KYBERNETIKA — VOLUME *42* (2006), NUMBER 2, PAGES 161 – 180

MONOTONICITY AND COMPARISON RESULTS FOR NONNEGATIVE DYNAMIC SYSTEMS
Part II: Continuous-Time Case

Nico M. van Dijk and Karel Sladký

This second Part II, which follows a first Part I for the discrete-time case (see [3]), deals with monotonicity and comparison results, as generalization of the pure stochastic case, for stochastic dynamic systems with arbitrary nonnegative generators in the continuous-time case.

In contrast with the discrete-time case the generalization is no longer straightforward. A discrete-time transformation will therefore be developed first. Next, results from Part I can be adopted.

The conditions, the technicalities and the results will be studied in detail for a reliability application that initiated the research. This concerns a reliability network with dependent components that can breakdown. A secure analytic performance bound is obtained.

Keywords: Markov chains, monotonicity, nonnegative matrices

AMS Subject Classification: 60J27, 90A16

1. INTRODUCTION

As a continuation of Part I for the discrete-time case (see [3]), this second Part II is devoted to the continuous-time case. For a more general introduction, for more examples of dynamic nonnegative systems and for a more general motivation of the research, the reader is referred to the introduction of Part I. In this section, a more specific motivation for the continuous-time case is given. In Section 2 the reliability model that initiated the research is described in detail.

1.1. Motivation: Stochastic case

Continuous-time Markov chains are widely known in the literature for modeling a variety of practical systems, most notably among which in the areas of queuing, telecommunication, computer performance evaluation and reliability. In continuous time the evolutionary characterization of and motivation for these continuous-time Markov chains and related performance measures have the mathematical (functional) form:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_t = \boldsymbol{Q}\,\boldsymbol{x}_t, & t \geq 0 \quad \text{and} \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{W}_t = r + \boldsymbol{Q}\,\boldsymbol{W}_t & t \geq 0 \end{cases}$$

where t represents a time parameter, x_t the state vector, and $\boldsymbol{W}_t$ a total or cumulative reward vector, while $\boldsymbol{Q}$ is the infinitesimal generator matrix (kernel). This latter generator matrix (or kernel) thus essentially determines the dynamic behavior of the system.

As the state process involved with the applications mentioned will generally be prohibitively large for computational purposes, also monotonicity and comparison results may become of interest such as to:

- study the qualitative nature as a function of time,
- compare the system with a simplified system, or
- provide a secure performance bound.

As discussed more detailed in Part I for the stochastic case a substantial literature, initiated by the pioneering work of Stoyan [9], on monotonicity and comparison results has appeared over the last decades (cf. Part I, references [1–6], [10], [12], [13], [15]). The results from these references can be applied to both the discrete- and continuous-time case as based upon the so-called technique of uniformization. This technique enables one to transform continuous-time Markov chains into discrete-time Markov chains ([5, 8]).

Particularly, in Keilson and Kester [6] and Massey [7] monotonicity results were established for reliability systems, under the condition that components can break down and be repaired *independently*.

1.2. Motivation: Nonnegative case

A natural mathematical extension of the Markovian continuous-time structure given above, that is with a stochastic generator $\boldsymbol{Q}$, and thus necessarily row sums equal to 0, is its generalization where $\boldsymbol{Q}$ is replaced by an arbitrary nonnegative matrix $\boldsymbol{A}$, that is with nonnegative off-diagonal elements without row sums equal to 0.

As in the discrete-time case, one classical example of interest is Leontieff's so-called input-output model in economic analysis. But also other examples in line with the discrete-time case can be thought of. Most notably, though, also the class of substochastic models is included with natural examples such as from reliability analysis.

In fact, our research has been motivated and initiated by a specific substochastic reliability application (as will be dealt with in Section 5). For this application, time monotonicity results were of interest.

Unfortunately, as in the discrete-time case no general monotonicity or comparison results appear to be available in the literature for this more general nonnegative case.

1.3. Results for Part II

The following results will be established.

- First, in order to extend the discrete-time results from Part I to the continuous-time case, a complication arises in that the uniformization technique mentioned for the stochastic case could no longer be applied. An extension of this uniformization technique will therefore be developed first (Section 3). This extension is of interest in itself.
- Next, based upon this extended uniformization the monotonicity and comparison results for the discrete-time case will be transformed into the continuous-time case (Section 4).
- Finally, the results will be applied to a general class of reliability models as of our primary interest. These include reliability systems with *dependent* components and a *sub*stochastic feature. Neither of these two aspects has been dealt with before in the literature in the setting of monotonicity results.

2. CONTINUOUS–TIME SYSTEMS OF INTEREST

2.1. General form

We are interested in continuous-time systems of the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_t = \boldsymbol{A}\,\boldsymbol{x}_t, & \text{(1a)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{W}_t = r + \boldsymbol{A}\,\boldsymbol{W}_t & \text{(1b)} \end{cases}$$

or more detailed:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_t(i) = \sum_j a(i,j)\,\boldsymbol{x}_t(j), & \text{(2a)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{W}_t(i) = r(i) + \sum_j a(i,j)\,\boldsymbol{W}_t(j), & \text{(2b)} \\ \quad \text{for all } i = 1,2,\dots \text{ and } t \geq 0, \end{cases}$$

where

$$\begin{cases} \boldsymbol{A} = (a(i,j)) & \text{is an arbitrary transition generator at a finite or countable state space } S \text{ with nonnegative off-diagonal elements,} \\ \boldsymbol{x}_t \text{ and } \boldsymbol{W}_t & \text{are vectors at } S, \text{ for all } t \geq 0, \text{ and} \\ r & \text{represents a reward rate vector.} \end{cases}$$

Stochastic case

For the special stochastic (Markovian) case the matrix $\boldsymbol{A}$ has the form:

$$\boldsymbol{A} \;=\; \boldsymbol{Q} \quad \text{with} \quad \sum_j q(i,j) = 0 \quad \text{for all} \quad i \qquad \text{probability generator matrix.}$$

In this case, $\boldsymbol{x}_t(i)$ represents a marginal expectation for given initial state i,

$$\boldsymbol{x}_t(i) \;=\; (\boldsymbol{P}_t f)(i) = \sum_j \boldsymbol{P}_t(i,j) f(j), \qquad \boldsymbol{x}_0(i) = f(i)$$

for some given initial vector function f, where $\boldsymbol{P}_t$ is the transition probability matrix over time t, while $\boldsymbol{W}_t(i)$ is the cumulative expected reward for given initial state i,

$$\boldsymbol{W}_t(i) \;=\; \int_0^t (\boldsymbol{P}_s r \mathrm{d}s)(i) = \int_0^t \left\{ \sum_j \boldsymbol{P}_s(i,j)\, r(j) \right\} \mathrm{d}s.$$

