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STATIONARY DISTRIBUTION OF ABSOLUTE AUTOREGRESSION

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A procedure for computation of stationary density of the absolute autoregression (AAR) model driven by white noise with symmetrical density is described. This method is used for deriving explicit formulas for stationary distribution and further characteristics of AAR models with given distribution of white noise. The cases of Gaussian, Cauchy, Laplace and discrete rectangular distribution are investigated in detail.

Keywords: absolute autoregression, stationary distribution, marginal distribution

AMS Subject Classification: 60G10

1. INTRODUCTION

Let $\{X_t\}$ be an ergodic Markov process with discrete time. Its stationary distribution π is given by integral equation

$$\pi(A) = \int_{-\infty}^{\infty} \mathsf{P}(A|x)\,\mathrm{d}\pi(x), \qquad A \in \mathcal{B} \tag{1}$$

where $\mathcal{B}$ is the σ-algebra of Borel sets and P is the conditional (i.e. transition) probability. Even in the simplest linear models such as AR(1) it is not easy to find a closed form solution π of (1). If the model for $\{X_t\}$ is non-linear, it is even more difficult to solve (1). One of the rare exceptions where a solution of (1) was found is so called absolute autoregression (AAR)

$$X_t = a|X_{t-1}| + \varepsilon_t \tag{2}$$

where $a \in (-1,1)$ and ε_t is a strict white noise (i.e., a sequence of i.i.d. random variables). Anděl et al. [4] proved that for $a \in (-1,0)$ and for $\varepsilon_t \sim \mathsf{N}(0,1)$ the stationary density of (2) is

$$h(x) = \sqrt{\frac{2(1-a^2)}{\pi}} \exp\{-(1-a^2)x^2/2\}\Phi(ax) \tag{3}$$

where Φ is the distribution function of $\mathsf{N}(0,1)$. It was derived that in this case

$$\mathsf{E}X_t = \sqrt{\frac{2}{\pi}}\frac{a}{\sqrt{1-a^2}}, \qquad \operatorname{var} X_t = \frac{\pi - 2a^2}{\pi(1-a^2)} \tag{4}$$

and the correlation coefficient between X_t and X_{t-1} is

$$\rho(a) = \frac{|a|\pi + 2a^2\sqrt{1-a^2} - 2a^2 - 2|a|\operatorname{arctg}\sqrt{a^{-2}-1}}{\pi - 2a^2}. \tag{5}$$

Let $\mathsf{C}(\alpha, \beta)$ be the Cauchy distribution with the density

$$f(x) = \frac{1}{\pi}\frac{\beta}{\beta^2 + (x-\alpha)^2}.$$

Consider the model (2) with $a \in (-1, 0)$ and $\varepsilon_t \sim \mathsf{C}(0,1)$. Define $A = |a|/(1-|a|)$. Anděl and Bartoň [3] proved that X_t in (2) has the stationary density

$$h(x) = \frac{2A}{\pi^2}\left\{\frac{(1+A)\pi}{2A[(1+A)^2 + x^2]} - \frac{x\ln[A^{-2}(1+x^2)] + (A^2 - 1 + x^2)\operatorname{arctg} x}{4A^2x^2 + (1 - A^2 + x^2)^2}\right\}. \tag{6}$$

Chan and Tong [5] and Tong [7, p. 141] simplified the methods used for derivation of (3) and (6). Their procedure can be summarized as follows. Let ε_t in (2) have a symmetric density f. Let g be the stationary density of the AR(1) process ξ_t given by

$$\xi_t = a\xi_{t-1} + \varepsilon_t. \tag{7}$$

Then the stationary density h of X_t in (2) is

$$h(y) = 2\int_0^\infty g(x) f(y - ax)\,\mathrm{d}x. \tag{8}$$

(The authors overlooked that the factor 2 must be introduced in the last formula.) Let us remark that if we have a guess that a function h could be a stationary density of X_t then it is easy to verify it from (2).

