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AGGREGATION OPERATORS ON PARTIALLY ORDERED SETS AND THEIR CATEGORICAL FOUNDATIONS

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In spite of increasing studies and investigations in the field of aggregation operators, there are two fundamental problems remaining unsolved: aggregation of L-fuzzy set-theoretic notions and their justification. In order to solve these problems, we will formulate aggregation operators and their special types on partially ordered sets with universal bounds, and introduce their categories. Furthermore, we will show that there exists a strong connection between the category of aggregation operators on partially ordered sets with universal bounds (**Agop**) and the category of partially ordered groupoids with universal bounds (**Pogpu**). Moreover, the subcategories of **Agop** consisting of associative aggregation operators, symmetric and associative aggregation operators and associative aggregation operators with neutral elements are, respectively, isomorphic to the subcategories of **Pogpu** formed by partially ordered semigroups, commutative partially ordered semigroups and partially ordered monoids in the sense of Birkhoff. As a justification of the present notions and results, some relevant examples for aggregations operators on partially ordered sets are given. Particularly, aggregation process in probabilistic metric spaces is also considered.

Keywords: category theory, aggregation operator, associative aggregation operator, partially ordered groupoid, partially ordered semigroup, partially ordered monoid

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1. INTRODUCTION

Aggregation operators are an essential mathematical tool for the combination of several data as a single outcome. Although they have been studied intensively in the last years [4, 11, 15], there are still some fundamental questions remaining unanswered. Because of the fact that aggregation operators have been formulated based on a particular partially ordered set $([0, 1], \leq)$ (or more generally, $([a, b], \leq)$ for some $a, b \in \mathbb{R}$) [4, 11, 15], the present literature does not provide an aggregation process for L-fuzzy set theoretic notions that are playing a central role in the development of various branches of fuzzy mathematics [10], where L is a partial ordered set with universal bounds (posetu, for short) in general. This is one of the fundamental problems. In order to solve this problem, we will formulate aggregation operators based on a posetu L, and extend the algebraic properties of symmetry, associativity and the neutral element of aggregation operators to this setting. Furthermore, we design some relevant examples, and show that aggregation operators on posetus are sufficiently rich and abstract notions for unifying various kinds of aggregation operator under the same framework.

In addition to the above problem, since aggregation operators are some kinds of generalization of binary operation on L with some further properties, it is natural to ask the connections between aggregation operators and such binary operations on L. This question can also be stated for the connection between the subclass of aggregation operators consisting of symmetric aggregation operators (resp. associative aggregation operators, aggregation operators with neutral elements) and binary operations on L with some suitable algebraic properties. These questions lead us to the necessity for considering the category of aggregation operators on posetus (**Agop**) and its subcategories of associative aggregation operators (**Asagop**), of symmetric and associative aggregation operators (**Smasagop**), and of associative aggregation operators with neutral elements (or simply monoidal aggregation operators) (**Monagop**). If we take into consideration the category of partially ordered groupoids with universal bounds (**Pogpu**) and its subcategories of partially ordered semigroups (**Posmgu**), of commutative partially ordered semigroups (**Cmposmgu**), and of partially ordered monoids (**Pomonu**), then the second main problem can be abstracted to the categorical relation between **Agop** (resp. **Asagop**, **Smasagop** and **Monagop**) and **Pogpu** (resp. **Posmgu**, **Cmposmgu** and **Pomonu**). All of these categories are summarized in Table 1. As an answer to this problem, we will prove that **Pogpu** is isomorphic to an isomorphism-closed and full subcategory of **Agop**. Furthermore, **Asagop**, **Smasagop** and **Monagop** are, respectively, isomorphic to **Posmgu**, **Cmposmgu** and **Pomonu**. In addition to these categorical connections, we will also look at subobjects and fibres in **Agop** towards the end of this paper, and establish a meaningful definition of subaggregation operator as a special subobject in **Agop**.

2. AGGREGATION OPERATORS ON PARTIALLY ORDERED SETS

Let $\mathbf{L} = (L, \leq, \perp, \top)$ denote a posetu, i. e. $(L, \leq)$ is a partially ordered set (poset, for brevity) with the least element $\perp$ and the greatest element $\top$. The ordering structure on L can be coordinatwisely extended to L^n ($n \in \mathbb{N}^+$), i. e. for the relation $\leq$ on L^n, defined by

$$[(\alpha_1, \ldots, \alpha_n) \leq (\beta_1, \ldots, \beta_n)] \Leftrightarrow [(\forall\, i = 1, \ldots, n)(\alpha_i \leq \beta_i)],$$

and for the elements $\perp^{(n)} = (\perp, \ldots, \perp)$, $\top^{(n)} = (\top, \ldots, \top)$ of L^n, the four-tuple $\mathbf{L}_n = (L^n, \leq, \perp^{(n)}, \top^{(n)})$ forms a poset with universal bounds $\perp^{(n)}$ and $\top^{(n)}$.

Definition 1. A mapping $A : \bigcup_{n \in \mathbb{N}^+} L^n \to L$ is called an aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$ if the following conditions are fulfilled:

(AG.1) A preserves the order on L^n for all $n \in \mathbb{N}^+$, i. e.

$$[(\alpha_1, \ldots, \alpha_n) \leq (\beta_1, \ldots, \beta_n)] \Rightarrow [A(\alpha_1, \ldots, \alpha_n) \leq A(\beta_1, \ldots, \beta_n)]$$

(AG.2) A is the identity mapping id_L on L, i. e. $A(\alpha) = \alpha$ for all $\alpha \in L$,

(AG.3) A preserves universal bounds, i. e.

$$A(\perp^{(n)}) = \perp \text{ and } A(\top^{(n)}) = \top \text{ for all } n \in \mathbb{N}^+.$$

For $n \geq 2$, a mapping $B : L^n \to L$ is called an n -ary aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$ iff the conditions (AG.1) and (AG.3) are satisfied. A 1-ary aggregation operator $B : L \to L$ is the identity mapping id_L on L.

An aggregation operator A on $\mathbf{L}$ can be identified by a family of n-ary aggregation operators $\{A_n \mid n \in \mathbb{N}^+\}$. This means that for a given aggregation operator A on $\mathbf{L}$, we may associate a family of n-ary aggregation operators $\{A_n \mid n \in \mathbb{N}^+\}$ to A, which is defined by $A_n(\alpha_1, \dots, \alpha_n) = A(\alpha_1, \dots, \alpha_n)$. Conversely, if $\{A_n \mid n \in \mathbb{N}^+\}$ is a family of n-ary aggregation operators on $\mathbf{L}$, then we can define an aggregation operator A on $\mathbf{L}$ by $A(\alpha_1, \dots, \alpha_n) = A_n(\alpha_1, \dots, \alpha_n)$. It is obvious that the connection between the aggregation operators and the families of n -ary aggregation operators are bijective. Thus an aggregation operator and its associated family of n-ary aggregation operators $\{A_n \mid n \in \mathbb{N}^+\}$ can be conceived as the same things.