Limits

As in the discrete-time case in Part I and in analogy with the stochastic case, also the transient and average reward case might be of interest, as defined by:

$$\boldsymbol{W} \;=\; \lim_{t\to\infty} \boldsymbol{W}_t \qquad \text{for the transient case} \tag{3a}$$

$$\boldsymbol{G} \;=\; \lim_{t\to\infty} \frac{1}{t} \boldsymbol{W}_t \qquad \text{for the average case.} \tag{3b}$$

The vectors $\boldsymbol{W}$ and $\boldsymbol{G}$ represent the cumulative expected reward up to some random (usual exit) time respectively the expected reward per unit of time. In the special irreducible case the vector $\boldsymbol{G}$ will be a constant vector, say with constant value g.

General conditions for these limits to exists can be adopted from the literature (e.g. [1] similar to the discrete-time case as in Section 2.4 of Part I. As these will not be used explicitly in this part, they are not listed here. Moreover, by Result 3.1 in Section 3 sufficient conditions to this end can be concluded from the discrete-time case (that is, Section 2.3 from Part I).

2.2. Motivational examples

2.2.1. Stochastic case and queuing

For the special stochastic case the areas of queuing and queuing networks received considerable attention in the literature as a main field of research by itself. In line with the objectives of this paper, several monotonicity and comparison results in the literature have been motivated by and applied to queuing systems (for details, cf. Part I and references therein). In this respect, the results that will come out in this second part, for the more general nonnegative case, may also expand the insights and results for the stochastic case and queuing applications.

2.2.2. Input-output models

In analogy with Section 2.2.1 from Part I, also continuous-time versions of the classical Leontieff's input-output models in economic analysis have been studied in the literature, as of the form:

$$\left\{\begin{array}{lcl} 0 & = & d + \boldsymbol{A}\,\boldsymbol{x} \quad \Longleftrightarrow \quad \boldsymbol{x} = \lim\limits_{t\to\infty} \boldsymbol{x}_t \qquad \text{(static case)} \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_t(i) & = & d(i) + \sum\limits_j a(i,j)\,\boldsymbol{x}_t(j), \quad (i = 1,\dots,N) \ \text{(dynamic case)} \\ \text{where} & & \\ \boldsymbol{x}_t(j) & = & \text{gross production of an industry (or sector) } j \text{ at time } t \\ d(j) & = & \text{net (or final) demand (or output) of industry (sector) } j \\ a(i,j) & = & \text{attribution rate of the production } \boldsymbol{x}_t(i) \text{ from industry } i \\ & & \text{for industry } j. \end{array}\right\}$$

With the interpretation that in the dynamic case the production (and consumption) quantities of N industries (or sectors) with internal exchanges (productions for one another) are in full balance as input and output (consumption and production) per industry separately while in the static case the system is in economic equilibrium.

2.2.3. Reliability models

Reliability models have become of growing practical interest over the last two decades. In such models the operability of a system is studied which consists of a number of components which can individually break down, as well as possibly be repaired, at rates which generally depend on the total state of the system, and thus the state of other components. For example, in the front-end data base system as shown below each of the individual components: front-end (FE), data base (DB), a processor, a memory or a switch or just the whole system can individually break down.

Let $\boldsymbol{x}_t$ denote the state of the system, which represents the components that are up (working). Then $\boldsymbol{x}_t$ is generally governed by a system of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}_t = \boldsymbol{A}\,\boldsymbol{x}_t$$

where $\boldsymbol{A}$ is a matrix determined by breakdown and repair rates, depending on the present state. Here this matrix $\boldsymbol{A}$ is not necessarily purely stochastic (i. e. with row sums equal to 0). It can be substochastic as the system may become totally inoperative. With $f(\boldsymbol{x})$ some operability function, for example counting the number of components that are up (working), a typical question of interest is the time-dependent behavior of $f(\boldsymbol{x}_t)$. More precisely, time-monotonicity results for $f(\boldsymbol{x}_t)$ may guarantee secure bounds, such as based on steady state bounds, for time-dependent performance measures like the availability of components or the operability of the system (see Figure 2 below).

In fact, for the pure stochastic case such questions have been analyzed by stochastic comparison results (e. g. [4, 6, 7, 9]). The extension of such results actually motivated the research for this paper (both Part I and II). The results will be presented in Section 5.

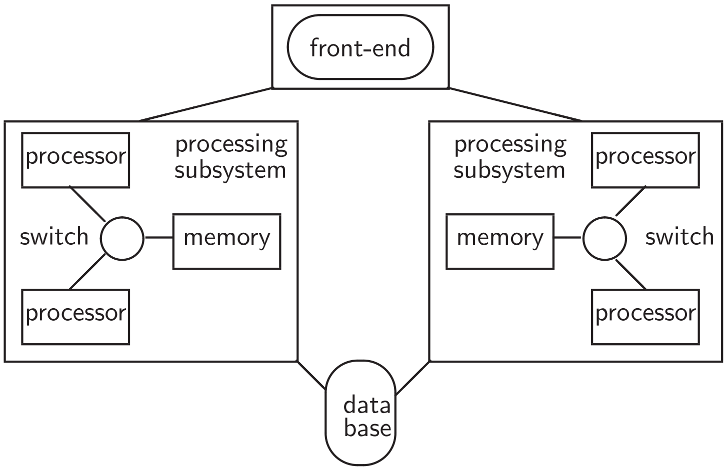

Fig. 1.

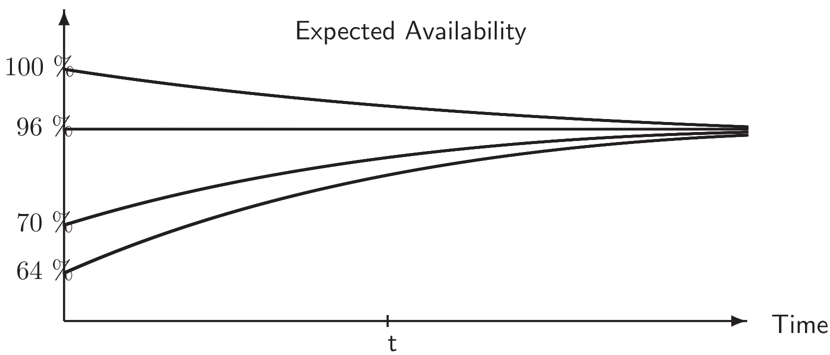

Fig. 2.