Problems in non-linear time series models are usually quite complicated and must be solved numerically or by simulations. It is important to have a few explicit solutions because they allow to compare accuracy of numerical methods with the exact results. One such solution for the first-order threshold autoregression with Laplace white noise has been deduced quite recently by Loges [6]. In this paper we derive some new stationary distributions of the AAR process $\{X_t\}$ given by (2).

2. NORMAL DISTRIBUTION

We mentioned above that formulas (3), (4), and (5) were derived under the assumptions that $\varepsilon_t \sim \mathsf{N}(0,1)$ and $a \in (-1,0)$. We generalize the results to $a \in (-1,1)$.

If $\varepsilon_t \sim \mathsf{N}(0,1)$ then ξ_t in (7) has the distribution $\mathsf{N}(0, \frac{1}{1-a^2})$. From (8) we get that the stationary density h of X_t is

$$h(y) = 2\int_0^\infty \sqrt{\frac{1-a^2}{2\pi}} \exp\{-(1-a^2)x^2/2\} \frac{1}{\sqrt{2\pi}} \exp\{-(y-ax)^2/2\}\,\mathrm{d}x.$$

Direct integration leads to formula (3) and we can see that (3) and (4) are valid for $a \in (-1,1)$.

Using the same procedures as in Theorem 4.3 in Anděl et al. [4], we can derive for $a \in (0,1)$ that the correlation coefficient $\rho(a)$ between X_t and X_{t-1} is also given by formula (5). This means that $\rho(-a) = \rho(a)$, $a \in (-1,1)$.

The density h is plotted in Figure 1 (for $a = -0.8$) and in Figure 2 (for $a = 0.8$). Expectation $\mathsf{E}X_t$ and variance $\operatorname{var} X_t$ as functions of a given by (4) are introduced in Figure 3 and Figure 4, respectively. In Figure 5 we can see $\rho(a)$, which is defined by (5).

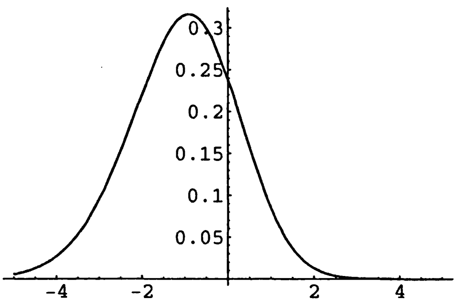

Fig. 1. Function h for $a = -0.8$.

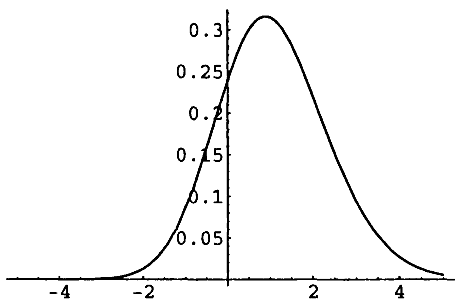

Fig. 2. Function h for $a = 0.8$.

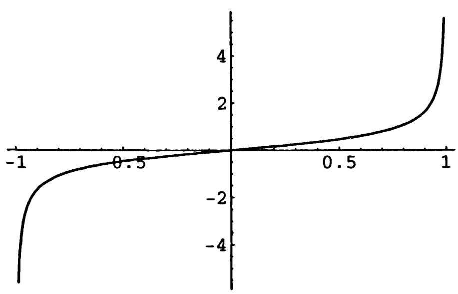

Fig. 3. Expectation $\mathsf{E}X_t$.

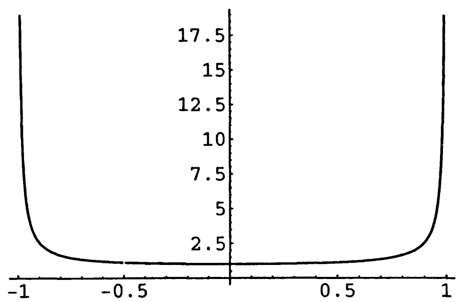

Fig. 4. Variance $\operatorname{var} X_t$.

