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MULTIPLICATION, DISTRIBUTIVITY AND FUZZY–INTEGRAL III[1]

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Based on the results of generalized additions, multiplications and differences proven in Part I and II of this paper a framework for a general integral is presented. Moreover it is shown that many results of the literature are contained as special cases in our results.

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11. A GENERAL FUZZY–INTEGRAL

We assume that the reader is familiar with the notations and results in Part I and II. In these two papers we introduced the necessary preparations for a framework for a generalized integral which we will call fuzzy integral.

Let us start by agreeing that in this paper now $[A, B] = [0, B]$, $0 < B \leq \infty$. Thus the assumption "Let $\triangle$ be a pseudo-addition and let $\diamond$ be a pseudo-multiplication" always means: let "$\triangle : [0, B]^2 \to [0, B]$ be a pseudo-addition and let $\diamond : [0, B]^2 \to [0, B]$ be a pseudo-multiplication". Moreover, we agree on some notations.

$(X, \mathcal{A}, \mu)$ is a fuzzy measure space, where $(X, \mathcal{A})$ is a measure space and $\mu : \mathcal{A} \to [0, B]$ is a fuzzy measure (see Section 2).

This is a general assumption in all following definitions and theorems.

For a simple function $f : X \to [0, B]$ we always assume the representation

$$f = \sum_{i=1}^{n} a_i 1_{A_i}, \quad a_0 = 0 \leq a_1 \leq \cdots \leq a_n \leq B, \tag{88}$$

where $A_1, \ldots, A_n \in \mathcal{A}$ are pairwise disjoint. Moreover, we define for a right unit e

$$e_A = e \cdot 1_A, \quad A \in \mathcal{A}. \tag{89}$$

We want to define the integral for simple functions. To do this we first show different representations for simple functions (see Section 2).

[1]This paper is a continuation of our papers Multiplication, Distributivity and Fuzzy-Integral I and II in Kybernetika No.3,4/2005. We continue the enumeration of formulas, definitions, lemmas and theorems.

Lemma 6. Let $\triangle, \amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL*), (RU) and (Z). If a simple function f has the form (88), then we get

$$\text{(a)} \qquad f = \amalg_{i=1}^{n}[\, a_i \diamond e_{A_i}\,]. \tag{90}$$

$$\text{(b)} \qquad f = \amalg_{i=1}^{n}[(a_i -_{\triangle} a_{i-1}) \diamond e_{B_i}], \quad \text{where} \quad B_i = \bigcup_{j=i}^{n} A_j. \tag{91}$$

$$\text{(c)} \qquad f = \sum_{i=1}^{n}[(a_i -_{\triangle} a_{i-1})\, 1_{B_i}]. \tag{92}$$

Proof. First we note that $\triangle = \amalg$ by Lemma 3 (b). Let $m \in \{1, 2, \ldots, n\}$ and let $x \in A_m$.

To prove (a) we use (RU) and (Z) to get $f(x) = a_m = a_m \diamond e = \amalg_{i=1}^{n}[a_i \diamond e_{A_i}(x)]$.

To prove (b) we use (RU), Theorem 11 (d) and (Z) to arrive at $f(x) = a_m = a_m \diamond e = \amalg_{i=1}^{m}[(a_i -_{\triangle} a_{i-1}) \diamond e] = \amalg_{i=1}^{n}[(a_i -_{\triangle} a_{i-1}) \diamond e_{B_i}(x)]$ (note that $e_{B_i}(x) = 0$ for $i > m$).

For the proof of (c) choose $\triangle = \amalg := +$, $\diamond := \cdot$, the usual addition and multiplication and $a -_{\triangle} b = a -_{\amalg} b = 0 \vee (a - b)$ (which is a special case of Example 2):

$$a \triangle B = a \amalg b = (a + b) \wedge \infty, \quad a \diamond b = (a \cdot b) \wedge \infty = a \cdot b,$$
$$a -_{\triangle} b = a -_{\amalg} b = 0 \vee (a - b), \quad a -_{\triangle} b = a -_{\amalg} b = 0 \vee (a - b)$$
$$(\text{if } a = b = \infty \text{ then } a - b := 0)).$$

Using part(b) we get (c). □

We now define the integral of a simple function (here no right unit is needed).