Parametrization. To specify this primary motivational application of our research more detailed, let us give a more detailed description and corresponding parametrization of the generator $\boldsymbol{A}$. This parametrization will used in Section 5.

Consider a reliability system with N components numbered $1, \ldots, N$, such as illustrated in Figure 1. Each component can be in an *up* and *down* state. Let the state

$$H = \{h_1, h_2, \dots, h_n\} \subset \{1, \dots, N\}$$

denote that components $h_1, h_2, \dots, h_n$ (say in increasing number order) are currently *down.* Then in that state H either one of the following events may occur at rates:

$\alpha(H)$: The total system may break down
$\delta(h|H)$: A down component $h \in H$ is repaired (for any $h \in H$)
$\beta(h|H)$: Another component $h \notin H$ also goes down (for any $h \notin H$).

When the system has broken down, it can no more become operable. In order to evaluate the operability level of the system at time t, we need to keep track of the state H. Its corresponding generator matrix, where we identify i and j with H and $H^{'}$, is then given by:

$$a(H, H^{'}) = \begin{cases} \beta(h|H), & H^{'} = H \cup h \quad \text{all } h \notin H \\ \delta(h|H), & H^{'} = H/h \quad \text{all } h \in H \\ -[\alpha(H) + \sum_{h \notin H} \beta(h|H) + \sum_{h \in H} \delta(h|H)], & H^{'} = H. \end{cases} \tag{4}$$

As we do not take into account the broken down state of the total system, we thus obtain a substochastic generator matrix with state dependent (nonnegative) row sums $-\alpha(H)$ in state H.

3. UNIFORMIZATION OF NONNEGATIVE SYSTEMS

In order to extend the monotonicity and comparison results for the discrete-time case to the continuous-time case, in this section we first present a generalized uniformization result. To this end, also referring to Remark 3.1 below for a brief discussion on its generality, we impose the following condition.

Condition 3.1. There exists a strictly positive eigenvector u for $\boldsymbol{A}$ with real eigenvalue $\lambda \neq 0$, that is:

$$\boldsymbol{A}u = \lambda u \quad \text{or equivalently} \tag{5a}$$

$$a(i,i)\,u(i) - \lambda\,u(i) = -\sum_{j \neq i} a(i,j)\,u(j) \quad \text{all } i. \tag{5b}$$

Remark 3.1. Except for special examples, Condition 3.1 is most naturally fulfilled. More precisely, for finite irreducible nonnegative matrix this condition is always trivially fulfilled by the Perron–Frobenius theorem (e. g. [1]). For the infinite or reducible case it need not be necessarily fulfilled. When $\boldsymbol{A}$ is finite and reducible, Condition 3.1 is fulfilled if and only if each irreducible class which has a real eigenvector less than λ has access to some irreducible class which has a real eigenvalue λ.

Remark 3.2. Normalized case $\lambda = 0$. The special case for which $\lambda = 0$, that is

$$a(i,i)\,u(i) = -\sum_{j\neq i} a(i,j)\,u(j), \quad \text{for all } i \tag{6}$$

includes the pure *stochastic case* (for which $u \equiv 1$) as well as several non-stochastic systems, most notably *open and closed economic input-output models.*

Now let the family of matrices $\{\boldsymbol{M}_t = (m_t(i,j)) | \, t \geq 0\}$ and the family of vectors $\{\boldsymbol{W}_t | \, t \geq 0\}$ be defined by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{M}_t = \boldsymbol{A}\,\boldsymbol{M}_t, & \text{with} \quad \boldsymbol{M}_0 = I, \quad \text{the identity matrix,} \qquad (7a) \\ \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{W}_t = r + \boldsymbol{A}\,\boldsymbol{W}_t, & \text{with} \quad \boldsymbol{W}_0 = 0, \quad \text{the null vector.} \qquad (7b) \end{cases}$$

The following lemma can then be regarded as a generalization of the standard uniformization (or randomization) technique in the stochastic case (e. g. [5, 8]).

Result 3.1. Under Condition 3.1, let $B < \infty$ such that

$$\sum_{j\neq i} a(i,j) \frac{u(j)}{u(i)} \leq B \tag{8}$$

and define the nonnegative matrix $\boldsymbol{M} = (m(i,j))$ by:

$$\boldsymbol{M} = \left(1 - \frac{\lambda}{B}\right) I + \frac{1}{B}\boldsymbol{A}, \tag{9}$$

i. e.

$$m(i,j) = \begin{cases} \frac{a(i,j)}{B}, & j \neq i \\ \left(1 - \frac{\lambda}{B}\right) + \frac{a(i,i)}{B}, & j = i. \end{cases} \tag{10}$$

Then, for all t:

$$\boldsymbol{M}_t = e^{\lambda t} \sum_{k=0}^{\infty} e^{-tB} \frac{(tB)^k}{k!} \boldsymbol{M}^k \tag{11}$$

and

$$\boldsymbol{W}_t = \sum_{k=0}^{\infty} e^{-t(B-\lambda)} \frac{(t(B-\lambda))^k}{k!} \boldsymbol{W}^k \tag{12}$$

where

$$\boldsymbol{W}^{k+1} = \frac{1}{B-\lambda} r + \frac{B}{B-\lambda} \boldsymbol{M}\boldsymbol{W}^k \qquad (k \geq 0), \quad \boldsymbol{W}^0 = 0. \tag{13}$$

Proof. The proof will follow in three steps. First, in Lemma 3.1 we show that the system can be stochasticized. Next, in Lemma 3.2 we will then apply the standard uniformization for Markov chains which will eventually lead to the expression (11). Finally, in Lemma 3.3 expression (12) is proven. □

Lemma 3.1. Under Condition 3.1 there exists a unique probability transition semigroup $\{\boldsymbol{P}_t\}$ of a continuous-time Markov chain with generator $\boldsymbol{Q} = (q(i,j))$ defined by:

$$q(i,j) = \begin{cases} a(i,j)\,\frac{u(j)}{u(i)}, & j \neq i \quad \text{(14a)} \\ a(i,i) - \lambda, & j = i. \quad \text{(14b)} \end{cases}$$

