KYBERNETIKA — VOLUME 42 (2006), NUMBER 2, PAGES 129 – 142

CONDITIONAL STATES AND JOINT DISTRIBUTIONS ON MV–ALGEBRAS

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In this paper we construct conditional states on semi-simple MV-algebras. We show that these conditional states are not given uniquely. By using them we construct the joint probability distributions and discuss the properties of these distributions. We show that the independence is not symmetric.

Keywords: semi-simple MV-algebra, conditional distribution, joint distribution

AMS Subject Classification: 06D35, 28E10, 60B99

1. INTRODUCTION

It is a well-known fact that the classical probability space $(\Omega, \mathcal{S}, P)$ is a measurable space with a fixed normalized measure (probability measure). This model was introduced by Kolmogoroff in [9]. Afterwards he published a paper [10] on conditional probability measures. These conditional probability measures are in fact derived from the original one, P, restricting the σ-algebra $\mathcal{S}$. This system has been studied by many authors till nowadays.

Except of this (Kolmogorovian) model, also normalized measures (states) on more general structures, such as von Neumann algebra, Hilbert space, quantum logic (orthomodular lattice with states), etc. have been studied, e. g., in [1, 5, 7, 8, 12, 13, 15, 18, 25]. In 1958 and 1959 Chang published his papers [2, 3] where he introduced the notion of an MV-algebra. An MV-algebra (or, more precisely, a semi-simple MV-algebra) is in fact an algebra of fuzzy sets.

One of the basic problems on all those general structures is a correct definition of a conditional state in such a way that, in the special case when the structures coincide with the Boolean algebra (i. e. if we work with the classical probability space), we would get the same results as those achieved in the Kolmogorovian models.

Our approach is based on a convex combination of some special states. This model, on a quantum logic, was utilized in the paper [16]. It was shown there that, in the case of a Boolean algebra, it gives the Full Probability Theorem. But notions as independence or joint state (joint probability distribution on $(\Omega, \mathcal{S}, P)$) have, in some sense, the time-axis incorporated in themselves. However, this is not possible in classical models.

In this paper we define the conditional states on semi-simple MV-algebras using the above mentioned convex combinations. Using these conditional states we define the joint states. It is shown in this paper that it is possible to define the joint states also on MV-algebras which are not necessarily closed under product (in fact, which are sub-MV-algebras of the so-called product MV-algebras). For results achieved on product MV-algebras, see e. g. [6, 11, 19, 20, 21, 22, 23].

2. PRELIMINARIES

Definition 1. An MV-algebra is a 5-tuple $(\mathcal{M}, \oplus, *, \emptyset, 1)$ such that $(\mathcal{M}, \oplus, \emptyset)$ is an Abelian monoid and moreover

- $x^{**} = x$
- $\emptyset^* = 1$
- $x \oplus 1 = 1$
- $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

Moreover for all $x, y \in \mathcal{M}$ we can define

$$\begin{aligned} x \odot y &= (x^* \oplus y^*)^* \\ x \wedge y &= (x \oplus y^*) \odot y \\ x \vee y &= (x \odot y^*) \oplus y \end{aligned}$$

and then $(\mathcal{M}, \vee, \wedge, \emptyset, 1)$ is a bounded distributive lattice with its top and bottom elements $1, \emptyset$, respectively.

Example 1 (system of functions). Denote by $\mathcal{M}$ a system of $[0;1]$-valued functions, which are closed under the following operations

$$\begin{aligned} (f \oplus g)(x) &= \min(1, f(x) + g(x)) \\ (f \odot g)(x) &= \max(0, f(x) + g(x) - 1) \\ f^*(x) &= 1 - f(x) \end{aligned}$$

and such that $0 \in \mathcal{M}$. Then $\mathcal{M}$ is an MV-algebra (see [3], also [24]).

Definition 2. Let $\mathcal{M}$ be an MV-algebra. A function $\nu : \mathcal{M} \to [0,1]$ is said to be a state iff $\nu(1) = 1$ and

$$\forall f, g \in \mathcal{M} : \ g \le 1 - f \quad \Rightarrow \quad \nu(f \oplus g) = \nu(f) + \nu(g).$$

Definition 3. Denote $\mathcal{M}$ an MV-algebra and let $\nu : \mathcal{M} \to [0,1]$ be a state. Events $f, g \in \mathcal{M}$ will be called ν-orthogonal if $\nu(f \wedge g) = 0$.

Definition 4. Let $\mathcal{M}$ be an MV-algebra. $f \in \mathcal{M}$ will be called crisp, if

$$f \wedge f^* = 0.$$

An element, which is not crisp, will be called unsharp.