The joint stationary density of (X_s, X_{s-1}) is

$$p_2(x_s, x_{s-1}) = \begin{cases} \frac{\sqrt{1-a^2}}{\pi} \exp\left\{-\frac{1-a^2}{2} x_{s-1}^2\right\} \Phi(a x_{s-1}) \exp\left\{-\frac{(x_s + a x_{s-1})^2}{2}\right\} & \text{for } x_{s-1} < 0, \\ \frac{\sqrt{1-a^2}}{\pi} \exp\left\{-\frac{1-a^2}{2} x_{s-1}^2\right\} \Phi(a x_{s-1}) \exp\left\{-\frac{(x_s - a x_{s-1})^2}{2}\right\} & \text{for } x_{s-1} > 0. \end{cases}$$

The joint stationary density of (X_s, X_{s-2}) is

$$\begin{aligned} p_3(x_s, x_{s-2}) =& \frac{1}{\pi}\sqrt{\frac{1-a^2}{1+a^2}}\Phi(ax_{s-2}) \\ & \times \left(\exp\left\{ -\frac{x_s^2 - 2a^2 x_s x_{s-2} + x_{s-2}^2}{2(1+a^2)} \right\} \Phi\left[\frac{a(x_s + x_{s-2})}{\sqrt{1+a^2}} \right] \right. \\ & \left. + \exp\left\{ -\frac{x_s^2 + 2a^2 x_s x_{s-2} + x_{s-2}^2}{2(1+a^2)} \right\} \Phi\left[\frac{a(x_s - x_{s-2})}{\sqrt{1+a^2}} \right] \right). \end{aligned}$$

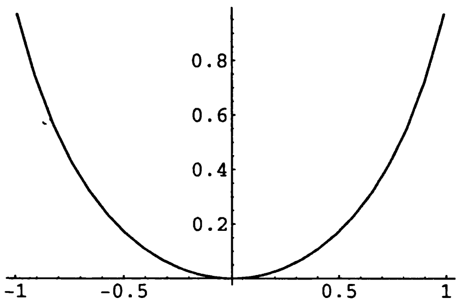

Fig. 5. Correlation coefficient $\rho(a)$.

The functions p_2 and p_3 for $a = 0.8$ are introduced in Figure 6 and Figure 7, respectively.

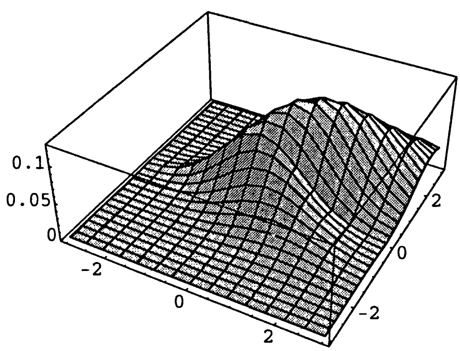

Fig. 6. Function $p_2(x_s, x_{s-1})$.

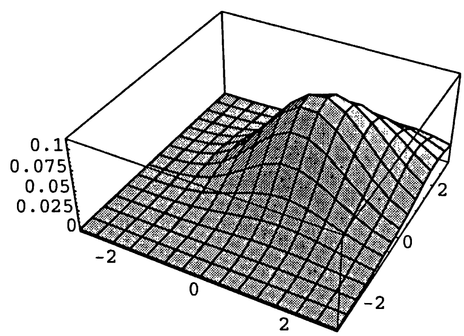

Fig. 7. Function $p_3(x_s, x_{s-2})$.

3. CAUCHY DISTRIBUTION

If $\varepsilon_t \sim \mathsf{C}(0,1)$ then the stationary distribution of the process $\xi_t = a\xi_{t-1} + \varepsilon_t$ with $|a| < 1$ is $\mathsf{C}(0,Q)$ where $Q = 1/(1-|a|)$. The corresponding density is

$$g(x) = \frac{1}{\pi}\frac{Q}{Q^2+x^2}.$$

The stationary density of X_t can be calculated from (8). We obtain

$$h(y) = \frac{2Q}{\pi^2}\int_0^\infty \frac{1}{Q^2+x^2}\frac{1}{1+(y-ax)^2}\,\mathrm{d}x.$$

Again define $A = |a|/(1-|a|)$. Let $a \in (0,1)$. After some computations, which can be simplified using program package Mathematica, we get

$$h(y) = \frac{2A}{\pi^2}\left\{\frac{(1+A)\pi}{2A[(1+A)^2+y^2]} + \frac{y\ln[A^{-2}(1+y^2)] + (A^2-1+y^2)\operatorname{arctg} y}{4A^2y^2+(1-A^2+y^2)^2}\right\}. \quad (9)$$

The density h for $a = -0.8$ defined by (6) is plotted in Figure 8. If $a = 0.8$ then h is defined by (9) and its graph can be found in Figure 9.