Furthermore, with U the diagonal matrix with elements $u(i,j) = 0$ for $j \neq i$ and $u(i,i) = u(i)$ for all i, for $\boldsymbol{M}_t$ (cf. (7a)) we have

$$\boldsymbol{M}_t = e^{\lambda t}\,(U\,\boldsymbol{P}_t\,U^{-1}). \tag{15}$$

Proof. That $\boldsymbol{Q}$ is a stochastic generator is an immediate consequence of Condition 3.1. Next, by (14), note that

$$\boldsymbol{Q} = U^{-1}\boldsymbol{A}U - \lambda I \tag{16a}$$

$$\boldsymbol{A} = U\boldsymbol{Q}U^{-1} + \lambda I. \tag{16b}$$

By substituting (15) we thus obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{M}_t) &= e^{\lambda\,t}\,U\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{P}_t\right)U^{-1} + \lambda\,e^{\lambda\,t}\,U\boldsymbol{P}_t\,U^{-1} \\ &= e^{\lambda\,t}\,(U\boldsymbol{Q}\boldsymbol{P}_t)\,U^{-1} + \lambda\,e^{\lambda\,t}\,U\boldsymbol{P}_t\,U^{-1} \\ &= (U\boldsymbol{Q}\,U^{-1})(e^{\lambda\,t}\,I)\,(U\boldsymbol{P}_t\,U^{-1}) + \lambda\,(e^{\lambda\,t}\,I)U\boldsymbol{P}_t\,U^{-1} \\ &= [(U\,\boldsymbol{Q}\,U^{-1}) + \lambda\,I]\,[e^{\lambda t}(U\,\boldsymbol{P}_t\,U^{-1})] = \boldsymbol{A}\,\boldsymbol{M}_t. \end{aligned} \tag{17}$$

By quoting the unique solution $\boldsymbol{M}_t$ of the equation $\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{M}_t) = \boldsymbol{A}\,\boldsymbol{M}_t$ for given initial matrix $\boldsymbol{M}_0 = I$, the proof is hereby completed. □

Lemma 3.2. If Condition 3.1 and (8) hold, then (11) is valid for all $t \geq 0$.

Proof. First note that by (6) and (8)

$$0 = \sum_{j \neq i} a(i,j)\frac{u(j)}{u(i)} + a(i,i) - \lambda \leq (B - \lambda) + a(i,i), \tag{18}$$

so that

$$\left(1 - \frac{\lambda}{B}\right) + \frac{1}{B}\,a(i,i) \geq 0. \tag{19}$$

Hence, $\boldsymbol{M}$ defined by (9) is a nonnegative matrix. Furthermore, with $\boldsymbol{Q}$ as by (14) we directly conclude from (8) that for all i:

$$\sum_{j \neq i} q(i,j) \leq B < \infty. \tag{20}$$

As a consequence, the corresponding stochastic semigroup $\{\boldsymbol{P}_t\}$ from Lemma 3.1 can be transformed into a discrete expansion by virtue of the standard uniformization (also called randomization) method (e. g. [5, 8]). This yields

$$\boldsymbol{P}_t = \sum_{k=0}^{\infty} e^{-tB} \frac{(tB)^k}{k!} \boldsymbol{P}^k \qquad t \geq 0 \tag{21}$$

where $\boldsymbol{P}$ is the stochastic matrix defined by $\boldsymbol{P} = I + \boldsymbol{Q}/B$ or equivalently

$$\boldsymbol{P}(i,j) = \begin{cases} q(i,j)/B & j \neq i \\ 1 - \sum_{j \neq i} q(i,j)/B & j = i. \end{cases} \tag{22}$$

Furthermore, by (9) and (14):

$$\begin{aligned} \boldsymbol{M} &= \left(1 - \frac{\lambda}{B}\right) I + \frac{1}{B} \boldsymbol{A} = \left(1 - \frac{\lambda}{B}\right) I + \frac{1}{B} \left(U\boldsymbol{Q}U^{-1} + \lambda I\right) \\ &= I + \frac{1}{B} U\boldsymbol{Q}U^{-1} = U\left(I + \frac{1}{B}\boldsymbol{Q}\right) U^{-1} = (U\boldsymbol{P}U^{-1}). \end{aligned} \tag{23}$$

Since also by repetition: $(U\boldsymbol{P}U^{-1})^k = U\boldsymbol{P}^k U^{-1}$, we can thus conclude from Lemma 3.1, (21) and (23)

$$\begin{aligned} \boldsymbol{M}_t &= e^{\lambda t} U \boldsymbol{P}_t U^{-1} \\ &= e^{\lambda t} U \left(\sum_{k=0}^{\infty} e^{-tB} \frac{(tB)^k}{k!} \boldsymbol{P}^k \right) U^{-1} = e^{\lambda t} \left(\sum_{k=0}^{\infty} e^{-tB} \frac{(tB)^k}{k!} U \boldsymbol{P}^k U^{-1} \right) \\ &= e^{\lambda t} \sum_{k=0}^{\infty} e^{-tB} \frac{(tB)^k}{k!} (U\boldsymbol{P}U^{-1})^k = e^{\lambda t} \sum_{k=0}^{\infty} e^{-tB} \frac{(tB)^k}{k!} \boldsymbol{M}^k. \end{aligned} \tag{24}$$

□

Lemma 3.3. Under the same conditions as in Lemma 3.2 (that is, Condition 3.1 and (8)), (12) holds for all $t \geq 0$.

P r o o f. To verify that $\boldsymbol{W}_t$ is a solution to (1b) or to (7b) written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{W}_t = r + \boldsymbol{A}\boldsymbol{W}_t \tag{25}$$

we calculate $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{W}_t$. From (12) we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{W}_t = -(B-\lambda)\boldsymbol{W}_t + (B-\lambda)\sum_{k=1}^{\infty} e^{-t(B-\lambda)}\frac{(t(B-\lambda))^k}{k!}\boldsymbol{W}^{k+1}. \tag{26}$$

By inserting (13) and (9) into (26) we conclude that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{W}_t &= -(B-\lambda)\boldsymbol{W}_t + (B-\lambda)\left\{\frac{1}{B-\lambda}r + \frac{B}{B-\lambda}\boldsymbol{M}\boldsymbol{W}_t\right\} \\ &= -(B-\lambda)\boldsymbol{W}_t + r + [(B-\lambda)I + \boldsymbol{A}]\boldsymbol{W}_t = r + \boldsymbol{A}\boldsymbol{W}_t. \end{aligned} \tag{27}$$

□

By Result 3.1 we are now able to transform the monotonicity and comparison results for the discrete-time case, as developed in Part I, into those for the continuous-time case. The various continuous-time results, with direct references to Part I for proofs, will be provided in Section 4. Herein we restrict our presentation to monotonicity and comparison results in one direction, leaving the other as obvious by reversing signs.