Aggregation operators on posetus in the sense of Definition 1 provide an abstract and a useful mathematical tool for unifying various kinds of aggregation operator under the same framework, and they also enable us to apply the notion of aggregation operator in several branch of mathematics. In order to clarify this, we design the following example:

Example 2. (a) For the particular posetu $\mathbf{I} = ([0, 1], \leq, 0, 1)$, an ($n$-ary) aggregation operator on $\mathbf{I}$ is called an (n-ary) aggregation operator in [4, 11, 15]. Thus an (n -ary) aggregation operator in the sense of Definition 1 is a straightforward extension of (n-ary) aggregation operator in the sense of [4, 11, 15] to a general posetu.

(b) Let $\mathcal{I}[0, 1]$ denote the set of all closed subintervals of $[0, 1]$, i. e. $\mathcal{I}[0, 1] = \{[a, b] \mid a, b \in [0, 1],\ a \leq b\}$. If we consider the partial ordering $\preceq_w$ on $\mathcal{I}[0, 1]$ (called the weak interval-ordering [12]) defined by

$$[a, b] \preceq_w [c, d] \Leftrightarrow ((a \leq c) \text{ and } (b \leq d)),$$

then $[0, 0]$ and $[1, 1]$ are obviously the least and greatest elements of the poset $(\mathcal{I}[0, 1], \preceq_w)$. An interval-valued aggregation operator (or an extended aggregation operator) in [12] is nothing else but an aggregation operator on $(\mathcal{I}[0, 1], \preceq_w, [0, 0], [1, 1])$.

(c) Consider the set $L^* = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}$. The relation $\leq_{L^*}$ on L^*, defined as

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow ((x_1 \leq y_1) \text{ and } (y_2 \leq x_2)),$$

is a partial ordering [6, 7]. $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$ are the least and greatest elements of the poset $(L^*, \leq_{L^*})$. An aggregation operator on $(L^*, \leq_{L^*}, 0_{L^*}, 1_{L^*})$ is called an aggregation operator on L^* in [6].

(d) Let $\mathcal{NC}[0,1]$ stand for the set of all normal convex fuzzy subsets of $[0,1]$ (i. e. a normal convex fuzzy subset μ of $[0,1]$ is a mapping $\mu : [0,1] \to [0,1]$ satisfying the conditions that $\bigvee_{x\in[0,1]} \mu(x) = 1$ and $\mu(x_1) \wedge \mu(x_3) \leq \mu(x_2)$ for all $x_1, x_2, x_3 \in [0,1]$ with $x_1 \leq x_2 \leq x_3$ [13]). Define the relation $\sqsubseteq$ on $\mathcal{NC}[0,1]$ by

$$\mu \sqsubseteq \nu \;\Leftrightarrow \mu \sqcap \nu = \mu,$$

where $\sqcap : \mathcal{NC}[0,1] \times \mathcal{NC}[0,1] \to \mathcal{NC}[0,1]$ is the extended minimum operation $\widetilde{\min}$ on $\mathcal{NC}[0,1]$ [8, 13], i. e.

$$[\sqcap(\mu,\nu)]\ (x) = \left[\widetilde{\min}(\mu,\nu)\right]\ (x) = \bigvee \{\mu(x_1) \wedge \nu(x_2) \mid x_1, x_2 \in [0,1],\ x = \min\{x_1, x_2\}\}.$$

Then $(\mathcal{NC}[0,1], \sqsubseteq)$ forms a poset [13] . Furthermore the fuzzy subsets $0_1, 1_1$ of $[0,1]$, defined by

$$0_1(x) = \left\{ \begin{array}{ll} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{array} \right\} \text{ and } 1_1(x) = \left\{ \begin{array}{ll} 1, & \text{if } x = 1 \\ 0, & \text{otherwise,} \end{array} \right\}$$

are the least and greatest elements of $(\mathcal{NC}[0,1], \sqsubseteq)$. We will show in Remark 3 that aggregation operators on $(\mathcal{NC}[0,1], \sqsubseteq, 0_1, 1_1)$ enable us to aggregate type 2 fuzzy sets with normal convex fuzzy grades.

(e) A non-decreasing function $F : [-\infty, \infty] \to [0,1]$, satisfying the properties that F is left-continuous on $(0, \infty)$, $F(0) = 0$ and $F(\infty) = 1$, is called a distance distribution function on $[-\infty, \infty]$ [11, 16]. The set of all distance distribution functions on $[-\infty, \infty]$ is denoted by Δ^+, and is an essential part of probabilistic metric spaces [11, 16]. The usual ordering $\leq$ on $[-\infty, \infty]$ can be carried to Δ^+ pointwisely, i. e. $F \leq G$ iff $F(x) \leq G(x)$ for all $x \in [-\infty, \infty]$ and $F, G \in \Delta^+$. Then the Dirac distributions $\varepsilon_\infty, \varepsilon_0 \in \Delta^+$, defined by

$$\varepsilon_\infty(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \in [-\infty, \infty) \\ 1, & \text{if } x = \infty \end{array} \right\} \text{ and } \varepsilon_0(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \in [-\infty, 0] \\ 1, & \text{if } x \in (0, \infty] \end{array} \right\},$$

are the least and greatest elements of the poset $(\Delta^+, \leq)$ [11]. Aggregation operators on $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$ provide a natural way for combining a finite number of distance distribution functions as a single one. Furthermore we will later expose that they can be used for aggregating of probabilistic metrics (see explanations just after Theorem 9 at the end of this section).

For a posetu $\mathbf{L} = (L, \leq, \perp, \top)$, let L^X stand for the set of all L-valued mappings $\mu : X \to L$ on X (L-fuzzy sets of X), and $\mathbf{1}_\emptyset, \mathbf{1}_X$ denote the characteristic functions of $\emptyset$ and X, i. e. $\mathbf{1}_\emptyset(x) = \perp$ and $\mathbf{1}_\emptyset(x) = \top$ for all $x \in X$ [10]. Consider the pointwise extension of $\leq$ to L^X, i. e. $\mu \leq \nu$ iff $\mu(x) \leq \nu(x)$ for all $x \in X$. Then the quadruple $(L^X, \leq, \mathbf{1}_\emptyset, \mathbf{1}_X)$ obviously forms a poset with the least element $\mathbf{1}_\emptyset$ and the greatest element $\mathbf{1}_X$. Therewith an aggregation operator A on $\mathbf{L}$ can be extended to an aggregation operator A_X on $(L^X, \leq, \mathbf{1}_\emptyset, \mathbf{1}_X)$ by the equality

$$[A_X(\mu_1, \ldots, \mu_n)]\,(x) = A(\mu_1(x), \ldots, \mu_n(x)). \tag{1}$$

Remark 3. For the particular choice of the underlying posetu $\mathbf{L} = (L, \leq, \perp, \top)$ in Example 2 (a) (resp. (b), (c) and (d)), an L-fuzzy set of X is named as a fuzzy set [17] (resp. an interval-valued fuzzy set [14], an intuitionistic fuzzy set [2] and a type 2 fuzzy set with normal convex fuzzy grades [13]). In other words, L^X is nothing but the set of all fuzzy sets (resp. interval-valued fuzzy sets, intuitionistic fuzzy sets, type 2 fuzzy sets with normal convex fuzzy grades) of X depending on the particular choice of $\mathbf{L} = (L, \leq, \perp, \top)$ in Example 2 (a–d). Therefore starting with an aggregation operator on the special poset stated in Example 2 (a) (resp. (b), (c) and (d)), and by making use of the equality (1), we can construct an aggregation operator on the posetu of fuzzy sets (resp. interval-valued fuzzy sets, intuitionistic fuzzy sets and type 2 fuzzy sets with normal convex fuzzy grades).

