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HOMOGENEOUS AGGREGATION OPERATORS

TATIANA RÜCKSCHLOSSOVÁ AND ROMAN RÜCKSCHLOSS

Recently, the utilization of invariant aggregation operators, i.e., aggregation operators not depending on a given scale of measurement was found as a very current theme. One type of invariantness of aggregation operators is the homogeneity what means that an aggregation operator is invariant with respect to multiplication by a constant. We present here a complete characterization of homogeneous aggregation operators. We discuss a relationship between homogeneity, kernel property and shift-invariance of aggregation operators. Several examples are included.

Keywords: aggregation operator, homogeneity, kernel property

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1. INTRODUCTION

Aggregation operators are known as tools for aggregation (fusion) processes where from several input values one output value is required. As an example it could be mentioned the aggregation of infinitely many inputs [7, 8, 14], of inputs from ordinal scales [6] and also of complex inputs as distribution functions [14] or fuzzy sets [14]. Application of these operators in mathematics, physics, engineering, economics or social and other sciences verifies their broad utilization.

Values incoming into some aggregation process are related to a certain scale of measurement as well as output should be. Many decisions are based on the results of an appropriate aggregation and there is a need to realize the same decisions independently of a chosen scale of measurement, i. e. to apply aggregation operators reflecting this requirement. Rescaling the values we are dealing with is modeled by the transformation of an applied operator. An operator invariant under appropriate transformation is then an operator not depending on a given scale. According to the type of scales we can speak about several types of invariantnesses of aggregation operators.

Invariantness with respect to any scale which is rather restrictive has been studied in [11]. If we fix the unit of measurement but not the beginning of our scale (i. e., "zero" is free), we have to deal with the shift-invariant aggregation operators (recall, e.g., the temperature measurement in degrees of Celsius and in degrees of Kelvin). These operators are completely characterized in [10], compare also [1]. Fixing the "zero" but letting free the unit (recall, e. g., the length measurement in meters and in yards), we come to the need of homogeneous aggregation operators. The aim of this paper is their complete description. The paper is organized as follows. In the next section we recall basic definitions. Section 3 is devoted to the characterization of homogeneous aggregation operators. Finally, in Section 4 we discuss the homogeneity of aggregation operators in relation to the kernel property and shift-invariantness of some related operators.

2. PRELIMINARIES

Following [3, 8] recall the notions of an aggregation operator and a homogeneous operator.

Definition 1. A mapping $A : \bigcup_{n\in N}[0,1]^n \to [0,1]$ is called an aggregation operator if it fulfils the following conditions:

(A1) $A(x) = x$ for each $x \in [0,1]$,

(A2) $A(x_1,\dots,x_n) \le A(y_1,\dots,y_n)$ whenever $x_i \le y_i \ \forall\ i = 1,\dots,n,\ n \in N$,

(A3) $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$.

Definition 2. An operator $A : \bigcup_{n\in N}[0,1]^n \to [0,1]$ is said to be homogeneous if $\forall\, n \in N$, $\forall\, b \in]0,1[$, $\forall\, x_1,\dots,x_n \in [0,1]$:

$$A(bx_1,\dots,bx_n) = bA(x_1,\dots,x_n).$$

By means of an arbitrary aggregation operator $C : \bigcup_{n\in N}[0,1]^n \to [0,1]$, a homogeneous aggregation operator $H^C : \bigcup_{n\in N}[0,1]^n \to [0,1]$ can be constructed as follows:

$$H^C(x_1,\dots,x_n) = bC\left(\frac{x_1}{b},\dots,\frac{x_n}{b}\right),$$

where $b = \max(x_1,\dots,x_n) > 0$ (and by convention we put $H^C(0,\dots,0) = 0$). Notice, that by this construction each homogeneous operator is idempotent. Typical examples of homogeneous operators are weighted arithmetic means, including the standard arithmetic mean. However, although the homogeneity of an operator H^C is satisfied, the property of monotonicity need not be ensured. So the operator H^C need not be an aggregation operator, see example below.