4. MONOTONICITY AND COMPARISON RESULTS

4.1. Monotonicity results

Similarly to the discrete-time case in Section 3.1 of Part I and with the family of matrices $\{\boldsymbol{M}_t |\, t \geq 0\}$ determined by (7), for given initial vector y define:

$$(\boldsymbol{M}_t f|y) := \sum_i y(i) m_t(i,j) f(j) = \sum_i y(i)\boldsymbol{M}_t f(i).$$

Then as a direct consequence of (11):

$$(\boldsymbol{M}_t f|y) = e^{\lambda t}\sum_{k=0}^{\infty} e^{-tB}\frac{(tB)^k}{k!}(\boldsymbol{M}^k f|y). \tag{28}$$

Now, as in Part I, let $\mathcal{M}$ be a monotonicity class which is closed under $\boldsymbol{M}$, i. e.

$$\boldsymbol{M}f \in \mathcal{M} \qquad \text{for all} \;\; f \in \mathcal{M}. \tag{29}$$

Result 4.1. (Monotonicity in time) Let y be such that with $\boldsymbol{M}$ defined by (9):

$$(\boldsymbol{M}f|y) \leq (f|y) \qquad \forall\, f \in \mathcal{M}. \tag{30}$$

Then for any $f \in \mathcal{M}$:

$$\left[\frac{(\boldsymbol{M}_t f\,|\,y)}{e^{\lambda t}}\right] \downarrow (t) \tag{31}$$

Proof. By Lemma 3.2 of Part I we directly conclude that

$$(\boldsymbol{M}^k f \mid y) \downarrow (k). \tag{32}$$

To complete the proof, by (28) note that

$$\begin{aligned} \frac{(\boldsymbol{M}_{t+s} f|y)}{e^{\lambda(t+s)}} &= \sum_{k=0}^{\infty} e^{-(t+s)B} \frac{[(t+s)B)]^k}{k!} (\boldsymbol{M}^k f|y) \\ &= \sum_{z=0}^{\infty} \sum_{u=0}^{\infty} e^{-tB} \frac{[tB]^z}{z!} e^{-sB} \frac{[sB]^u}{u!} (\boldsymbol{M}^{z+u} f|y), \end{aligned}$$

by virtue of its Poissonian expansion and conditioning up on epoch t. By (32) the right hand side can be estimated from above by:

$$\begin{aligned} &\leq \sum_{z=0}^{\infty} \sum_{u=0}^{\infty} e^{-tB} \frac{(tB)^z}{z!} e^{-sB} \frac{(sB)^u}{u!} (\boldsymbol{M}^z f|y) \\ &= \sum_{z=0}^{\infty} e^{-tB} \frac{(tB)^z}{z!} (\boldsymbol{M}^z f|y) \left\{ \sum_{u=0}^{\infty} e^{-sB} \frac{(sB)^u}{u!} \right\} = \frac{(\boldsymbol{M}_t f|y)}{e^{\lambda t}}. \end{aligned} \tag{33}$$

□

Remark 4.1.1. For the special normalizable case (that is with $\lambda = 0$), as naturally satisfied for instance for economic input-output systems, note that the monotonicity in (31) also holds for $(\boldsymbol{M}_t f|y)$.

Remark 4.1.2. Clearly, for $\lambda > 0$ we also conclude from (31): $(\boldsymbol{M}_t f \mid y) \downarrow t$. This case corresponds to the 'instable' case for which a limit as $t \to \infty$ does not generally exists.

Corollary 4.1. Suppose that conditions (29) and (30) hold for some initial vector y while in addition for some given function $r \in \mathcal{M}$ and value r^∞:

$$(\boldsymbol{M}^k r|y) \to r^\infty.$$

Then

$$\frac{(\boldsymbol{M}_t r|y)}{e^{\lambda t}} \downarrow r^\infty.$$

Proof. Immediate by (28) and Result 4.1. □

4.2. Comparison results

By virtue of the discrete Poissonian expressions (11) and (12), also comparison results for the continuous-time case can directly be expressed in terms of the one-step matrix $\boldsymbol{M}$ as in line with the discrete-time case in Part I, Section 3.

Consider two continuous-time dynamic nonnegative systems as in (1) with generators $\boldsymbol{A}$ and $\overline{\boldsymbol{A}}$, and suppose that condition (5) is satisfied for both $\boldsymbol{A}$ and $\overline{\boldsymbol{A}}$ with possibly different eigenvalues λ and $\overline{\lambda}$. Furthermore, let B be such that (8) is also satisfied with $a(i,i)$ and $u(i)$ replaced by $\overline{a}(i,i)$ and $\overline{u}(i)$. Define $\boldsymbol{M}$ and $\overline{\boldsymbol{M}}$ as by (9), that is

$$\boldsymbol{M} = \left(1 - B^{-1}\lambda\right) + B^{-1}\boldsymbol{A} \tag{34a}$$
$$\overline{\boldsymbol{M}} = \left(1 - B^{-1}\overline{\lambda}\right) + B^{-1}\overline{\boldsymbol{A}} \tag{34b}$$

and let $\mathcal{M} \subset \{f : |\overline{S} \to \mathbb{R}\}$ (with $\overline{S} \subset S$) such that $\mathcal{M}$ is closed under both $\boldsymbol{M}$ and $\overline{\boldsymbol{M}}$, i. e.:

$$\boldsymbol{M}f \in \mathcal{M} \quad \text{for all} \quad f \in \mathcal{M}$$
$$\overline{\boldsymbol{M}}f \in \mathcal{M} \quad \text{for all} \quad f \in \mathcal{M}.$$

The following two results are then an immediate consequence of the expression (9), and a combination of Result 4.1 and Corollary 4.1, from Part I (Result 4.2) respectively Results 4.2 and 4.3 from Part I (Result 4.3).