Aggregation operators have some important properties leading to useful subclasses as introduced below.

Definition 4. Let A be an aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$.

(i) A is said to be associative iff

$$A(\alpha_1, \ldots, \alpha_k, \ldots \alpha_n) = A_2(A_k(\alpha_1, \ldots, \alpha_k), A_{n-k}(\alpha_{k+1}, \ldots, \alpha_n))$$

for all $n \geq 2$, $k = 1, \ldots, n-1$ and $\alpha_i \in L$ $(i = 1, \ldots, n)$.

(ii) A is said to be symmetric iff

$$A(\alpha_1, \ldots, \alpha_n) = A(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)})$$

for all $n \in \mathbb{N}^+$, $\alpha_i \in L$ $(i = 1, \ldots, n)$ and for all permutations $\pi(1),\ldots,\pi(n)$ of $\{1, \ldots, n\}$.

(iii) A has the neutral element $e \in L$ iff for all $n \geq 2$ and $\alpha_i \in L$ $(i = 1, \ldots, n)$, if $\alpha_k = e$ for some $k \in \{1, \ldots, n\}$, then

$$A(\alpha_1, \ldots, \alpha_n) = A(\alpha_1, \ldots, \alpha_{k-1}, \alpha_{k+1}, \ldots, \alpha_n).$$

(iv) An associative aggregation operator with the neutral element is called a monoidal aggregation operator.

It should be noted here that Definition 4 (i), (ii) and (iii) are straightforward generalization of associative aggregation operators, symmetric aggregation operators and aggregation operators with neutral elements in [4, 11] to the present approach.

Aggregation operators with neutral elements are a generalization of binary operations with identity elements, and they provide some bounds for aggregation operators as follows:

Proposition 5. Let A be an aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$ with the neutral element $e \in L$. Then for the binary operation $\otimes_A = A_2$ on L, e is the identity element of the groupoid $(L, \otimes_A)$, i. e. $e \otimes_A \alpha = \alpha \otimes_A e = \alpha$ for all $\alpha \in L$. Furthermore, if the underlying poset $(L, \leq)$ is a lattice, and if we denote the meet and join operations on it by the usual notations $\bigwedge$ and $\bigvee$ respectively, then for the case $e = \top$ $(e = \perp)$, for all $n \geq 2$ and $\alpha_i \in L$, $i = 1, \ldots, n$, $A(\alpha_1, \ldots, \alpha_n)$ is bounded above (below) by $\bigwedge_{i=1}^n \alpha_i$ $(\bigvee_{i=1}^n \alpha_i)$, i. e.

$$A(\alpha_1, \ldots, \alpha_n) \leq \bigwedge_{i=1}^n \alpha_i \quad \left(\bigvee_{i=1}^n \alpha_i \leq A(\alpha_1, \ldots, \alpha_n) \right).$$

Proof. The first part of proposition is clear from Definition 4 (iii). To see the remaining part of proposition, assume that $(L, \leq)$ is a lattice. For the case $e = \top$, using the condition (AG.1) together with Definition 4 (iii), we easily observe that for all $n \geq 2$, $i = 1, \ldots, n$ and $\alpha_i \in L$,

$$A(\alpha_1, \ldots, \alpha_n) \leq A(\top, \ldots, \top, \alpha_i, \top, \ldots, \top) = \alpha_i, \text{ i. e. } A(\alpha_1, \ldots, \alpha_n) \leq \bigwedge_{i=1}^n \alpha_i.$$

If $e = \perp$, we similarly see that

$$\alpha_i = A(\perp, \ldots, \perp, \alpha_i, \perp, \ldots, \perp) \leq A(\alpha_1, \ldots, \alpha_n), \text{ i. e. } \bigvee_{i=1}^n \alpha_i \leq A(\alpha_1, \ldots, \alpha_n). \quad \square$$

In this section, we will demonstrate the connections between aggregation operators and partially ordered groupoids in the sense of Birkhoff [3]. For this purpose, it is useful to recall the definition of partially ordered groupoid and the relevant notions in [3] at first:

Definition 6. [3] Let $(L, \leq)$ be a poset, and $\otimes$ a binary operation on L.

(i) A triple $(L, \leq, \otimes)$ is called a partially ordered groupoid, in brevity po-groupoid iff $\alpha \leq \beta$ implies $\alpha \otimes \gamma \leq \beta \otimes \gamma$ and $\gamma \otimes \alpha \leq \gamma \otimes \beta$ for all $\alpha, \beta, \gamma \in L$.

(ii) A po-groupoid $(L, \leq, \otimes)$ is called a partially ordered semigroup, in short po-semigroup iff $(L, \otimes)$ forms a semigroup.

(iii) A po-semigroup $(L, \leq, \otimes)$ is called a partially ordered monoid, in short po-monoid iff there exist an element $e \in L$ (called the identity of $(L, \otimes)$) such that $(L, \otimes, e)$ forms a monoid, i. e. $e \otimes \alpha = \alpha \otimes e = \alpha$ for all $\alpha \in L$.

(iv) A po-groupoid (semigroup, monoid) $(L, \leq, \otimes)$ is said to be commutative iff $\otimes$ is commutative.

If the poset $(L, \leq)$ in Definition 6 has the least element $\perp$ and the greatest element $\top$, then for the sake of simplicity, we denote a po-groupoid (a po-semigroup or a po-monoid) $(L, \leq, \otimes)$ by the notation $(L, \leq, \otimes, \perp, \top)$. Considering Definition 1 and Definition 6, one can easily note that a binary operation $\otimes$ on L is a 2 -ary aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$ iff $(L, \leq, \otimes, \perp, \top)$ is a po-groupoid such that $\perp \otimes \perp = \perp$ and $\top \otimes \top = \top$. If $(L, \leq, \otimes, \perp, \top)$ is a po-monoid, then the equalities $\perp \otimes \perp = \perp$ and $\top \otimes \top = \top$ are obviously satisfied, so $\otimes$ is a 2-ary aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$. For some particular choices of $\mathbf{L} = (L, \leq, \perp, \top)$, the operation $\otimes$ in a commutative po-monoid $(L, \leq, \otimes, \perp, \top)$ has a special interest in various fields:

Example 7.