Definition 5. Denote $\mathcal{M}$ an MV-algebra and let $\nu : \mathcal{M} \to [0,1]$ be a state. We say that $\gamma : \mathcal{M} \times \mathcal{M} \to [0,1]$ is a conditional state if and only if the following conditions are fulfilled for all $f, g \in \mathcal{M}$:

(C1) if $\nu(g) > 0$, then $\gamma(.|g)$ is a state

(C2) $\nu(f) = \nu(g)\gamma(f|g) + \nu(g^*)\gamma(f|g^*)$

(C3) if f, g are ν-orthogonal to each other, then $\gamma(f|g) = 0$.

Definition 5 immediately implies the following

Lemma 1. Any $g \in \mathcal{M}$, such that $\nu(g) > 0$, holds $\gamma(1|g) = 1$.

Lemma 2. Denote $\mathcal{M}$ an MV-algebra and let $\nu : \mathcal{M} \to [0,1]$ be a state. Let $g \in \mathcal{M}$ be a crisp element. Then $\gamma(\cdot|g)$ is given uniquely.

P r o o f. If $\nu(g) = 0$, then f and g are ν-orthogonal and hence $\gamma(f|g) = 0$ by property (C3) of Definition 5.

If $\nu(g^*) = 0$, then $\nu(g) = 1$ and we get $\gamma(f|g) = \nu(f)$ by property (C2) of Definition 5.

Otherwise, $g \in \mathcal{M}$ is a crisp element, hence g and g^* are orthogonal and we get for each $f \in \mathcal{M}$

$$f = (f \wedge g) \oplus (f \wedge g^*).$$

The orthogonality of g and g^* implies

$$\gamma(f \wedge g|g^*) = \gamma(f \wedge g^*|g) = 0.$$

Hence

$$\gamma(f \wedge g|g) = \frac{\nu(f \wedge g)}{\nu(g)}, \qquad \gamma(f \wedge g^*|g^*) = \frac{\nu(f \wedge g^*)}{\nu(g^*)}.$$

The additivity of $\gamma(\cdot|g)$ implies the uniqueness. □

Denote $(\Omega, \mathcal{S}, \mu)$ a space with measure. In the whole paper $\mathcal{M}$ will denote an MV-algebra of $\mathcal{S}$-measurable functions from Ω to $[0,1]$, containing some unsharp elements.

Denote $\nu(f) = \int f \, \mathrm{d}\mu$, then ν is a state on $\mathcal{M}$, i. e.

$$\nu(f \oplus g) = \nu(f) + \nu(g) \qquad \text{if} \;\; f \leq (1 - g). \tag{1}$$

3. CONSTRUCTION OF CONDITIONAL STATES ON MV-ALGEBRAS

Notation. Denote $\mathcal{F}$ the system of all $\mathcal{S}$-measurable functions $f : \Omega \to [0,1]$ (i. e., $\mathcal{F}$ is an MV-algebra with product) and $\mathcal{T}$ the system of all transformations $\tau : \mathcal{M} \to \mathcal{F}$ such that for each $f \in \mathcal{M}$

1. $\int f \, \mathrm{d}\mu = \int \tau(f) \, \mathrm{d}\mu$
2. for any $x \in \Omega$ such that $f(x) = 0$ or $f(x) = 1$ the following holds
$$(\tau(f))(x) = f(x)$$
3. $(\tau(f^*))(x) = 1 - (\tau(f))(x)$.

We will assume that there is some element $f \in \mathcal{M}$ and some transformation $\tau \in \mathcal{T}$ such that $\tau(f) \neq f$.

Theorem 1. Let $\tau \in \mathcal{T}$. Define for any $f, g \in \mathcal{M}$

$$\gamma(f|g) = \begin{cases} \dfrac{\int f \cdot \tau(g) \, \mathrm{d}\mu}{\int \tau(g) \, \mathrm{d}\mu} & \text{if } 0 < \nu(g) \leq 1 \\ 0 & \text{if } \nu(g) = 0. \end{cases} \tag{2}$$

Then γ is a conditional state.

Comment 1 (to Theorem 1). It might seem that the condition

$$\gamma(f|g) = 0 \text{ if } \nu(g) = 0$$

is not necessary. However, it is a consequence of the condition (C3) in Definition 5.

Proof of Theorem 1. First, assume that $\nu(g) = 1$. Then, by the properties of $\tau \in \mathcal{T}$ we get $\tau(g) = g$ ν-almost surely and hence

$$\gamma(f|g) = \int f \cdot \tau(g) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu.$$

This means $\gamma(.|g)$ fulfils all properties of a conditional state from Definition 5.