Example 1. Take a product Π as an aggregation operator. Then the homogeneous operator $H^\Pi : \bigcup_{n\in N}[0,1]^n \to [0,1]$ is given by

$$H^\Pi(x_1,\dots,x_n) = b\prod_{i=1}^{n}\frac{x_i}{b} = \frac{\prod_{i=1}^{n} x_i}{b^{n-1}}$$

with convention $\frac{0}{0} = 0$, i. e.,

$$H^\Pi_{(n)} = \frac{\Pi_{(n)}}{(\max_{(n)})^{n-1}}.$$

Then $H^{\Pi}_{(3)}(0.5,0.5,0.5) = 0.5 > H^{\Pi}_{(3)}(0.5,0.5,1) = 0.25$, what contradicts the monotonicity of the operator $H^{\Pi}_{(n)}$.

Remark 1. In the class of quasi-arithmetic means (weighted quasi-arithmetic means), homogeneous aggregation operators form a 1-parameter subclass $(A_p)_{p\in]-\infty,\infty[}$ of so called power-root operators [5],

$$A_p(x_1,\dots,x_n) = \left(\frac{1}{n}\sum_{n=1}^{n} {x_i}^p\right)^{\frac{1}{p}} \quad \text{or} \quad A_p(x_1,\dots,x_n) = \left(\sum_{n=1}^{n} w_i {x_i}^p\right)^{\frac{1}{p}}$$

for $p \neq 0$ and $A_0 = G$ is the geometric mean (weighted geometric mean). For more details see [12, 13].

Remark 2. An interesting class of homogeneous aggregation operators can be derived by minimalization of sum of $|x_i - a|^p$ obtained in $a = B_p(x_1,\dots,x_n)$, $p \in [1,\infty]$, see [4]. Note that $B_1 = \mathit{median}$, $B_2 = M$ (arithmetic mean), $B_\infty = \frac{\min + \max}{2}$ and $B_p(x_1,x_2) = \frac{x_1+x_2}{2}$ for all $p \in [1,\infty]$.

3. CHARACTERIZATION OF HOMOGENEOUS AGGREGATION OPERATORS

The open problem to characterize all aggregation operators C such that H^C is the aggregation operator was already stated in [3]. We give here a necessary and sufficient condition for an operator H^C to be an aggregation operator (to be monotonic).

Theorem 1. Let $C : \bigcup_{n\in N}[0,1]^n \to [0,1]$ be an aggregation operator. Operator $H^C : \bigcup_{n\in N}[0,1]^n \to [0,1]$ is an aggregation operator if and only if for all $n \in N$, $\boldsymbol{x} = (x_1,\dots,x_n), \boldsymbol{y} = (y_1,\dots,y_n) \in [0,1]^n$, $\boldsymbol{x} \le \boldsymbol{y}$, such that there is $i \in \{1,\dots,n\}$ and $x_i = y_i = 1$, it holds (with convention $\frac{0}{0} = 1$)

$$\frac{C(\boldsymbol{x})}{C(\boldsymbol{y})} \ge \min_k \frac{x_k}{y_k}. \tag{1}$$

P r o o f. By construction, $H^C(0,\dots,0) = 0$, $H^C(1,\dots,1) = 1$ and $H^C(x) = x$, for $x \in [0,1]$. To see the sufficiency it is enough to show the monotonicity of the operator H^C. Let $\boldsymbol{x} = (x_1,\dots,x_n), \boldsymbol{y} = (y_1,\dots,y_n) \in [0,1]^n$ and $\boldsymbol{x} \le \boldsymbol{y}$. Without loss of generality we may suppose that both $\boldsymbol{x}$ and $\boldsymbol{y}$ are non-zero vectors. Denote $\max_k x_k = x_i$ and $\max_k y_k = y_j$. Put $z_k = \min(x_i, y_k)$. Then evidently $\boldsymbol{x} \le \boldsymbol{z} \le \boldsymbol{y}$ while $\max z_k = z_i = x_i$. Let an aggregation operator C fulfil the condition (1). Following the definition of the homogeneous operator we have

$$\begin{aligned} H^C(x_1,\dots,x_n) &= x_i C\left(\tfrac{x_1}{x_i},\dots,\tfrac{x_n}{x_i}\right) = x_i C\left(\tfrac{\boldsymbol{x}}{x_i}\right), \\ H^C(z_1,\dots,z_n) &= x_i C\left(\tfrac{\min(x_i,y_1)}{x_i},\dots,\tfrac{\min(x_i,y_n)}{x_i}\right) = x_i C\left(\min\left(1,\tfrac{\boldsymbol{y}}{x_i}\right)\right), \end{aligned}$$

$$H^C(y_1,\dots,y_n) = y_j C\left(\frac{y_1}{y_j},\dots,\frac{y_n}{y_j}\right) = y_j C\left(\frac{\boldsymbol{y}}{y_j}\right).$$