(a) If $([0, 1], \leq, \otimes, 0, 1)$ is a commutative po-monoid with the identity $e \in [0, 1]$, then the operation $\otimes$ is called a uninorm [9]. For $e = 1$ ($e = 0$), the uninorm $\otimes$ is also called a t-norm (t-conorm) [11, 16].

(b) Consider the posetu $(L^*, \leq_{L^*}, 0_{L^*}, 1_{L^*})$ defined in Example 2 (c). In case $(L^*, \leq_{L^*}, \otimes, 0_{L^*}, 1_{L^*})$ is a commutative po-monoid with the identity $e \in L^*$ (resp. $e = 1_{L^*}$, $e = 0_{L^*}$), $\otimes$ is named as a uninorm (resp. t-norm, t-conorm) on L^* [7].

(c) In an analogous manner to (b), if $(\mathcal{I}[0, 1], \preceq_w, \otimes, [0, 0], [1, 1])$ is a commutative po-monoid with the identity, then we call the binary operation $\otimes$ a uninorm on $\mathcal{I}[0, 1]$. Since the mapping $[a, b] \mapsto (a, 1 - b)$ is an isomorphism between $(\mathcal{I}[0, 1], \preceq_w, [0, 0], [1, 1])$ and $(L^*, \leq_{L^*}, 0_{L^*}, 1_{L^*})$, it is clear that $\otimes$ is a uninorm on $\mathcal{I}[0, 1]$ iff the binary operation $\oplus$ on L^*, defined by

$$(x_1, x_2) \oplus (y_1, y_2) = (z_1, z_2) \Leftrightarrow [x_1, 1 - x_2] \otimes [y_1, 1 - y_2] = [z_1, 1 - z_2],$$

is a uninorm on L^*.

(d) Let us recall the posetu $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$ stated in Example 2 (d). If $(\Delta^+, \leq, \tau, \varepsilon_\infty, \varepsilon_0)$ is a commutative po-monoid with the identity ε_0, then the binary operation τ is known as a triangle function [11, 16].

In the following theorem, we now show that for a given po-groupoid, we may construct an aggregation operator, and vice versa.

Theorem 8. If $(L, \leq, \otimes, \perp, \top)$ is a po-groupoid provided that $\perp \otimes \perp = \perp$ and $\top \otimes \top = \top$, then the mapping $A^{\otimes} : \bigcup_{n \in \mathbb{N}^+} L^n \to L$, defined by

$$A^{\otimes}(\alpha_1, \ldots, \alpha_n) = (\ldots(\alpha_1 \otimes \alpha_2) \otimes \alpha_3 \ldots) \otimes \alpha_n \tag{2}$$

for all $n \geq 2$, and $A^{\otimes}(\alpha) = \alpha$ for $n = 1$, is an aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$.

Conversely, if A is an aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$, then for the binary operation $\otimes_A = A_2$, the five-tuple $(L, \leq, \otimes_A, \perp, \top)$ forms a po-groupoid with the property that $\perp \otimes \perp = \perp$ and $\top \otimes \top = \top$. Furthermore $\otimes_{A^{\otimes}} = \otimes$.

P r o o f . Let $(L, \leq, \otimes, \perp, \top)$ be a po-groupoid such that $\perp \otimes \perp = \perp$ and $\top \otimes \top = \top$. From the definition of the mapping $A^{\otimes}$, and by using the assumption that $\perp \otimes \perp = \perp$ and $\top \otimes \top = \top$, $A^{\otimes}$ obviously satisfies the properties (AG.2) and (AG.3). For $n \geq 2$ and $i = 1, \ldots, n$, let us pick α_i, $\beta_i \in L$ with $\alpha_i \leq \beta_i$. To verify (AG.3), we apply induction on n, and show the inequality

$$A^{\otimes}(\alpha_1, \ldots, \alpha_n) \leq A^{\otimes}(\beta_1, \ldots, \beta_n). \tag{3}$$

For $n = 2$, we have

$$\alpha_1 \otimes \alpha_2 \leq \beta_1 \otimes \alpha_2 \leq \beta_1 \otimes \beta_2,$$

and so (3) is true. Suppose that (3) is true for $n - 1$, i. e.

$$A^{\otimes}(\alpha_1, \ldots, \alpha_{n-1}) \leq A^{\otimes}(\beta_1, \ldots, \beta_{n-1}).$$

Then

$$\begin{aligned} A^{\otimes}(\alpha_1, \ldots, \alpha_n) &= A^{\otimes}(\alpha_1, \ldots, \alpha_{n-1}) \otimes \alpha_n \leq A^{\otimes}(\beta_1, \ldots, \beta_{n-1}) \otimes \alpha_n \\ &\leq A^{\otimes}(\beta_1, \ldots, \beta_{n-1}) \otimes \beta_n = A^{\otimes}(\beta_1, \ldots, \beta_n). \end{aligned}$$

The converse part of theorem can be easily seen by using the definitions of aggregation operator and po-groupoid, so it is skipped here. □

Theorem 8 establishes a connection between po-groupoids and aggregation operators. Note here that Theorem 8 does not guarantee the equality $A = A^{\otimes_A}$. This means that this connection is not bijective in general. Nevertheless, we may establish a bijective connection between some special po-groupoids and some special aggregation operators:

Theorem 9. If $(L, \leq, \otimes, \perp, \top)$ is a po-semigroup (resp. a commutative po-semigroup, a po-monoid) with the property that $\perp \otimes \perp = \perp$ and $\top \otimes \top = \top$, then the mapping $A^{\otimes} : \bigcup_{n \in \mathbb{N}^+} L^n \to L$, defined by

$$A^{\otimes}(\alpha_1, \ldots, \alpha_n) = \alpha_1 \otimes \alpha_2 \otimes \alpha_3 \otimes \ldots \otimes \alpha_n \tag{4}$$

for all $n \geq 2$, and $A^{\otimes}(\alpha) = \alpha$ for $n = 1$, is an associative (resp. a symmetric and an associative, a monoidal) aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$.

Conversely, if A is an associative (resp. a symmetric and associative, a monoidal) aggregation operator on $\mathbf{L} = (L, \leq, \perp, \top)$, then for the binary operation $\otimes_A = A_2$, the five-tuple $(L, \leq, \otimes_A, \perp, \top)$ forms a po-semigroup (resp. a commutative po-semigroup, a po-monoid) with the property that $\perp \otimes_A \perp = \perp$ and $\top \otimes_A \top = \top$. Moreover $\otimes_{A^{\otimes}} = \otimes$ and $A = A^{\otimes_A}$.