Assume now $0 < \nu(g) < 1$. We must prove that condition (C2) holds for $\gamma(.|g)$. The right-hand-side of Condition (C2) gives

$$\nu(g) \frac{\int f \cdot \tau(g) \, \mathrm{d}\mu}{\int \tau(g) \, \mathrm{d}\mu} + \nu(g^*) \frac{\int f \cdot \tau(g^*) \, \mathrm{d}\mu}{\int \tau(g^*) \, \mathrm{d}\mu} = \int f \cdot \tau(g) \, \mathrm{d}\mu + \int f \cdot \tau(g^*) \, \mathrm{d}\mu$$

since the transformation τ preserves the state ν. Then we get

$$\int f \cdot \tau(g) \, \mathrm{d}\mu + \int f \cdot \tau(g^*) \, \mathrm{d}\mu = \int f \cdot \tau(g) \, \mathrm{d}\mu + \int f \cdot (1 - \tau(g)) \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu = \nu(f)$$

Property 2 of transformations from $\mathcal{T}$ gives $\gamma(f|g) = 0$, or $\gamma(f|g^*) = 0$, if f and g, or f and g^* are ν-orthogonal, respectively. This concludes the proof. □

Throughout this paper we will always denote by $\gamma(.|.)$ the state given by formula (2).

Comment 2. The transformation τ can be understood as the way how we handle the vague information (we do not transform values 0 and 1 of g in $\gamma(\cdot|g)$) we are apriori given by the condition g.

Definition 6. We say that event f is independent of g with respect to a conditional state γ iff $\nu(f) = \gamma(f|g)$.

Comment 3. As we will see in the next example, the independence of event f on g does not imply the independence of the event g on the event f. This nonsymmetric relation of independence allows us to distinguish between a cause and its effects. Similar results concerning the orthomodular lattices have been achieved also by O. Nánásiová in [14, 15, 17, 16].

Next example shows how the independence works.

Example 2. Let $\Omega = [0;1]$ and μ be the Lebesgue measure. Let τ be the transformation given by

$$(\tau(f))(x) = \begin{cases} \dfrac{1}{\mu(A(f))} \displaystyle\int_{A(f)} f \,\mathrm{d}\mu & \text{iff } f(x) \in\,]0.5;1[\\ \dfrac{1}{\mu(B(f))} \displaystyle\int_{B(f)} f \,\mathrm{d}\mu & \text{iff } f(x) \in\,]0;0.5[\\ f(x) & \text{otherwise} \end{cases}$$

where $A(f) = \{x \in \Omega; f(x) \in\,]0.5;1[\}$ and $B(f) = \{x \in \Omega; f(x) \in\,]0;0.5[\}$, provided $\mu(A(f)) \neq 0$, $\mu(B(f)) \neq 0$. If e. g. $\mu(A(f)) = 0$, we can put any value to $(\tau(g))(x)$ for $x \in A(f)$.

Take $f(x) = x$ and $g(x) = \frac{1}{2}x$. Then we get

$$(\tau(f))(x) = \begin{cases} 0.25 & \text{iff } x \in\,]0;0.5[\\ 0.75 & \text{iff } x \in\,]0.5;1[\\ x & \text{otherwise} \end{cases} \qquad (\tau(g))(x) = \begin{cases} 0.25 & \text{iff } x \in\,]0;1[\\ x & \text{otherwise.} \end{cases}$$

Now, compute the conditional state

$$\begin{aligned} \gamma(f|g) &= \frac{\int_0^1 f \cdot \tau(g) \,\mathrm{d}\mu}{\int_0^1 g \,\mathrm{d}\mu} = 0.5 = \nu(f) \\ \gamma(g|g) &= \frac{\int_0^1 g \cdot \tau(g) \,\mathrm{d}\mu}{\int_0^1 g \,\mathrm{d}\mu} = 0.25 = \nu(g) \\ \gamma(g|f) &= \frac{\int_0^1 g \cdot \tau(f) \,\mathrm{d}\mu}{\int_0^1 f \,\mathrm{d}\mu} = \frac{5}{16} \neq \nu(g) = 0.25 \\ \gamma(f|f) &= \frac{\int_0^1 f \cdot \tau(f) \,\mathrm{d}\mu}{\int_0^1 f \,\mathrm{d}\mu} = \frac{5}{8} \neq \nu(f) = 0.5. \end{aligned}$$

Hence we get that f is independent of g and also g is independent of itself. On the other hand, g is dependent on f and f is also dependent on f.

The definition of the conditional state γ, formula (2), immediately gives the following:

Theorem 2. (a) Let $f \in \mathcal{M}$ be μ-almost surely a constant function. Then $\gamma(f|f) = \nu(f)$.

(b) Let $\tau \in \mathcal{T}$. Then the following holds: if $\tau(f)$ is μ-almost surely a constant function, then $\gamma(f|f) = \nu(f)$.

Comment 4 (to Theorem 2). Event f depends on g, roughly speaking, if the knowledge that g has occurred, gives an additional information on f. If f is constant then, whatever g is, we have no additional information on f – even if g and f are the same elements! In fact, it is just a generalisation of that what happens in a Boolean algebra. 0 and 1 are the only constants there and they are independent of themselves.