Result 4.2. (Marginal and total reward case) If

$$\overline{\boldsymbol{M}}f \geq \boldsymbol{M}f \qquad \text{for all} \quad f \in \mathcal{M}. \tag{35}$$

Then, for any $f \in \mathcal{M}$

$$\overline{\boldsymbol{M}}_t f \geq \boldsymbol{M}_t f, \qquad t \geq 0. \tag{36}$$

$$\overline{\boldsymbol{W}}_t \geq \boldsymbol{W}_t, \qquad t \geq 0. \tag{37}$$

Result 4.3. (Total reward case) Result (37) also holds if for the given reward function r and all $k \geq 0$:

$$(\overline{r} - r) + (\overline{\boldsymbol{M}} - \boldsymbol{M})\boldsymbol{W}^k \geq 0 \tag{38}$$

or for all i,

$$\begin{cases} \sum_j \overline{m}(i,j) = \sum_j m(i,j) \qquad \text{and} & \text{(39a)} \\ (\overline{r} - r)(i) + \sum_j [\overline{m}(i,j) - m(i,j)][\boldsymbol{W}^k(j) - \boldsymbol{W}^k(i)] \geq 0. & \text{(39b)} \end{cases}$$

Clearly under either the strong condition (35) or its relaxations (38) or (39) and provided the limits exist, as in Corollary 4.1 of Part I, we can also conclude that

$$\overline{\boldsymbol{W}} \geq \boldsymbol{W} \tag{40a}$$

$$\overline{\boldsymbol{G}} \geq \boldsymbol{G}. \tag{40b}$$

However, in parallel with the discrete-time case in Section 4.3 of Part I a more practical form for the average reward case, in analogy with the stochastic case, can be given. To this end, similarly to the discrete-time case in Part I, assume that there is a function $\mu(i)$ such that

$$\sum_j m(i,j)\mu(j) = \mu(i) \qquad \forall\, i \tag{41a}$$

$$\sum_j \overline{m}(i,j)\mu(j) = \mu(i) \qquad \forall\, i. \tag{41b}$$

The following result is then immediate from the expansion (12) and Result 4.4 from Part I.

Result 4.4. (Average reward case) Under (41):

$$\overline{\boldsymbol{G}} \geq \boldsymbol{G}, \qquad \text{provided the limits exist,} \tag{42}$$

if for all i and k:

$$\left(\frac{\overline{r}}{\mu} - \frac{r}{\mu}\right)(i) + \sum_j [\overline{m}(i,j) - m(i,j)] \frac{\mu(j)}{\mu(i)} \left[\frac{\boldsymbol{W}^k(j)}{\mu(j)} - \frac{\boldsymbol{W}^k(i)}{\mu(i)}\right] \geq 0. \tag{43}$$

5. RELIABILITY MODEL

In this section we will study the reliability model, which initially motivated our research, as introduced in Section 2.2.3. Although our primary interest concerned the time monotonicity, as will be dealt with in Section 5.1, in Section 5.2 we will also illustrate the possibility of comparison results for the purpose of illustration as well as of possible practical interest by itself. A combination is given by Corollary 5.3.

5.1. Time monotonicity

Recall the notation and rate parametrization as given in Section 2.2.3.

Interest. We aim to investigate the behavior of $(\boldsymbol{M}_t r|y)$ for appropriate initial condition y and some given function $r \in \mathcal{M}$. For example, by

$$r(H) = N - |H|,$$

with $|H|$ the cardinality of H, we would simply count the number of up components, as *availability measure* of the system. By $y = 1_{\{\emptyset\}}$ we would represent a 'perfect' starting state of the system.

In order to study the monotonicity of this availability measure in time, we aim to apply Result 4.1. The essential step to this end, is the verification of condition (30) for a proper set of availability functions. Under Condition 5.1 below, this will be established in Lemma 5.1.

First, we need to define some notation and an appropriate monotonicity class.

Let $\mathcal{S}$ be the set of states: $\mathcal{S} = \{H | H \subset \{1, \ldots, N\}\}$ and for convenience write $H + h = H \cup \{h\}$, $H - h = H \setminus \{h\}$. As the operability status is our measure of interest, it is natural to look at the subclass $\mathcal{M}$:

$$\mathcal{M} = \{f : \mathcal{S} \to \mathbb{R} | \; f(H+h) \le f(H) \;\text{ for all }\; H, H+h \in \mathcal{S}; f \ge 0\}.$$

Furthermore, note that necessarily there exists a negative eigenvalue λ (see e. g. [1]) with $\min_H \alpha(H) < -\lambda < \max_H \alpha(H)$ and some corresponding positive eigenvalue u, satisfying (5), and let B be some sufficiently large number satisfying (8). (In fact, the actual values of λ and B are not required but in the proof of Lemma 5.1 we will choose B sufficiently large.) Let $\boldsymbol{M}$ be the corresponding matrix as by (9). The following lemma is crucial. This lemma shows that $\mathcal{M}$ is closed under $\boldsymbol{M}$ under the natural monotonicity conditions as given by Condition 5.1.

Condition 5.1.

$$\beta(h|H+s) \ge \beta(h|H) \;\text{ for any } H, s \text{ and } \; h \notin (H+s) \tag{44a}$$

$$\delta(h|H+s) \le \delta(h|H) \;\text{ for any } H, s \text{ and } \; h \in H \tag{44b}$$

$$\alpha(H+s) \ge \alpha(H) \;\text{ for any } H, s. \tag{44c}$$

Lemma 5.1. Under Condition 5.1, $\mathcal{M}$ is closed under $\boldsymbol{M}$, i. e.:

$$\boldsymbol{M} f \in \mathcal{M} \qquad \text{for any } \; f \in \mathcal{M}.$$

Proof. By substituting (4) and (10) we obtain:

$$\begin{aligned} \boldsymbol{M} f(H) \;\; = \;\; & B^{-1} \sum_{s \notin H} \beta(s|H) f(H+s) + \\ & B^{-1} \sum_{s \in H} \delta(s|H) f(H-s) + \left(1 - \tfrac{\lambda}{B}\right) f(H) - \\ & B^{-1} \left[\sum_{s \notin H} \beta(s|H) + \sum_{s \in H+h} \delta(s|H) + \alpha(H) \right] f(H) \end{aligned} \tag{45}$$

and

$$\boldsymbol{M}f(H+h) = \begin{array}{l} B^{-1} \sum_{s \notin H+h} \beta(s|H+h) f(H+h+s) + \\ B^{-1} \sum_{s \in H+h} \delta(s|H+h) f(H+h-s) + \left(1 - \frac{\lambda}{B}\right) f(H+h) - \\ B^{-1} \left[\sum_{s \notin H+h} \beta(s|H+h) + \sum_{s \in H+h} \delta(s|H+h) + \alpha(H+h) \right] f(H+h) \end{array} \tag{46}$$