P r o o f . Let $(L, \leq, \otimes, \perp, \top)$ be a po-semigroup provided that $\perp \otimes \perp = \perp$ and $\top \otimes \top = \top$. Since $\otimes$ is an associative binary operation, the mapping $A^{\otimes} : \bigcup_{n \in \mathbb{N}^+} L^n \to L$, given by the equality (2), can be written in the form (4). Thus we have from Theorem 8 that the mapping $A^{\otimes} : \bigcup_{n \in \mathbb{N}^+} L^n \to L$, defined by the equality (4), forms an aggregation operator on $\mathbf{L} = (L, \leq, \bot, \top)$. Then considering the fact that $(A^{\otimes})_2 = \otimes$, and by making use of the associativity of $\otimes$, we may write

$$\begin{aligned} A^{\otimes}(\alpha_1, \ldots, \alpha_k, \ldots \alpha_n) &= (\alpha_1 \otimes \alpha_2 \otimes \ldots \otimes \alpha_k) \otimes (\alpha_{k+1} \otimes \ldots \otimes \alpha_n) \\ &= (A^{\otimes})_k(\alpha_1, \ldots, \alpha_k) \otimes (A^{\otimes})_{n-k}(\alpha_{k+1}, \ldots, \alpha_n) \\ &= (A^{\otimes})_2((A^{\otimes})_k(\alpha_1, \ldots, \alpha_k), (A^{\otimes})_{n-k}(\alpha_{k+1}, \ldots, \alpha_n)), \end{aligned}$$

and therefore the associativity of $A^{\otimes}$ is achieved. Because of the fact that commutativity of $\otimes$ obviously implies the symmetry of $A^{\otimes}$, if $(L, \leq, \otimes, \bot, \top)$ is a commutative po-semigroup, then $A^{\otimes}$ will be symmetric and associative. Furthermore, it is easy to see that if e is the identity element of the monoid $(L, \otimes)$, then $A^{\otimes}$ has the neutral element e. Thus if $(L, \leq, \otimes, \bot, \top)$ is a po-monoid, then $A^{\otimes}$ will be a monoidal aggregation operator.

In order to prove the converse part of theorem, let us assume that A is an associative aggregation operator on $\mathbf{L} = (L, \leq, \bot, \top)$. Then by considering the definition $\otimes_A = A_2$, we obtain from the associativity of A and the property (AG.2) that for all $\alpha_1, \alpha_2, \alpha_3 \in L$,

$$\begin{aligned} (\alpha_1 \otimes_A \alpha_2) \otimes_A \alpha_3 &= A_2(A_2(\alpha_1, \alpha_2), A_1(\alpha_3)) \\ &= A(\alpha_1, \alpha_2, \alpha_3) = A_2(A_1(\alpha_1), A_2(\alpha_2, \alpha_3)) \\ &= \alpha_1 \otimes_A (\alpha_2 \otimes_A \alpha_3), \end{aligned}$$

i. e. $(L, \otimes_A)$ is a semigroup, and therefore it follows from Theorem 8 that $(L, \leq, \otimes_A, \bot, \top)$ forms a po-semigroup with the property that $\bot \otimes_A \bot = \bot$ and $\top \otimes_A \top = \top$. Since the symmetry of A implies the commutativity of $\otimes_A$, it is clear that if A is symmetric and associative, then $(L, \leq, \otimes_A, \bot, \top)$ will be a commutative po-semigroup. Furthermore, if A is a monoidal aggregation operator, i. e. A has the neutral element e, then since e will be the identity element of the monoid $(L, \otimes)$, $(L, \leq, \otimes_A, \bot, \top)$ is obviously a po-monoid. To complete the proof, we now prove the equality $A = A^{\otimes_A}$, i. e.

$$A(\alpha_1, \ldots, \alpha_n) = \alpha_1 \otimes_A \alpha_2 \otimes_A \alpha_3 \otimes_A \ldots \otimes_A \alpha_n. \tag{5}$$

For the confirmation of (5), we use induction on n. For $n = 2$, the equality (5) is clear. Assume that (5) is true for $n - 1$, i. e.

$$A(\alpha_1, \ldots, \alpha_{n-1}) = \alpha_1 \otimes_A \alpha_2 \otimes_A \alpha_3 \otimes_A \ldots \otimes_A \alpha_{n-1}.$$

Then since $A(\alpha_1, \ldots, \alpha_{n-1}) = A_n(\alpha_1, \ldots, \alpha_{n-1})$, and by making use of the associativity of A, we observe that

$$\begin{aligned} A(\alpha_1, \ldots, \alpha_n) &= A_2(A_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}), A_1(\alpha_n)) \\ &= A_{n-1}(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \otimes_A \alpha_n \\ &= A(\alpha_1, \ldots, \alpha_{n-1}) \otimes_A \alpha_n \\ &= \alpha_1 \otimes_A \alpha_2 \otimes_A \alpha_3 \otimes_A \ldots \otimes_A \alpha_n. \end{aligned}$$

$\square$

As a direct application of Theorem 9, starting with a uninorm in Example 7 (a–c), one can easily construct a symmetric and monoidal aggregation operator on the corresponding posetu considered in Example 2 (a–c). Similarly, given a triangle function τ , ,we can put the aggregation operator A^{τ} on $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$. Now let us recall the notion of probabilistic metric space [11, 16]: A probabilistic metric space is a triple $(X, \mathcal{F}, \tau)$ such that X is a nonempty set, τ is a triangle function and $\mathcal{F} : X^2 \to \Delta^+$ is a mapping provided with the following properties:

(PM1) $\quad \mathcal{F}(x, x) = \varepsilon_0,$

(PM2) $\quad \mathcal{F}(x, y) = \varepsilon_0 \Rightarrow x = y$,

(PM3) $\quad \mathcal{F}(x, y) = \mathcal{F}(y, x),$

(PM4) $\quad \mathcal{F}(x, y)\tau\mathcal{F}(y, z) \leq \mathcal{F}(x, z)$

for all $x, y, z \in X$. If the mapping $\mathcal{F} : X^2 \to \Delta^+$ satisfies the conditions (PM1), (PM3) and (PM4), then the triple $(X, \mathcal{F}, \tau)$ is called a probabilistic pseudometric space [16]. For fixed X and τ, we call the mapping $\mathcal{F} : X^2 \to \Delta^+$ a probabilistic metric (pseudometric) on X w.r.t. τ iff $(X, \mathcal{F}, \tau)$ forms a probabilistic metric (pseudometric) space, and denote by $PM(X, \tau)$ $(PPM(X, \tau))$ the set of all probabilistic metrics (pseudometrics) on X w.r.t. τ. Define the mappings $\mathcal{F}_s, \mathcal{F}_g : X^2 \to \Delta^+$ by

$$\mathcal{F}_s(x, y) = \left\{ \begin{array}{ll} \varepsilon_0, & \text{if } x = y \\ \varepsilon_\infty, & \text{otherwise} \end{array} \right\} \text{ and } \mathcal{F}_g(x, y) = \varepsilon_0,$$

and consider the pointwise ordering $\leq$ on $PPM(X, \tau)$, i. e. for $\mathcal{F}, \mathcal{G} \in PPM(X, \tau)$ we have $\mathcal{F} \leq \mathcal{G}$ iff $\mathcal{F}(x, y) \leq \mathcal{G}(x, y)$ for all $x, y \in X$. It is clear that $\mathcal{F}_s, \mathcal{F}_g$ are the least and greatest elements of the poset $(PPM(X, \tau), \leq)$. Furthermore, it is not difficult to see that the aggregation operator A^{τ} on $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$ can be extended to an aggregation operator $P_{(X,\tau)}$ on $(PPM(X, \tau), \leq, \mathcal{F}_s, \mathcal{F}_g)$ defined by

$$\left[P_{(X,\tau)}\left(\mathcal{F}_1, \ldots, \mathcal{F}_n\right)\right](x, y) = A^{\tau}\left(\mathcal{F}_1(x, y), \ldots, \mathcal{F}_n(x, y)\right).$$