Once having defined the values $\gamma(f|g)$ for any pair $f,\ g$ of elements of the MV-algebra $\mathcal{M}$, we can define also the two-dimensional joint distribution on $\mathcal{M} \times \mathcal{M}$ – the measure (probability) of occurrence of this pair $f,\ g$.

Definition 7. The joint distribution of a pair $f,\ g \in \mathcal{M}$ will be denoted by $p(f, g)$ and defined as

$$p(f, g) = \gamma(f|g)\gamma(g|1) \tag{3}$$

If γ is defined as in Theorem 1 we get from formulas (2) and (3)

$$p(f, g) = \int f\tau(g)\,\mathrm{d}\mu, \tag{4}$$

where τ is a given transformation from $\mathcal{T}$. Particularly, if τ is the identity, we get

$$p(f, g) = \int fg\,\mathrm{d}\mu$$

which is the joint distribution, additive in both variables. In general $f \cdot g \notin \mathcal{M}$. However, it is possible to compute that value, since $f \cdot g \in \mathcal{F}$.

Comment 5. The joint distribution is not an intersection of f and g, since in formula (4) we use a transformation τ. In fact, it represents the interaction of f and g. And the interaction can be different if we change the order.

Theorem 3 (basic properties of p). Let p be a joint distribution on the MV-algebra $\mathcal{M}$ and f, g be any elements of $\mathcal{M}$. Then

1. $p(f,1) = p(1,f) = \nu(f)$
2. $p(f,g) = p(g,f) = 0$, if f and g are ν-orthogonal
3. $\max\{0, \nu(f) + \nu(g) - 1\} \le p(f,g) \le \min\{\nu(f), \nu(g)\}$.
   Particularly
   $$\max\{0, 2\nu(f) - 1\} \le p(f,f) \le \nu(f)$$
4. $p(f_1 \oplus f_2, g) = p(f_1, g) + p(f_2, g)$ iff $f_1(x) + f_2(x) \le 1$ for any $x \in \Omega$.

Proof. We show the property 3. $p(f,g) = \int f \cdot \tau(g)\, d\mu$, hence

$$p(f,g) \le \min\{\nu(f), \nu(\tau(g))\} = \min\{\nu(f), \nu(g)\}$$

since the transformation τ is measure preserving. Similarly,

$$p(f,g) \ge \nu(f) + \nu(\tau(g)) - 1 = \nu(f) + \nu(g) - 1$$

since p is a joint probability distribution of f and $\tau(g)$.

The proof of the other properties is straightforward given by formula (4) and by the properties of the transformation τ. □

The next example shows that the variables of p need not commute, i. e., for a suitably chosen transformation τ there exist elements $f, g \in \mathcal{M}$ such that $p(f,g) \neq p(g,f)$.

Example 3. Assume that $\Omega = [0;1]$ and μ is the Lebesgue measure. For any element f of $\mathcal{M}$ let the transformation τ be defined by the following

$$(\tau(f))(x) = \begin{cases} 0, & \text{if } f(x) = 0 \\ 1, & \text{if } f(x) = 1 \\ \frac{1}{\mu(A(f))} \int_{A(f)} f(x)\,\mathrm{d}\mu(x) & \\ \quad \text{otherwise} & \end{cases}$$

where $A(f) = \{x; 0 < f(x) < 1\}$ (of course, the last item holds provided $\mu(A(f)) > 0$). Let $f(x) = x$ and $g(x) = \max\{0, x - 0.5\}$. Then

$$(\tau(f))(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x = 1 \\ 0.5 & \text{otherwise} \end{cases} \qquad (\tau(g))(x) = \begin{cases} 0, & \text{if } x \le 0.5 \\ 0.25 & \text{if } x > 0.5. \end{cases}$$

Then

$$\begin{aligned} p(g,f) &= \int_0^1 0.5g\,\mathrm{d}\mu = 0.5\int_{0.5}^1 (0.5 - x)\,\mathrm{d}\mu = \frac{1}{16} \\ p(f,g) &= \int_{0.5}^1 0.25x\,\mathrm{d}\mu = \frac{1}{4}\frac{3}{8} = \frac{3}{32} \\ p(f,f) &= \int_0^1 0.5x\,\mathrm{d}\mu = \frac{1}{4} \\ p(g,g) &= \int_{0.5}^1 0.25(x - 0.5)\,\mathrm{d}\mu = \frac{1}{32}. \end{aligned}$$

4. CONDITIONING WITH MORE CONDITIONS

If we add one dimension to the conditional state, we get three-dimensional conditional states. They can be of two different types. Either we condition two elements of the MV-algebra $\mathcal{M}$ by a third one, or we condition one element by two other ones. These two, in fact different states are linked to each other. This is the reason, why they are defined in one batch. Their definition is the following:

Definition 8. Three-dimensional conditional states $\gamma_{2,1}:\mathcal{M}\times\mathcal{M}\times\mathcal{M}\to[0,1]$ and $\gamma_{1,2}:\mathcal{M}\times\mathcal{M}\times\mathcal{M}\to[0,1]$ are defined by the following formulas ($f_1,f_2,f_3\in\mathcal{M}$):

$$\begin{aligned}\gamma_{2,1}(f_1,f_2|1) &= \gamma_{2,1}(f_1,f_2|f_3)\nu(f_3)+\gamma_{2,1}(f_1,f_2|f_3^*)\nu(f_3^*)\\ \gamma_{1,2}(f_1|1,f_3) &= \gamma_{1,2}(f_1|f_2,f_3)\gamma_{2,1}(1,f_2|f_3)+\gamma_{1,2}(f_1|f_2^*,f_3)\gamma_{2,1}(1,f_2^*|f_3)\\ \gamma_{2,1}(f_1,f_2|f_3) &= \gamma_{1,2}(f_1|f_2,f_3)\gamma_{2,1}(1,f_2|f_3)\end{aligned}$$

with the following properties:

1. $\gamma_{1,2}(\cdot|f_2,f_3):\mathcal{M}\to[0,1]$ is a state
2. $\gamma_{2,1}(\cdot,\cdot|1):\mathcal{M}\times\mathcal{M}\to[0,1]$ has the properties of two-dimensional joint distribution
3. if $\nu(f_1\wedge f_2\wedge f_3)=0$, then $\gamma_{2,1}(f_1,f_2|f_3)=\gamma_{1,2}(f_1|f_2,f_3)=0$.

Theorem 4. Let $\tau_1,\tau_2\in\mathcal{T}$ be some transformations. Then $\gamma_{1,2}:\mathcal{M}\times\mathcal{M}\times\mathcal{M}\to[0,1]$ and $\gamma_{2,1}:\mathcal{M}\times\mathcal{M}\times\mathcal{M}\to[0,1]$ defined by

$$\gamma_{1,2}(g|f_1,f_2) = \begin{cases}\frac{\int g\cdot\tau_1(f_1)\cdot\tau_2(f_2)\,\mathrm{d}\mu}{\int \tau_1(f_1)\cdot\tau_2(f_2)\,\mathrm{d}\mu}, & \text{if } \int\tau_1(f_1)\cdot\tau_2(f_2)\,\mathrm{d}\mu\neq 0\\ 0, & \text{otherwise}\end{cases} \tag{5}$$

$$\gamma_{2,1}(g,f_1|f_2) = \begin{cases}\frac{\int g\cdot\tau_1(f_1)\cdot\tau_2(f_2)\,\mathrm{d}\mu}{\int \tau_2(f_2)\,\mathrm{d}\mu}, & \text{if } \int\tau_2(f_2)\,\mathrm{d}\mu\neq 0\\ 0, & \text{otherwise}\end{cases} \tag{6}$$

are three-dimensional conditional states.

P r o o f of this statement is just a slight modification of that of Theorem 1. That is way it is omitted.

For the conditional state $\gamma_{1,2}(\cdot|\cdot,\cdot)$ the following holds

Lemma 3. Let $\gamma_{1,2}$ be a three-dimensional conditional state defined by formula 5 and let g,f_1,f_2 be any elements of $\mathcal{M}$. Then, in general, $\gamma_{1,2}(g|f_1,f_2)\neq\gamma_{1,2}(g|f_2,f_1)$. Particularly, there exist transformations τ_1,τ_2 and elements g,f_1 such that

$$\gamma_{1,2}(g|1,f_1)\neq\gamma_{1,2}(g|f_1,1).$$

We have $\gamma_{1,2}(g|1,f_1)\neq\gamma_{1,2}(g|f_1,1)$. Of course, in each of these two cases we can omit 1 and get (two-dimensional) conditional states (defined by Definition 5). However, these conditional states are different. By using the three-dimensional conditional states, we can define a three-dimensional joint distribution

Example 4. Assume that $\Omega = [0;1]$ and μ is the Lebesgue measure. For any element f of $\mathcal{F}$ let the transformation τ_2 be defined by the following

$$(\tau_2(f))(x) = \begin{cases} f(x), & \text{if } f(x) \in \{0, 0.5, 1\} \\ \frac{1}{\mu(A(f))} \int_{A(f)} f(x)\, \mathrm{d}\mu(x), & \text{if } 0 < f(x) < 0.5 \\ \frac{1}{\mu(B(f))} \int_{B(f)} f(x)\, \mathrm{d}\mu(x), & \text{if } 0.5 < f(x) < 1, \end{cases}$$

where $A(f) = \{x; 0 < f(x) < 0.5\}$, $B(f) = \{x; 0.5 < f(x) < 1\}$ (of course, the last two items hold provided $\mu(A(f)) > 0$ and $\mu(B(f)) > 0$).