From the monotonicity of the aggregation operator C we have $H^C(\boldsymbol{x}) \le H^C(\boldsymbol{z})$. Further, $\frac{y_k}{y_j} \le \min\left(1, \frac{y_k}{x_i}\right)$ for all $k = 1,\dots,n$ and thus from the property (1) we have

$$\frac{C(\frac{\boldsymbol{y}}{y_j})}{C(\min(1,\frac{\boldsymbol{y}}{x_i}))} \ge \min_k \frac{\frac{y_k}{y_j}}{\min(1,\frac{y_k}{x_i})},$$

what means that

$$\frac{C(\frac{\boldsymbol{y}}{y_j})}{C(\min(1,\frac{\boldsymbol{y}}{x_i}))} \ge \min_k \max\left(\frac{y_k}{y_j},\frac{x_i}{y_j}\right) \ge \frac{x_i}{y_j},$$

and therefore

$$y_j C\left(\frac{\boldsymbol{y}}{y_j}\right) = H^C(\boldsymbol{y}) \ge H^C(\boldsymbol{z}) = x_i C\left(\min_k\left(1,\frac{y_k}{x_i}\right)\right).$$

Finally we have

$$H^C(\boldsymbol{x}) \le H^C(\boldsymbol{z}) \le H^C(\boldsymbol{y}) \Rightarrow H^C(\boldsymbol{x}) \le H^C(\boldsymbol{y}),$$

and the monotonicity of the operator H^C is proved. Thus H^C is an aggregation operator.

To see the necessity we prove the converse assertion. Assume that an aggregation operator C has not property (1), i.e., there is an index $i \in \{1,\dots,n\}$ and points $\boldsymbol{u},\ \boldsymbol{v} \in [0,1]^n$, $\boldsymbol{u} \le \boldsymbol{v}$, $u_i = v_i = 1$ such that

$$\frac{C(\boldsymbol{u})}{C(\boldsymbol{v})} < \min_k \frac{u_k}{v_k} \le 1.$$

Create a point $\boldsymbol{w}$, for $j = 1,\dots,n$, $w_j = v_j \min_k \frac{u_k}{v_k}$. Then evidently $\boldsymbol{w} \le \boldsymbol{u}$. As the operator H^C is homogeneous we obtain that $H^C(\boldsymbol{u}) = C(\boldsymbol{u})$ and $H^C(\boldsymbol{w}) = \min_k \frac{u_k}{v_k} C(\boldsymbol{v})$, what implies $H^C(\boldsymbol{u}) < H^C(\boldsymbol{w})$, i.e., the operator H^C is not monotonic. □

Example 2. An example of a class of homogeneous aggregation operators is the class of weighted geometric means. For fixed n a weighted geometric mean G^W is given by

$$G^W(x_1,\dots,x_n) = \prod_{i=1}^{n} x_i^{w_i},$$

where $\boldsymbol{w} = (w_1,\dots,w_n) \in [0,1]^n$ denotes so called weighting vector fulfilling $\sum_{i=1}^n w_i = 1$, $i \in \{1,\dots,n\}$, $n \in N$.

For a symmetric weighting vector $\boldsymbol{w} = (\frac{1}{n},\dots,\frac{1}{n})$ we obtain the standard geometric mean G.

For the illustration in the case of $n = 2$ it is given by $G(x_1,x_2) = \sqrt{x_1 x_2}$, see Figure 1. For a non symmetric weighting vector $\boldsymbol{w} = (\frac{1}{4},\frac{3}{4})$ we have corresponding weighted geometric mean $G^W(x_1,x_2) = {x_1}^{\frac{1}{4}} {x_2}^{\frac{3}{4}}$ illustrated by Figure 2.