If we are interested in probabilistic metrics, then it is easy to observe that the poset $(PM(X, \tau), \leq)$ has the least element $\mathcal{F}_s$, but not the greatest element. In this case, although the poset $(PM(X, \tau), \leq)$ is inadequate to define aggregation operator on it, the restriction of the mapping $P_{(X,\tau)} : \bigcup_{n\in\mathbb{N}^+}^{n} [PPM(X, \tau)]^n \to PPM(X, \tau)$ to $\bigcup_{n\in\mathbb{N}^+}^{n} [PM(X, \tau)]^n$, denoted by $P^*_{(X,\tau)}$, defines a mapping from $\bigcup_{n\in\mathbb{N}^+}^{n} [PM(X, \tau)]^n$ from $PM(X, \tau)$. Furthermore, $P^*_{(X,\tau)}$ holds the conditions (AG1), (AG2) and the property $P^*_{(X,\tau)}\left(\mathcal{F}_s, \ldots, \mathcal{F}_s\right) = \mathcal{F}_s$. This means that eventhough $P^*_{(X,\tau)}$ is not an aggregation operator, $P^*_{(X,\tau)}$ is still a reasonable tool for combining a finite number of probabilistic metrics as a singe one.

3. CATEGORY OF AGGREGATION OPERATORS

In the previous section, we established the connections between aggregation operators and po-groupoids. In this section, we will formulate these connections in the formalism of category theory. For the background materials in the category theory (categories, morphisms, functors, subcategories, etc.), we refer to [1], and use the same notations and terminology in [1]. We start with the definition of the category of aggregation operators:

Definition 10. (The category of aggregation operators (**Agop**)) **Agop** consists of the following items:

(i) Objects: Each object is an ordered pair $(\mathbf{L}, A)$, where A is an aggregation operator on $\mathbf{L} = (L, \leq_L, \perp_L, \top_L)$.

(ii) Morphisms: A morphism $(\mathbf{L}, A) \xrightarrow{f} (\mathbf{K}, B)$ is an order-preserving function $f : L \to K$ such that $f(\perp_L) = \perp_K$, $f(\top_L) = \top_K$ and the following diagram commutes for all $n \in \mathbb{N}^+$:

$$\begin{tikzcd} L^n \arrow[r, "f^n"] \arrow[d, "A_n"'] & K^n \arrow[d, "B_n"] \\ L \arrow[r, "f"] & K \end{tikzcd}$$

i. e. $f \circ A_n = B_n \circ f^n$ for all $n \in \mathbb{N}^+$. Here the function $f^n : L^n \to K^n$ is defined by $f^n(\alpha_1, \ldots, \alpha_n) = (f(\alpha_1), \ldots, f(\alpha_n))$.

(iii) The composition of two morphisms $(\mathbf{L}, A) \xrightarrow{f} (\mathbf{K}, B)$ and $(\mathbf{K}, B) \xrightarrow{g} (\mathbf{M}, C)$ is the morphism $(\mathbf{L}, A) \xrightarrow{g \circ f} (\mathbf{M}, C)$, where $g \circ f : L \to M$ is the usual composition of the functions g and f as in the category of sets and functions (**Set**) [1]. The identity morphism $(\mathbf{L}, A) \xrightarrow{id_L} (\mathbf{L}, A)$ of $(\mathbf{L}, A)$ is the identity function id_L of L as in **Set.**

For a subfamily $\Omega = (n_i)_{i \in J}$ of $\mathbb{N}^+$, for a set X and for mappings $A_i : X^{n_i} \to X$ $(i \in J)$, the ordered pair $(X, (A_i)_{i \in J})$ is called an Ω-algebra [1]. Given two Ω-algebras $(X, (A_i)_{i \in J})$ and $(Y, (B_i)_{i \in J})$, a function $f : X \to Y$ is called an Ω -homomorphism iff the diagram

$$\begin{tikzcd} X^{n_i} \arrow[r, "f^{n_i}"] \arrow[d, "A_i"'] & Y^{n_i} \arrow[d, "B_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes, i. e. $f \circ A_i = B_i \circ f^{n_i}$ for all $i \in J$, where $f^{n_i} : X^{n_i} \to Y^{n_i}$ is defined by $f^{n_i}(x_1, \ldots, x_{n_i}) = (f(x_1), \ldots, f(x_{n_i}))$. The category of Ω-algebras **Alg(**Ω**)** comprises all Ω -algebras and Ω-homomorphisms [1]. The composition and identities in **Alg(**Ω**)** are the same as in **Set** . If we take into account the functor U :**Agop**$\to$ **Alg(**$\mathbb{N}^+$**)** defined by

$$U((\mathbf{L}, A) \xrightarrow{f} (\mathbf{K}, B)) = (L, (A_n)_{n \in \mathbb{N}^+}) \xrightarrow{f} (K, (B_n)_{n \in \mathbb{N}^+}),$$

it is easy to note that (**Agop**,U) is a concrete category over **Alg(**$\mathbb{N}^+$**)**. On the other hand, if we recall the category **Pos** of posets and order-preserving functions [1], and define the forgetful functor V :**Agop**→**Pos** by

$$V((\mathbf{L},A)\xrightarrow{f}(\mathbf{K},B))=(L,\leq_L)\xrightarrow{f}(K,\leq_K),$$

then we also see that (**Agop**,V) is a concrete category over **Pos**.

In order to point out the categorical relations between aggregation operators and po-groupoids, we now introduce the category of po-groupoids with universal bounds:

Definition 11. (The category of po-groupoids with universal bounds (**Pogpu**)) The category **Pogpu** comprises the following data:

(i) Objects: Po-groupoids $(L,\leq_L,\otimes_L,\perp_L,\top_L)$ satisfying the properties $\perp_L\otimes\perp_L=\perp_L$ and $\top_L\otimes\top_L=\top_L$.

(ii) Morphisms: All **Set** morphisms, between the objects of **Pogpu**, which preserve $\leq$, $\otimes$, $\perp$ and $\top$, i. e. a **Pogpu**-morphism $(L,\leq_L,\otimes_L,\perp_L,\top_L)\xrightarrow{f}(K,\leq_K,\otimes_K,\perp_K,\top_K)$ is a function $f:L\to K$ satisfying the following conditions:

(a) $\alpha\leq_L\beta\Rightarrow f(\alpha)\leq_K f(\beta),\ \forall\,\alpha,\beta\in L,$

(b) $f(\alpha\otimes_L\beta)=f(\alpha)\otimes_K f(\beta),\ \forall\,\alpha,\beta\in L,$

(c) $f(\perp_L)=\perp_K$ and $f(\top_L)=\top_K$.