Further, put

$$(\tau_1(f))(x) = \begin{cases} f(x), & \text{if } f(x) \in \{0, 1\} \\ \frac{1}{\mu(C(f))} \int_{C(f)} f(x)\, \mathrm{d}\mu(x), & \text{if } 0 < f(x) < 1 \end{cases}$$

where $C(f) = \{x; 0 < f(x) < 1\}$ (the second item holds provided $\mu(C(f)) > 0$).

Let $f, g, h \in \mathcal{M}$ be the following functions: $g(x) = x$,

$$f(x) = \begin{cases} x, & \text{if } x \le 0.5 \\ 1 - x, & \text{if } x > 0.5, \end{cases} \qquad h(x) = \begin{cases} 0, & \text{if } x \le 0.5 \\ 2(x - 0.5), & \text{if } x > 0.5. \end{cases}$$

Then

$$(\tau_2(h))(x) = \begin{cases} 0.25, & \text{if } 0.5 < x < 0.75 \\ 0.75, & \text{if } 0.75 < x < 1 \\ h(x), & \text{otherwise} \end{cases}$$

$$(\tau_2(g))(x) = \begin{cases} 0.25, & \text{if } 0 < x < 0.5 \\ 0.75, & \text{if } 0.5 < x < 1 \\ g(x), & \text{otherwise} \end{cases}$$

$$(\tau_1(g))(x) = \begin{cases} 1, & \text{if } x = 1 \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

$$(\tau_1(h))(x) = \begin{cases} 0, & \text{if } x \le 0.5 \\ \frac{1}{2}, & \text{if } 0.5 < x < 1 \\ 1, & \text{if } x = 1. \end{cases}$$

Hence

$$\gamma_{1,2}(f|g,h) = \frac{3}{16}, \qquad \gamma_{1,2}(f|h,g) = \frac{1}{4} = \nu(f),$$

i. e. f is dependent on the pair (g, h), but independent of (h, g).

Definition 9. The three-dimensional joint probability distribution, $p_3 : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0, 1]$, will be defined by

$$p_3(f_1, f_2, f_3) = \gamma_{1,2}(f_1|f_2, f_3) \cdot \gamma_{2,1}(1, f_2|f_3) \cdot \gamma_{1,2}(f_3|1, 1).$$

Basic properties of p_3 are the following:

Theorem 5. Let p_3 be a three-dimensional joint probability distribution and let f_1, f_2, f_3 be any elements of $\mathcal{M}$. Then the following hold

1. $p_3(f_1, 1, 1) = p_3(1, f_1, 1) = p(1, 1, f_1)$.
2. In general, any two permutations of arguments in p_3 may have different values. Particularly, there exist transformations τ_1, τ_2 and elements f_1, f_2 such that

$$p_3(1, f_1, f_2) \neq p_3(f_1, 1, f_2) \neq p_3(f_1, f_2, 1).$$

3. p_3 is additive in the first variable.
4. If any two elements out of f_1, f_2, f_3 are ν-orthogonal, $p_3(f_1, f_2, f_3) = 0$.

Proof. In fact, $p_3(f_1, f_2, f_3) = \int f_1 \cdot \tau_1(f_2) \cdot \tau_2(f_3)\, \mathrm{d}\mu$ where $\tau_1, \tau_2 \in \mathcal{T}$. If we choose elements f_1, f_2 and transformations τ_1, τ_2 in such a way that

$$f_1 \neq \tau_1(f_1), \quad f_2 \neq \tau_1(f_2) \neq \tau_2(f_2),$$

we get

$$p_3(1, f_1, f_2) \neq p_3(f_1, 1, f_2) \neq p_3(f_1, f_2, 1).$$

All the other properties are straightforward, implied by the properties of transformations from $\mathcal{T}$. □

Comment 6. In Theorem 5 we have $p_3(1, f_1, f_2) \neq p_3(f_1, 1, f_2) \neq p_3(f_1, f_2, 1)$. We can omit 1 in $p_3(f_1, 1, f_2)$ and $p_3(f_1, f_2, 1)$, and get two-dimensional joint distributions. But if $\tau_1 \neq \tau_2$, in each case we get a different two-dimensional joint distribution. However, if we put $\widetilde{p}(f_1, f_2) = p_3(1, f_1, f_2)$, then $\widetilde{p}$ is not additive in the first variable.

Comment 7. The fact that the order of the conditions can influence the value of the corresponding conditional state – and hence also the dependence or independence – may be important by time series analysis.