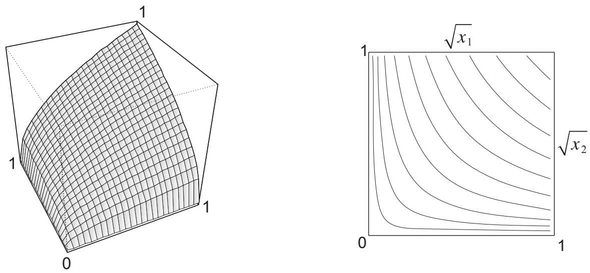

Fig. 1. 3D graph and contour plot of $G(x_1, x_2)$.

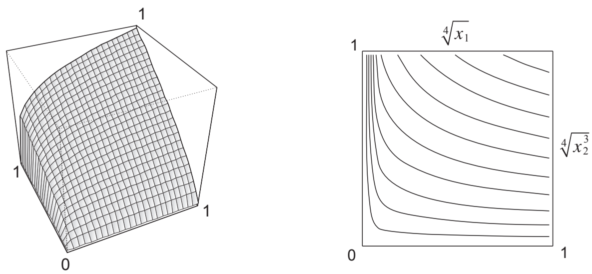

Fig. 2. 3D graph and contour plot of $G^W(x_1, x_2)$.

Example 3. Another class of homogeneous aggregation operators is the class of ordered weighted geometric means. Before applying an ordered weighted geometric mean $\bar{G}^W$, the input n-tuple $(x_1, \ldots, x_n)$ is first rearranged into a non-decreasing permutation $(\bar{x}_1, \ldots, \bar{x}_n)$ and then we have

$$\bar{G}^W(x_1, \ldots, x_n) = \prod_{i=1}^{n} \bar{x}_i^{w_i},$$

with corresponding weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$, $\sum_{i=1}^n w_i = 1$, $i \in \{1, \ldots, n\}$, $n \in N$. Having for $n = 2$ the same weighting vector $\boldsymbol{w} = (\frac{1}{4}, \frac{3}{4})$ we obtain an operator $\bar{G}^W$ different from G^W, see Figure 3.

4. HOMOGENEITY AND KERNEL PROPERTY

A similar open problem concerning the characterization of all shift-invariant aggregation operators has been solved in [10], where the close relation between shift-invariance of aggregation operators and the kernel property was shown. The kernel property has been recently introduced in [2, 9], see the next definition.

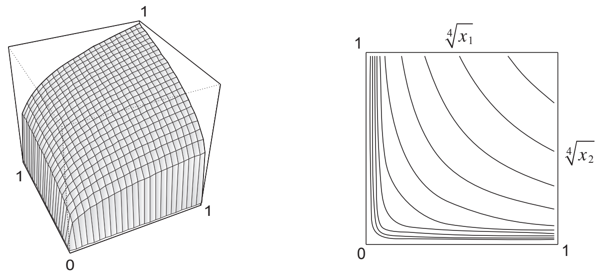

Fig. 3. 3D graph and contour plot of $\bar{G}^W(x_1, x_2)$.

Definition 3. An aggregation operator $A : \bigcup_{n\in N}[0,1]^n \to [0,1]$ is called a kernel aggregation operator if for all $n \in N$, $(x_1,\dots,x_n),(y_1,\dots,y_n) \in [0,1]^n$, it holds

$$|A(x_1,\dots,x_n) - A(y_1,\dots,y_n)| \le \max_k |x_k - y_k|. \tag{2}$$

This property of an aggregation operator A is linked to the Chebychev norm $\|A\|_\infty$ of an aggregation operator A, where

$$\|A\|_\infty = \sup\left\{\frac{|A(x_1,\dots,x_n) - A(y_1,\dots,y_n)|}{\max_k |x_k - y_k|}\right\},$$

for all $(x_1,\dots,x_n)$, $(y_1,\dots,y_n) \in [0,1]^n$, corresponding namely with the case $\|A\|_\infty = 1$.