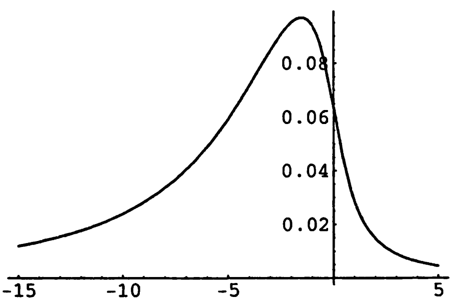

Fig. 8. Function h for $a = -0.8$.

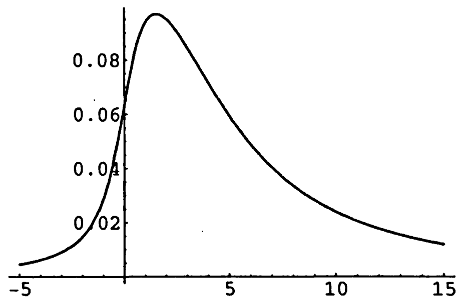

Fig. 9. Function h for $a = 0.8$.

4. A DISCRETE WHITE NOISE

Assume that $a = \frac{1}{2n}$ and

$$\varepsilon_t = \frac{2i-1}{2n}b \text{ with probability } \frac{1}{2n}$$

for $i = -n+1, \ldots, n$ where $b > 0$ and $n = 1, 2, \ldots$. Then the rectangular distribution $\mathsf{R}(-b, b)$ is the stationary distribution of the process ξ_t in (7) (see Anděl [1,2]). Let χ_B be the characteristic function of the set B and let $\delta_c(x)$ be the Dirac δ-function, i. e. $\int_{-\infty}^{\infty} \delta_c(x)\mathrm{d}x = 1$ where

$$\delta_c(x) = \begin{cases} \infty & \text{for } x = c, \\ 0 & \text{otherwise.} \end{cases}$$

The distribution of ε_t can be described by the generalized density

$$f(x) = \frac{1}{2n} \sum_{i=-n+1}^{n} \delta_{\frac{2i-1}{2n}b}(x).$$

A random variable with the density f is discrete and it reaches values $\frac{2i-1}{2n}b$, $i=-n+1,\dots,n$, each with probability $\frac{1}{2n}$. A straightforward calculation gives

$$h(y) = \frac{1}{b}\int_0^\infty \chi_{[-b,b]}(x) f\left(y - \frac{1}{2n}x\right) \mathrm{d}x = \frac{1}{b}\sum_{i=-n+1}^{n} \chi_{[\frac{b}{n}i - \frac{b}{2n}, \frac{b}{n}i]}(y).$$

It is easy to verify that h is really the stationary density of the AAR process (2). Further we obtain

$$\mathsf{E}X_t = \frac{b}{4n}, \qquad \mathsf{E}X_t^2 = \frac{b^2}{3}, \qquad \operatorname{var} X_t = \frac{b^2(16n^2-3)}{48n^2}.$$

Since

$$X_t = \frac{1}{2n}|X_{t-1}| + \varepsilon_t, \qquad \mathsf{E}\varepsilon_t = 0,$$

we have

$$\mathsf{E}X_tX_{t-1} = \frac{1}{2n}\mathsf{E}|X_{t-1}|X_{t-1} = -\frac{1}{2n}\int_{-\infty}^0 x^2 h(x)\,\mathrm{d}x + \frac{1}{2n}\int_0^\infty x^2 h(x)\,\mathrm{d}x = \frac{b^2}{8n^2}$$

and thus

$$\rho = \operatorname{corr}(X_t, X_{t-1}) = \frac{\mathsf{E}X_tX_{t-1} - (\mathsf{E}X_t)^2}{\operatorname{var} X_t} = \frac{3}{16n^2-3}.$$