4.1. Some special transformations

In this section we show some special transformations by using of which the corresponding conditional states $\gamma_{2,1}$ and the three-dimensional joint distributions are monotone. In what follows, for each $f \in \mathcal{M}$, we denote $\widetilde{f} : \Omega \times [0, 1]$ the two-place function given by

$$\widetilde{f}(x, y) = f(x).$$

λ will denote the usual Lebesgue measure.

Notation. Let $\mathcal{F}_2$ denote the system of all functions from $\Omega \times [0, 1]$ to $[0, 1]$. We denote $\mathcal{E}$ the family of all transformations $\eta : \mathcal{M} \to \mathcal{F}_2$ fulfilling the following properties for all $f \in \mathcal{M}$ and all $x \in \Omega$:

1. $(\eta(f))\,(x, \cdot) : [0, 1] \to [0, 1]$ is a Lebesgue measurable function;
2. $f(x) = \int_0^1 (\eta(f))\,(x, y)\, \lambda(\mathrm{d}y)$;

3. if for any $g \in \mathcal{M}$ $f(x) \geq g(x)$ then for all $y \in [0,1]$

$$(\eta(f))\,(x,y) \geq (\eta(g))\,(x,y);$$

4. if $f(x) \in \{0, 0.5, 1\}$, then $(\eta(f))\,(x,y) = f(x)$;
5. if $f(x) < 0.5$ then for all $y \in [0,1]$

$$(\eta(f))\,(x,y) = 1 - (\eta(f^*))\,(x,y).$$

Theorem 6. Let $\eta_2, \eta_3 \in \mathcal{E}$ and $\gamma_{1,2} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0,1]$ and $\gamma_{2,1} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0,1]$ be defined by

$$\gamma_{1,2}\,(f_1|f_2,f_3) = \begin{cases} \frac{\int \int_0^1 \widetilde{f}_1 \cdot \eta_2(f_2) \cdot \eta_3(f_3)\, \mathrm{d}\lambda\, \mathrm{d}\mu}{\int \eta_2(f_2) \cdot \eta_3(f_3)\, \mathrm{d}\mu}, & \text{if } \int \eta_2\,(f_2) \cdot \eta_3\,(f_3)\, \mathrm{d}\mu \neq 0 \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

$$\gamma_{2,1}\,(f_1,f_2|f_3) = \begin{cases} \frac{\int \int_0^1 \widetilde{f}_1 \cdot \eta_2(f_2) \cdot \eta_3(f_3)\, \mathrm{d}\lambda\, \mathrm{d}\mu}{\int f_3\, \mathrm{d}\mu}, & \text{if } \int f_3\, d\mu \neq 0 \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

Then $\gamma_{1,2}$ and $\gamma_{2,1}$ are three-dimensional conditional states with the following properties:

$$\begin{aligned} \gamma_{1,2}(f_1|f_2,1) &= \gamma_{1,2}(f_1|1,f_2) \\ \gamma_{2,1}(f_1,f_2|1) &= \gamma_{2,1}(f_2,f_1|1) \\ \gamma_{2,1}(f_1,f_2|f_3) &\neq \gamma_{2,1}(f_2,f_1|f_3) \end{aligned}$$

$\gamma_{2,1}(\cdot,\cdot|\cdot)$ is additive in the first and monotone in the second variable.

P r o o f. In fact, we have embeded the MV-algebra $\mathcal{M}$ into an MV-algebra $\widetilde{\mathcal{M}}$ by putting $\widetilde{\Omega} = \Omega \times [0,1]$. I.e., $\gamma_{1,2}$ and $\gamma_{2,1}$, just defined, are conditional states in the sense of Definition 8. The monotonicity of $\gamma_{2,1}$ in the second coordinate is given by choosing the transformation $\eta_2 \in \mathcal{E}$. Furthermore, for any $f_1, f_2 \in \mathcal{M}$ and any $x \in \Omega$ we get

$$\int_0^1 \widetilde{f}_1(x,y) \cdot (\eta_2(f_2))(x,y)\, \lambda(\mathrm{d}y) = f_1(x) \cdot f_2(x).$$