(iii) Composition and identities: As in **Set**.

As stated in Definition 6, we may consider po-groupoids on posets that do not necessarily have universal bounds. Thus one may also define a category of po-groupoids whose underlying posets may not have universal bounds (**Pogp**). Although the category **Pogpu** can also be introduced as a subcategory of the category **Pogp** , **Pogp** has no use for establishing the categorical connections between aggregation operators and po-groupoids. For this reason, we are not interested in the category **Pogp**. The letter "u" at the end of the abbreviation **Pogpu** emphasizes the existence of universal bounds in the underlying posets of po-groupoids. There exists an important categorical connection between **Pogpu** and **Agop**:

Theorem 12. **Pogpu** is isomorphic to an isomorphism-closed full subcategory of **Agop.**

P r o o f . Using Theorem 8, it is not difficult to see that the mapping F : **Pogpu**→**Agop**, defined by

$$\begin{aligned}F((L,\ \leq\ &_L,\otimes_L,\perp_L,\top_L)\xrightarrow{f}(K,\leq_K,\otimes_K,\perp_K,\top_K))\\ =\ &(\mathbf{L},A^{\otimes_L})\xrightarrow{f}(\mathbf{K},A^{\otimes_K}),\end{aligned}\qquad(6)$$

where $\mathbf{L} = (L, \leq_L, \perp_L, \top_L)$, $\mathbf{K} = (K, \leq_K, \perp_K, \top_K)$, and $A^{\otimes_L}$, $A^{\otimes_K}$ are given by the equality (2) in Theorem 8, is a full and faithful functor. Thus, the image F(**Pogpu**) of **Pogpu** under F forms a full subcategory of **Agop**. Furthermore since F is injective on objects of **Pogpu**, the functor F : **Pogpu**$\rightarrow$ F(**Pogpu**) is obviously an isomorphism, i. e. **Pogpu** is isomorphic to F(**Pogpu**). On the other hand, one can also easily show that F(**Pogpu**) is an isomorphism-closed full subcategory of **Agop.** $\square$

Theorem 12 shows that **Pogpu** can be fully embedded into **Agop**, so aggregation operators are some kinds of generalization of po-groupoid. Let us denote the full subcategories of **Agop** consisting of associative aggregation operators, symmetric and associative aggregation operators by **Asagop** and **Smasagop**, respectively. Now if we consider the full subcategory of **Pogpu** formed by po-semigroups (**Posmgu**) and the full subcategory of **Pogpu** formed by commutative po-semigroups (**Cmposmgu**), then we will prove in the following theorem that **Asagop (Smasagop)** is isomorphic to **Posmgu (Cmposmgu).**

Theorem 13. The categories **Asagop** and **Smasagop** are, respectively, isomorphic to the categories **Posmgu**, **Cmposmgu**.

P r o o f. If we reconsider the functor F : **Posmgu**$\rightarrow$ **Asagop** (F : **Cmposmgu**$\rightarrow$**Smasagop**) defined by the rule (6) in Theorem 12, then by virtue of Theorem 9, F will be an isomorphism. Hence the assertion follows. $\square$

We will conclude this section by establishing an isomorphism between the category of monoidal aggregation operators and the category of po-monoids with universal bounds. This will also justify why an associative aggregation operator with the neutral element is called a monoidal aggregation operator. For this purpose, we first give explicit definitions of the categories of monoidal aggregation operators, and of po-monoids with universal bounds:

Definition 14. (i) The category of monoidal aggregation operators (**Monagop**) is defined by the following items: Each object of **Monagop** is an **Asagop**-object $(\mathbf{L}, A)$ such that A has the neutral element $e_A \in L$. A morphism of **Monagop** is an **Agop**-morphism $(\mathbf{L}, A) \xrightarrow{f} (\mathbf{K}, B)$ preserving neutral elements, i. e. $f(e_A) = e_B$. As in **Agop**, composition and identities in **Monagop** are taken from **Set**.

(ii) The category of po-monoids with universal bounds (**Pomonu**) consists of the following items: A **Pomonu**-object is a po-monoid $(L, \leq_L, \otimes_L, \perp_L, \top_L)$ with the identity element $e_L \in L$. A **Pomonu**-morphism is a **Pogpu** -morphism $(L, \leq_L, \otimes_L, \perp_L, \top_L) \xrightarrow{f} (K, \leq_K, \otimes_K, \perp_K, \top_K)$ preserving identities, i. e. $f(e_L) = e_K$. Composition and identities in **Pomonu** are the same as **Set**.

Note here that **Monagop** (**Pomonu**) is a non-full subcategory of **Asagop** (**Posmgu**). In a similar fashion to Theorem 13, the mapping F : **Pomonu**$\rightarrow$**Monagop**, given by the equality (6) in Theorem 12, defines a functor. Furthermore, by making use of Theorem 9 , we easily observe that F is an isomorphism. This fact proves the isomorphism between **Monagop** and **Pomonu**.

4. FIBRES AND SUBOBJECTS IN **Agop**

4.1. Subaggregation operators and subobjects in Agop

Given a posetu $\mathbf{L} = (L, \leq_L, \perp_L, \top_L)$ and a nonempty subset K of L with the property $\perp_L, \top_L \in K$, if we denote the restriction of $\leq_L$ on K by $\leq_K$, then $\mathbf{K} = (K, \leq_K, \perp_L, \top_L)$ will be obviously a poset with the universal bounds $\perp_L$ and $\top_L$. For a given aggregation operator A on $\mathbf{L}$ taking the values in K over $\bigcup_{n\in\mathbb{N}^+} K^n$, we easily observe that the restriction $A_{|_K} : \bigcup_{n\in\mathbb{N}^+} K^n \to K$ of A to $\bigcup_{n\in\mathbb{N}^+} K^n$, i. e. $A_{|_K}(\alpha_1, \ldots, \alpha_n) = A(\alpha_1, \ldots, \alpha_n)$ for all $\alpha_i \in K$ $(i = 1, \ldots, n)$, is an aggregation operator on $\mathbf{K}$. We call $A_{|_K}$ the subaggregation operator of A on $\mathbf{K}$. We now show that subaggregation operators are nothing but special subobjects of **Agop** -objects. For this purpose, we first need to clarify the subobjects of **Agop**-objects. Monomorphisms of the category **Agop** are obviously morphisms of **Agop** that are injective functions. Thus a subject of a given **Agop**-object $(\mathbf{L}, A)$ is a pair $((\mathbf{K}, B), m)$, where $(\mathbf{K}, B)$ is an **Agop**-object and $(\mathbf{K}, B) \xrightarrow{m} (\mathbf{L}, A)$ is an **Agop** -monomorphism. For a given aggregation operator A on $\mathbf{L}$, if $A_{|_K}$ is the subaggregation operator of A on $\mathbf{K}$, then it is clear that $((\mathbf{K}, A_{|_K}), i)$ is a subobject of $(\mathbf{L}, A)$, where $i : K \hookrightarrow L$ is the inclusion function.