In expressions (45) and (46) now artificially add and subtract the following terms:

$$B^{-1}[\delta(h|H+h) + \sum_{s \notin H+h} [\beta(s|H+h) - \beta(s|H+h)] \, f(H) \quad \text{in (45)}$$
$$B^{-1}[\beta(h|H) + \sum_{s \in H} [\delta(s|H) - \delta(s|H+h)] \, f(H+h). \quad \text{in (46)}.$$

In addition, rewrite:

$$\begin{aligned} \alpha(H) &= \alpha(H+h) + [\alpha(H) - \alpha(H+h)] \\ \beta(s|H+h) &= \beta(s|H) + [\beta(s|H+h) - \beta(s|H)] \\ \delta(s|H) &= \delta(s|H+h) + [\delta(s|H) - \delta(s|H+h)]. \end{aligned}$$

Then by subtracting (46) from (45), we find

$$\begin{aligned} \boldsymbol{M}f(H) \; &- \boldsymbol{M}f(H+h) \\ &= B^{-1}[\alpha(H+h) - \alpha(H)] f(H) \\ &+ B^{-1} \sum_{s \notin H+h} \beta(s|H)[f(H+s) - f(H+h+s)] \\ &+ B^{-1} \sum_{s \notin H+h} [\beta(s|H+h) - \beta(s|H)][f(H) - f(H+h+s)] \\ &+ B^{-1} \sum_{s \in H} \delta(s|H+h)[f(H-s) - f(H+h-s)] \\ &+ B^{-1} \sum_{s \in H} [\delta(s|H) - \delta(s|H+h)][f(H-s) - f(H+h)] \\ &+ [(1 - \tfrac{\lambda}{B}) - B^{-1}\{\alpha(H+h) + \sum_{s \notin H+h} \beta(s|H+h) + \beta(h|H) \\ &+ \sum_{s \in H} \delta(s|H) + \delta(h|H+h)\}] \, [f(H) - f(H+h)]. \end{aligned} \tag{47}$$

Now note that so far we have only assumed, that a value B can be found such that in state $H+h$:

$$\left(1 - \frac{\lambda}{B}\right) + \frac{1}{B} a(H+h, H+h) \geq 0.$$

However, we can also enlarge B such that

$$\left(1 - \frac{\lambda}{B}\right) - B^{-1} \left\{ a(H+h, H+h) + B^{-1} \left[\beta(h|H) + \sum_{s \in H} [\delta(s|H) - \delta(s|H+h)] \right] \right\} \geq 0$$

which corresponds exactly to the coefficient for $[f(H) - f(H+h)]$ in (47). By also writing

$$f(H-s) - f(H+h) = [f(H-s) - f(H+h-s)] + [f(H+h-s) - f(H+h)]$$
$$f(H) - f(H+h+s) = [f(H) - f(H+h)] + [f(H+h) - f(H+h+s)]$$

and recalling that

$$f(H) - f(H+h) \geq 0 \text{ and } f(\cdot) \geq 0, \quad \text{for all } f \in \mathcal{M}$$

by substitution in (47) we have proven that for any H and $H+h$:

$$\boldsymbol{M}f(H) - \boldsymbol{M}f(H+h) \geq 0.$$

□

Result 5.1. (Time monotonicity) Under the natural monotonicity Condition 5.1, for any monotone reward function $r \in \mathcal{M}$ and starting with a *'perfect'* system, i. e. $y = 1_{\{\emptyset\}}$, we have

$$(\boldsymbol{M}_t r|y) \downarrow (t). \tag{48}$$

P r o o f . This is a direct consequence of Result 3.1 by also recalling that $\lambda < 0$ (so that in fact even stronger results than (48) can also be achieved) provided equation (29) is satisfied with $y = 1_{\{\emptyset\}}$ and $\leq$ sign. To this end, let $f \in \mathcal{M}$. Then by (4):

$$(\boldsymbol{M}f|y) = \boldsymbol{M}f(\emptyset) = B^{-1}\sum_h \beta(h|\emptyset)f(h)$$

$$+\left\{\left(1-\frac{\lambda}{B}\right) - \frac{1}{B}\left[\sum_h \beta(h|\emptyset) + \alpha(\emptyset)\right]\right\} f(\emptyset) \leq f(\emptyset) = (f|y = 1_{\{\emptyset\}}).$$

where the inequality follows by virtue of the fact that $f(\emptyset) \geq 0$ while $\sum_h \beta(h|\emptyset)[f(h) - f(\emptyset)] \leq 0$. As condition (29) is hereby proven, the proof is completed. □

Remark 5.1. (Dependent components) Results of the form (48) for reliability systems have been reported in the literature Keilson and Kester [6], Stoyan [9] and Massey [7])but only for the case of *in*dependent components, that is with $\delta(h|H)$ and $\beta(h|H)$ *not* depending on H.

Remark 5.2. (Essence of initial condition) Note here the essence of the specific appropriate initial condition y, as announced earlier (see Part I, Section 3.1).

Special stochastic case

As a special case, consider the situation that $\alpha(H) \equiv 0$ for all H. In other words, the situation in which the system will always continue to operate. Under rather general stability assumptions a steady state value r^∞ will then exist. This value can be obtained by either

$$r^\infty = \sum_H \pi(H) r(H) \tag{49}$$

and *numerically solving* the steady state equations:

$$\pi(H)\left[\sum_{s\notin H} \beta(s|H) + \sum_{s\in H} \delta(s|H)\right]$$

$$= \sum_{s\notin H} \pi(H+s)\delta(s|H+s)) + \sum_{s\in H} \pi(H-s)\beta(s|H-s)) \tag{50}$$

or by standard *simulation*. By virtue of Result 5.1 we would thus have established a secure lower bound for the reward over time, as shown below.

Here the reward function might typically represent the 'availability' or 'operability level' of the system.

