neutral elements e and e_d are both in $]0, 1[$. In this case, we can similarly prove equivalent lemmata as Lemma 1, Lemma 2 and Lemma 3, showing that U can not be representable and when U is idempotent necessarily U_c also is idempotent. With respect to the case when U lies in $\mathcal{U}_{\max}$ we have now the following result.

Lemma 7. Let $U = (e, T, S)$ be a uninorm in $\mathcal{U}_{\max}$ and $U_c = (e_c, T_c, S_c)$ be a conjunctive uninorm. Then, they never satisfy equation (21).

Proof. Suppose that U and U_c satisfy equation (21). First we prove that necessarily $e < e_c$. Effectively, if on the contrary we suppose that $e_c \leq e$ then we have $U(0, U_c(e_c, 1)) = U(0, 1) = 1$ whereas

$$U_c(U(0, e_c), U(0, 1)) = U_c(0, 1) = 0$$

obtaining a contradiction. Thus $e < e_c$, and following the proofs of Lemmas 4 and 5 we can similarly show that U_c must be idempotent, say $U_c = (e_c, g_c)$, and that $g_c(e) = e_c$ and $g_c(x) = 0$ for all $x > e_c$. However, since U_c is conjunctive we have $U_c(x, 0) = 0$ for all $x \in [0, 1]$ and taking x such that $0 < e < e_c = g_c(e) < x < 1$ we obtain:

$$U(0, U_c(e,x)) = U(0, x) = x \quad \text{whereas} \quad U_c(U(0, e), U(0, x)) = U_c(0, x) = 0$$

obtaining again a contradiction. This completes the proof. □

Now, we can give the solutions of equation (21).

Theorem 9. Let $U = (e, T, S)$ be a disjunctive uninorm, N a strong negation, I the strong implication defined by $I(x, y) = U(N(x), y)$, $U_d = (e_d, T_d, S_d)$ a disjunctive uninorm and $U_c = (e_c, T_c, S_c)$ a conjunctive one. Then, I, U_d and U_c satisfy equation (18) if and only if U_d and U_c are N-dual, $U = (e, g)$ and $U_c = (e_c, g_c)$ are idempotent such that functions g and g_c are as in case i)-a) of Theorem 4.

P r o o f. Straightforward from the previous results. □

Dually we also have

Theorem 10. Let $U = (e, T, S)$ be a disjunctive uninorm, N a strong negation, I the strong implication defined by $I(x, y) = U(N(x), y)$, $U_c = (e_c, T_c, S_c)$ and $U_c = (e'_c, T'_c, S'_c)$ two conjunctive uninorms. Then, I, U_c and U'_c satisfy equation (19) if and only if $U_c = U'_c$, $U = (e, g)$ and $U_c = (e_c, g_c)$ are idempotent such that functions g and g_c are as in case i) − a) of Theorem 4.

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