4.2. Fibres in Agop

In Section 3, we stated that (**Agop**,U) is a concrete category over **Alg(**$\mathbb{N}^+$**)**, where the underlying functor U :**Agop**→**Alg(**$\mathbb{N}^+$ **)** is defined by

$$U((\mathbf{L}, A) \xrightarrow{f} (\mathbf{K}, B)) = (L, (A_n)_{n\in\mathbb{N}^+}) \xrightarrow{f} (K, (B_n)_{n\in\mathbb{N}^+}).$$

The fibre of an **Alg(**$\mathbb{N}^+$**)**-object $(L, (A_n)_{n\in\mathbb{N}^+})$ is the class $\mathcal{F}((L, (A_n)_{n\in\mathbb{N}^+}))$ of all **Agop**-objects $(\mathbf{L}, A)$ such that $U((\mathbf{L}, A)) = (L, (A_n)_{n\in\mathbb{N}^+})$, and is categorically ordered by the preorder relation $\precsim$ (see [1, 5.4 Definition (1) in pp. 54]) as follows:

$$\begin{aligned}(\mathbf{L}, A) \quad \precsim \quad & (\mathbf{L}, B) \Leftrightarrow id_{(L,(A_n)_{n\in\mathbb{N}^+})} : (L, (A_n)_{n\in\mathbb{N}^+}) \to (L, (B_n)_{n\in\mathbb{N}^+}) \\ & \text{is an } \mathbf{Agop}\text{-morphism.}\end{aligned}$$

More clearly, the preorder relation $\precsim$ on $\mathcal{F}((L, (A_n)_{n\in\mathbb{N}^+}))$ can also be expressed by

$$\begin{aligned}((L, \leq_1, \perp_{(1)}, \top_{(1)}), A) \precsim \quad & ((L, \leq_2, \perp_{(2)}, \top_{(2)}), A) \\ \Leftrightarrow \quad & [(\leq_1 \subseteq \leq_2) \text{ and } (\perp_{(1)} = \perp_{(2)}) \text{ and } (\top_{(1)} = \top_{(2)})]\end{aligned}$$

for all $((L, \leq_1, \perp_{(1)}, \top_{(1)}), A)$, $((L, \leq_2, \perp_{(2)}, \top_{(2)}), A) \in \mathcal{F}((L, (A_n)_{n\in\mathbb{N}^+}))$. Since $\precsim$ is obviously a partial ordering relation on $\mathcal{F}((L, (A_n)_{n\in\mathbb{N}^+}))$, (**Agop**,$U$) is amnestic (see [1, 5.4 Definition (3) in pp. 54]). On the other hand, (**Agop,**V) forms a concrete category over **Pos** where the forgetful functor V :**Agop**→**Pos** is defined by

$$V((\mathbf{L},A)\xrightarrow{f}(\mathbf{K},B))=(L,\leq_L)\xrightarrow{f}(K,\leq_K).$$

The fibre $\mathcal{F}((L,\leq))$ of a **Pos**-object $(L,\leq)$ is the class of all **Agop**-objects $(\mathbf{L},A)$ satisfying the condition $V((\mathbf{L},A))=(L,\leq)$. The preorder relation $\precsim$ on $\mathcal{F}((L,\leq))$, defined by

$$(\mathbf{L},A) \precsim (\mathbf{L},B) \Leftrightarrow id_{(L,\leq_L)}:(L,\leq)\to(L,\leq) \text{ is an } \mathbf{Agop}\text{-morphism}$$

for all $(\mathbf{L},A),(\mathbf{L},B)\in\mathcal{F}((L,\leq))$, is obviously the equality relation on $\mathcal{F}((L,\leq))$. Thus (**Agop,**V) is fibre-discrete (see [1, 5.7 Definition (2) in pp. 56]). Now let us consider the class $\mathcal{F}^*(\mathbf{L})=\{A\mid(\mathbf{L},A)\in\mathcal{F}((L,\leq))\}$ of all aggregation operators on a fixed poset $\mathbf{L}=(L,\leq,\perp,\top)$. The partial ordering relation $\leq$ on L can be pointwisely extended to $\mathcal{F}^*(\mathbf{L})$, i. e. for $A,B\in\mathcal{F}^*(\mathbf{L})$,

$$A\leq B\Leftrightarrow\left(\forall x\in\bigcup_{n\in\mathbb{N}^+}L^n\right)(A(x)\leq B(x)).$$

Defining the aggregation operators $S,G:\bigcup_{n\in\mathbb{N}^+}L^n\to L$ by

$$S(\alpha_1,\ldots,\alpha_n) = \left\{\begin{array}{ll}\top, & \text{if } \alpha_1=\ldots=\alpha_n=\top\\ \perp, & \text{otherwise}\end{array}\right\}$$
$$\text{and } G(\alpha_1,\ldots,\alpha_n) = \left\{\begin{array}{ll}\perp, & \text{if } \alpha_1=\ldots=\alpha_n=\perp\\ \top, & \text{otherwise}\end{array}\right\}$$

for $n\geq 2$, and $S(\alpha)=G(\alpha)=\alpha$ for $n=1$, we observe that S and G are the least and greatest elements of $(\mathcal{F}^*(\mathbf{L}),\leq)$.

5. CONCLUSIONS

In this paper, starting with the generalization of aggregation operators to posetus, we have established the connections between aggregation operators and po-groupoids in the sense of [3]. With the help of these connections, we have pointed out that the category **Pogpu** of po-groupoids with universal bounds can be fully embedded into the category of aggregation operators, and so aggregation operators are some kinds of generalization of po-groupoid with universal bounds. Furthermore, we have proven that the categories of associative aggregation operators, symmetric and associative aggregation operators and monoidal aggregation operators are, respectively, isomorphic to the subcategories of **Pogpu** consisting of po-semigroups, commutative po-semigroups and po-monoids. This means that associative aggregation operators, symmetric and associative aggregation operators and monoidal aggregation operators are, respectively and essentially, the same as po-semigroups, commutative po-semigroups and po-monoids in the sense of [3].

Table 1. List of categories.

Abbreviation	Category
Agop	Aggregation operators
Asagop	Associative aggregation operators
Smasagop	Symmetric and associative aggregation operators
Monagop	Monoidal aggregation operators
Pogp	Partially ordered groupoids
Pogpu	Partially ordered groupoids with universal bounds
Posmgu	Partially ordered semigroups with universal bounds
Pomonu	Partially ordered monoids with universal bounds
Cmposmgu	Commutative partially ordered semigroups with universal bounds
Alg(Ω)	Ω-algebras
Pos	Partially ordered sets
Set	Sets

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