Observe that the condition (1) for an aggregation operator C can be reformulated as

$$\frac{\min(C(\boldsymbol{x}),C(\boldsymbol{y}))}{\max(C(\boldsymbol{x}),C(\boldsymbol{y}))} \ge \min_k \frac{\min(x_k,y_k)}{\max(x_k,y_k)} \tag{3}$$

or, equivalently,

$$\left|\log\frac{C(\boldsymbol{x})}{C(\boldsymbol{y})}\right| \le \max_k \left|\log\frac{x_k}{y_k}\right|,$$

i. e.,

$$|\log C(x) - \log C(y)| \le \max_k |\log x_k - \log y_k|. \tag{4}$$

The property (4) of an aggregation operator C will be called log-kernel property and as far as the property (1) deals with one coordinate fixed in one, it can be denoted as one-log-kernel property. Let us define an operator $D : \bigcup_{n\in N}[0,\infty]^n \to [0,\infty]$ as follows:

$$D(u_1,\dots,u_n) = -\log(C(\exp(-u_1),\dots,\exp(-u_n))).$$

Then (4) is equivalent to the kernel property of D,

$$|D(u_1,\dots,u_n) - D(v_1,\dots,v_n)| \le \max_k |u_k - v_k|. \tag{5}$$

Thus we can use kernel aggregation operator D (it is sufficient to deal with so called zero-kernel property , fixing one coordinate to be zero) acting on $[0,\infty]$ to define a homogeneous aggregation operator H^C, where

$$C(x_1,\ldots,x_n)=\exp(D(-\log(x_1),\ldots,-\log(x_n))).$$

Moreover, if D is shift-invariant on $[0,\infty]$ then already C is homogeneous aggregation operator on $[0,1]$.

Example 4.

a) If D is a weighted arithmetic mean, i.e., $D(u_1,\ldots,u_n)=\sum_{i=1}^n w_i x_i$ then

$$C(x_1,\ldots,x_n)=\exp\Bigg(-\sum_{i=1}^n w_i-\log x_i\Bigg)=\prod_{i=1}^n x_i^{w_i}=H^C(x_1,\ldots,x_n)$$

is the weighted geometric mean. Similarly OWA operator D leads to $C=H^C$ which is ordered weighted geometric mean.

b) Let D be any Choquet-integral aggregation operator based on fuzzy measures. For example, for $n=2$,

$$D(u_1,u_2)=\begin{cases}\frac{1}{3}u_1+\frac{2}{3}u_2 & \text{if } u_1\le u_2,\\ \frac{3}{4}u_1+\frac{1}{4}u_2 & \text{otherwise.}\end{cases}$$

Then

$$C(x_1,x_2)=H^C(x_1,x_2)=\begin{cases}\sqrt[3]{x_1{x_2}^2} & \text{if } x_1\ge x_2,\\ \sqrt[4]{{x_1}^3x_2} & \text{otherwise,}\end{cases}$$

is a homogeneous binary aggregation operator.

c) Finally, consider $D(u_1,u_2)=\sqrt{1+|u_1-u_2|}+\min(u_1,u_2)-1$. Then $C=H^C$,

$$\begin{aligned}C(x_1,x_2)&=\exp\Big(1-\min(-\log x_1,-\log x_2)-\sqrt{1+|-\log x_1+\log x_2|}\Big)\\&=\max(x_1,x_2)\exp\Big(1-\sqrt{1+\big|\log\tfrac{x_2}{x_1}\big|}\Big).\end{aligned}$$

5. CONCLUSION

We have investigated homogeneous aggregation operators. Especially, we have characterized all aggregation operators leading to homogeneous aggregation operators by showing the relationship between homogeneity and kernel (exactly one-log-kernel) property of an aggregation operator. According to characterization of shift-invariant aggregation operators we have stressed the fact that it is sufficient to take into account shift-invariant aggregation operators acting on $[0,\infty]$ to obtain homogeneous aggregation operators. Several classes of aggregation operators as examples of classes of homogeneous aggregation operators were presented.

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Tatiana Rückschlossová and Roman Rückschloss, Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology, Radlinského 11, 813 68 Bratislava. Slovak Republic.

e-mails: tatiana@math.sk, romanr@math.sk