5.2. Comparison results

To also illustrate the application of the general comparison results, as well as for its interest in itself as indicated below, assume that the rates α, β and δ are modified into $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\delta}$ such that for all H and s:

Condition 5.2. (Monotonicity condition)

$$\overline{\alpha}(H) \geq \alpha(H) \tag{51a}$$

$$\overline{\beta}(h|H) \geq \beta(h|H) \quad (h \notin H) \tag{51b}$$

$$\overline{\delta}(h|H) \leq \delta(h|H) \quad (h \in H). \tag{51c}$$

Intuitively, one may thus expect that the modified system will be less 'available' or 'operable' than the original system. However, one can construct counter-intuitive examples at sample path basis. Formal results, therefore, have been provided on an expectational basis by means of stochastic ordering results (see [8] and [9]. These however, do not cover the general nonnegative case under investigation nor the situation with component dependent components, as allowed above. A formal result is therefore of interest.

To this end, we choose $\overline{B} = B$ by the general breakdown and repair rates as given in Section 2.2.3, and with some restriction of generality, we also assume that $\overline{\lambda} = \lambda$. For example, when $\overline{\alpha}(H) = \alpha(H) = \alpha$ independent of H (a state independent system breakdown) this is easily verified, while $\overline{\alpha} = \alpha = 0$ (no system breakdown) leads to the pure stochastic case with $\overline{\lambda} = \lambda = 0$. The essential comparison Result 3.3 then applies as by

Result 5.2. Under Condition 5.2 for any $r \in \mathcal{M}$ and all $t \geq 0$:

$$\overline{\boldsymbol{M}}_t r \leq \boldsymbol{M}_t r. \tag{52}$$

P r o o f . We need to prove the comparison condition (35). To this end, by recalling (45) for the original and modified system we find

$$\begin{aligned}
&(\overline{\boldsymbol{M}}f - \boldsymbol{M}f)(H)\\
&= B^{-1}\sum_{s\notin H}\left[\overline{\beta}(s|H) - \beta(s|H)\right] f(H+s)\\
&+B^{-1}\sum_{s\in H}\left[\overline{\delta}(s|H) - \delta(s|H)\right] f(H-s)\\
&-B^{-1}\left\{\sum_{s\notin H}\left[\overline{\beta}(s|H) - \beta(s|H)\right] + \sum_{s\in H}\left[\overline{\delta}(s|H) - \delta(s|H)\right]\right\} f(H).
\end{aligned}$$

By reorganizing these terms and using that $f \in \mathcal{M}$, i.e. $[f(H-s) - f(H)] \geq 0$ while $f(H+s) - f(H) \leq 0$, and by applying the monotonicity condition (51) we easily verify $(\overline{\boldsymbol{M}}f - \boldsymbol{M}f) \leq 0$; i.e. condition (35). Application of Result 4.2 now completes the proof. □

By combining the monotonicity Result 5.1 and the comparison Result 5.2 and assuming the existence of an asymptotic value $\overline{r}^{\infty}$ for the modified system, the following corollary is achieved. This corollary can be regarded as a main result for practical interest, which motivated the research, as illustrated also by Figure 2.

Corollary 5.3. (Long run availability bound) Under the monotonicity and comparison Conditions 5.1 and 5.2, and under the assumption of the existence of an asymptotic value $\overline{r}^{\infty}$ for the modified system:

$$\boldsymbol{M}_t r(0) \downarrow \overline{r}^{\infty}.$$

Example. For example, for the stochastic case with $\alpha(H) = 0$ for all H, and with S the set of possible (admissible) states H, we could modify the original dependent system into an *independent* system by:

$$\begin{aligned}
\overline{\beta}(h|H) &= \max_{H\in S}\{\beta(h|H)\} =: \overline{\beta}(h) \qquad \text{all } h\\
\overline{\delta}(h|H) &= \min_{H\in S}\{\delta(h|H)\} =: \overline{\delta}(h) \qquad \text{all } h
\end{aligned} \tag{53}$$

for which the steady state equation (50) for the modified system directly leads to the solution:

$$\overline{\pi}(H) = c\prod_{h\in H}\left[\frac{\overline{\beta}_h}{\overline{\delta}_h}\right] \qquad \text{for all} \quad H \in S. \tag{54}$$

The value $\overline{r}^{\infty}$ is then directly computed as in (49). In fact, the form (54) also holds if the components have individual repair and breakdown rates as in (53) but with (natural) dependence, provided specific priorities are imposed that preserve the state space of admissible states S (see [2], Chapter 7).

Accordingly, various modifications of the original unsolvable reliability model can so be studied in order to provide simple secure lower bounds for the system 'availability' as based upon Corollary 5.3.

ACKNOWLEDGEMENT

This work was supported by the Grant Agency of the Czech Republic under Grants 402/05/0115 and 402/04/1294.

(Received May 19, 2005.)

REFERENCES

[1] A. Berman and R. J. Plemmons: Nonnegative Matrices in the Mathematical Sciences. Academic Press, New York 1979.
[2] N. M. van Dijk: Queuing Networks and Product Forms. Wiley, New York 1993.
[3] N. M. van Dijk and K. Sladký: Monotonicity and comparison results for nonnegative dynamic systems. Part I: Discrete-time case. Kybernetika *42* (2006), 37–56.
[4] N. M. van Dijk and P. G. Taylor: On strong stochastic comparison for steady state measures of Markov chains with a performability application. Oper. Res. *36* (2003), 3027–3030.
[5] D. Gross and D. R. Miller: The randomization technique as a modelling tool and solution procedure over discrete state Markov processes. Oper. Res. *32* (1984), 343–361.
[6] J. Keilson and A. Kester: Monotone matrices and monotone Markov processes. Stoch. Process. Appl. *5* (1977), 231–241.
[7] W. A. Massey: Stochastic ordering for Markov processes on partially ordered spaces. Math. Oper. Res. *12* (1987), 350–367.
[8] B. Melamed and N. Yadin: Randomization procedures in the computation of cumulative-time distributions over discrete state Markov processes. Oper. Res. *32* (1984), 926–943.
[9] D. Stoyan: Comparison Methods for Queues and Other Stochastic Models. Wiley, New York 1983.

Nico M. Van Dijk, University of Amsterdam, Department of Economic Sciences and Econometrics, Amsterdam. The Netherlands.
e-mail: N.M.vanDijk@uva.nl

Karel Sladký, Institute of Information Theory and Automation – Academy of Sciences of the Czech Republic, Pod Vodárenskou věží 4, 182 08 Praha 8. Czech Republic.
e-mail: sladky@utia.cas.cz