A simple case arises for $n=1$ when we have the process $X_t = \frac{1}{2}|X_{t-1}| + \varepsilon_t$ where

$$\varepsilon_t = \begin{cases} -\frac{b}{2} & \text{with probability } \frac{1}{2}, \\ \frac{b}{2} & \text{with probability } \frac{1}{2}. \end{cases}$$

The stationary density of X_t is

$$h(y) = \frac{1}{b}\left\{\chi_{[-\frac{b}{2},0]}(y) + \chi_{[\frac{b}{2},b]}(y)\right\}$$

and $\rho = 3/13 = 0.231$.

Now, we consider the case $a = \frac{1}{2n+1}$ and

$$\varepsilon_t = \frac{2i}{2n+1}b \text{ with probability } \frac{1}{2n+1}$$

for $i=-n,\dots,n$ where $b>0$ and $n=1,2,\dots$. The stationary distribution of the process ξ_t in (7) is again the rectangular distribution $\mathsf{R}(-b,b)$ (see Anděl [1, 2]). The generalized density of ε_t is

$$f(x) = \frac{1}{2n+1}\sum_{i=-n}^{n} \delta_{\frac{2i}{2n+1}b}(x)$$

and we obtain

$$h(y)=\frac{1}{b}\int_0^\infty \chi_{[-b,b]}(x)f\left(y-\frac{1}{2n+1}x\right)\,\mathrm{d}x=\frac{1}{b}\sum_{i=-n}^{n}\chi_{\left[\frac{2i}{2n+1}b,\frac{2i+1}{2n+1}b\right]}(y).$$

It can be verified that h is the stationary density of the process $\{X_t\}$ in (2). Further we obtain

$$\mathsf{E}X_t=\frac{b}{2(2n+1)},\qquad \mathsf{E}X_t^2=\frac{b^2}{3},\qquad \operatorname{var}X_t=b^2\frac{16n^2+16n+1}{12(2n+1)^2}$$

and

$$\mathsf{E}X_tX_{t-1}=b^2\frac{6n^2+6n+1}{3(2n+1)^4},\qquad \rho=\frac{12n^2+12n+1}{(2n+1)^2(16n^2+16n+1)}.$$

5. LAPLACE DISTRIBUTION

Laplace distribution $\mathsf{La}(b)$ has the density

$$p(x)=\frac{1}{2b}\exp\left\{-\frac{|x|}{b}\right\}$$

where $b>0$ is a parameter. Assume that $a\in(-1,1)$ and that $\{Z_t\}$ are i.i.d. $\mathsf{La}(b)$ random variables. Let the strict white noise be defined by

$$\varepsilon_t=\begin{cases}0 & \text{with probability } a^2,\\ Z_t & \text{with probability } 1-a^2.\end{cases}$$

Then $\xi_t=a\xi_{t-1}+\varepsilon_t$ has the stationary density $p(x)$ (see Anděl [1,2]) and

$$h(y)=2\int_0^\infty p(x)[a^2\delta_0(y-ax)+(1-a^2)p(y-ax)]\,\mathrm{d}x.$$

From here we obtain that for $a\in(-1,1)$ the stationary density of the process $\{X_t\}$ is given by

$$h(y)=\begin{cases}\dfrac{1+a}{2b}\exp\left\{-\dfrac{y}{b}\right\} & \text{for } y>0,\\[2mm] \dfrac{1-a}{2b}\exp\left\{\dfrac{y}{b}\right\} & \text{for } y<0.\end{cases}$$

Further we get

$$\mathsf{E}X_t=ab,\qquad \mathsf{E}X_t^2=2b^2,\qquad \operatorname{var}X_t=b^2(2-a^2)$$

and

$$\mathsf{E}X_tX_{t-1}=2a^2b^2,\qquad \rho=\frac{a^2}{2-a^2}.$$

If $a=0$ then $\rho=0$ and $\rho\to1$ as $a\to\pm1$.

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