This implies the propertis of $\gamma_{1,2}$ and $\gamma_{2,1}$. □

Example 5. Let $\Omega = [0,1]$ and μ the Lebesgue measure. Put $f_1(x) = 0.5,\ f_2(x) = 0.75,\ f_3(x) = \frac{1}{2}(x+1)$. Let the transformations $\eta_2, \eta_3 \in \mathcal{E}$ be the following

$$(\eta_2(g))\,(x,y) = \begin{cases} g(x), & \text{if } g(x) \in \{0, 0.5, 1\}, \\ \frac{1}{2}\left(1 + (1-y)^z\right), & \text{if } g(x) \in]0.5, 1[, \\ 1 - \frac{1}{2}\left(1 + (1-y)^z\right), & \text{if } g(x) \in]0, 0.5[, \end{cases}$$

$$(\eta_3(g))\,(x,y) = \begin{cases} g(x), & \text{if } g(x) \in \{0, 0.5, 1\}, \\ \frac{1}{2}\left(1 + y^z\right), & \text{if } g(x) \in]0.5, 1[, \\ 1 - \frac{1}{2}\left(1 + y^z\right), & \text{if } g(x) \in]0, 0.5[, \end{cases}$$

where
$$z = \frac{2-2g(x)}{2g(x)-1}.$$

Then
$$\begin{aligned}\gamma_{2,1}(f_1, f_2|1) &= \gamma_{2,1}(f_2, f_1|1) = \frac{3}{8}\\ \gamma_{2,1}(f_1, f_2|f_3) &= \frac{3}{16} + \frac{\ln 2}{8} \doteq 0.274\\ \gamma_{2,1}(f_2, f_1|f_3) &= \frac{3}{8} = \gamma_{2,1}(f_2, f_1|1).\end{aligned}$$
□

As a corollary to Theorem 6 we get the following:

Theorem 7. Let $\eta_2, \eta_3 \in \mathcal{E}$, $\gamma_{2,1} : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0, 1]$ be the three-dimensional conditional state, defined by formula (8) and let $p_3 : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to [0, 1]$ be defined by
$$p_3(f_1, f_2, f_3) = \gamma_{2,1}(f_1, f_2|f_3) \cdot \gamma_{2,1}(1, f_3|1).$$
Then p_3 is additive in the first variable and monotone in the second and third variables. Moreover, the following hold for the two-dimensional marginal distributions

- $p_3(f_1, f_2, 1) = p_3(f_2, f_1, 1) = p_3(f_1, 1, f_2) = p_3(f_2, 1, f_1)$
- $p_3(1, \cdot, \cdot)$ is monotone in both variables, but in general, they do not commute.

5. SOME COMMENTS CONCERNING OBSERVABLES AND THEIR JOINT DISTRIBUTION

First we recall the definition of a tribe and of an observable (see [4]).

Definition 10. An MV-algebra $\mathcal{M}$ will be called a tribe iff for any non-decreasing sequence of elements $\{f_i\}_{i=1}^{\infty}$ the following holds
$$\bigvee_{i=1}^{\infty} f_i = f \in \mathcal{M}.$$

From now on we will assume the MV-algebra to be a tribe.

Definition 11. An observable is a mapping λ from Borel sets $\mathcal{B}$ into the MV-algebra $\mathcal{M}$ such that

- $\lambda(R) = 1$
- If $A \cap B = \emptyset$, then $\lambda(A \cup B) = \lambda(A) \oplus \lambda(B)$ and $\lambda(A) \leq \lambda(B)^*$
- If $A_n \nearrow A$, then $\lambda(A_n) \nearrow \lambda(A)$.

In a natural way for each observable λ we can define also its cumulative distributive function F_λ and its expectation $E(\lambda)$ by
$$F_\lambda(x) = \nu\left(\lambda(]-\infty; x])\right)$$

$$E(\lambda) = \int_{-\infty}^{\infty} x F_\lambda(\mathrm{d}x)$$

(provided that it exists.)

Making a parallel to the joint distribution p from Definition 7, we can define the joint probability distribution $P_{\lambda,\kappa}$ for any pair of observables λ and κ by

$$P_{\lambda,\kappa}(A, B) = p(\lambda(A), \kappa(B))$$

where A, B are Borel sets. This can be interpreted as the measure of interaction of the observables λ and κ. The basic properties of $P_{\lambda,\kappa}$ can be just rewritten from Theorem 3. It is also possible to define the mean interaction of the observables λ and κ, denoted by $\mathcal{C}(\lambda, \kappa)$, as follows

$$\mathcal{C}(\lambda, \kappa) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x - E(\lambda))(y - E(\kappa))\, F_{\lambda,\kappa}(\mathrm{d}x, \mathrm{d}y)$$

where $F_{\lambda,\kappa}(x, y) = P_{\lambda,\kappa}(]-\infty; x],]-\infty; y])$ is the joint cummulative probability distribution. The investigation of the joint probability distributions (the interactions) of observables and of the corresponding mean interactions will be the topic of a next paper. Here we would like to point just to one very important property of the introduced notions, namely to the non-commutativity of the variables (observables) in the joint distribution $F_{\lambda,\kappa}$ and in the mean interaction $\mathcal{C}(\lambda, \kappa)$.

ACKNOWLEDGEMENT

This work was supported by Science and Technology Assistance Agency under the contract No. APVT-20-003204, and by the VEGA grant agency, grant numbers 1/3006/06 and 1/3014/06.

(Received May 11, 2005.)

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