Definition 7. Let $\triangle, \amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL*) and (Z). Then the fuzzy-integral of a simple function f (see (88)) with respect to μ is defined by

$$(\diamond) \int f \, \mathrm{d}\mu := (\diamond) \int_X f \, \mathrm{d}\mu := \amalg_{i=1}^{n}[(a_i -_{\triangle} a_{i-1}) \diamond \mu(B_i)]. \tag{93}$$

(see (90) for the sets B_i). If $U \subset X, U \in \mathcal{A}$, then we define

$$(\diamond) \int_U f \, \mathrm{d}\mu := \amalg_{i=1}^{n}[(a_i -_{\triangle} a_{i-1}) \diamond \mu(B_i \cap U)]. \tag{94}$$

Note that the fuzzy-integral is defined exactly like the t-conorm integral.

Because of the last statement in Theorem 11, the fuzzy integral for simple functions is well-defined:

— If $a_i = a_{i+1}$ for some $i \in \{1, 2, \ldots, n-1\}$ then the $(i+1)$th summand can be omitted (using Lemma 5 (e) and (Z)): $(a_{i+1} -_{\triangle} a_i) \diamond \mu(B_{i+1}) = 0 \diamond \mu(B_{i+1}) = 0$.

— If $a_m = 0$ for some $m \in \{1,2,\ldots,n\}$ then $a_i = 0$ for all $i \in \{1,2,\ldots,m\}$. Again by Lemma 5 (e) and (Z) we arrive at $(a_i -_\triangle a_{i-1}) \diamond \mu(B_i) = 0, i \in \{1,2,\ldots,m\}$.

— If $A_i = \emptyset$ for some $i \in \{1,2,\ldots,n\}$ then we consider two cases:

If $i = n$ then $(a_n -_\triangle a_{n-1}) \diamond \mu(B_n) = (a_n -_\triangle a_{n-1}) \diamond \mu(A_n) = 0$.

If $i < n$ then we get (because of $\mu(B_{i+1}) = \mu(B_i)$ and Theorem 11 (a))

$$[(a_{i+1} -_\triangle a_i) \diamond \mu(B_{i+1})] \amalg [(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)] = (a_{i+1} -_\triangle a_{i-1}) \diamond \mu(B_{i+1}).$$

The following result shows that the fuzzy integral satisfies some expected properties.

Lemma 7. Let $\triangle, \amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL*) and (Z). Then the fuzzy-integral of a simple function f with respect to μ satisfies

$$\text{(a)} \qquad U \in \mathcal{A} \Longrightarrow (\diamond) \int_U f \, \mathrm{d}\mu = (\diamond) \int_X f 1_U \mathrm{d}\mu. \tag{95}$$

$$\text{(b)} \qquad U \in \mathcal{A} \wedge (\mu(U) = 0) \Longrightarrow (\diamond) \int_U f \, \mathrm{d}\mu = 0. \tag{96}$$

$$\text{(c)} \qquad f \le g \Longrightarrow (\diamond) \int f \, \mathrm{d}\mu \le (\diamond) \int g \, \mathrm{d}\mu. \tag{97}$$

$$\text{(d)} \qquad (\mathrm{LU}) \wedge (M \in \mathcal{A}) \Longrightarrow (\diamond) \int \tilde{e} \, 1_M \mathrm{d}\mu = \mu(M). \tag{98}$$

P r o o f. (a) Let f have the representation (88), so that the representation (90) is valid, too. Using $1_U \cdot f = \sum_{i=1}^n a_i \cdot 1_{A_i \cap U}$ we get

$$\begin{aligned}(\diamond) \int f 1_U \mathrm{d}\mu &= \amalg_{i=1}^n \Big[(a_i -_\triangle a_{i-1}) \diamond \mu\Big(\bigcup_{j=i}^n (A_j \cap U)\Big)\Big] \\ &= \amalg_{i=1}^n [(a_i -_\triangle a_{i-1}) \diamond \mu(B_i \cap U)] = (\diamond) \int_U f \, \mathrm{d}\mu.\end{aligned}$$

Using that μ is isotone and (Z), we get (b).

To prove (c) we prove first the following statement:

(1) $\bigwedge_{M \in \mathcal{A}} [f|_M = a \in [0, B] \wedge b \in [a, B] \Rightarrow (\diamond) \int f \, \mathrm{d}\mu \le (\diamond) \int [f + (b-a) \, 1_M] \, \mathrm{d}\mu]$.

W.l.o.g. let $f \neq 0$ (otherwise (1) is trivial)).

Thus we have for f a representation of the form (88) with $a_0 := 0 < a_1 < a_2 < \cdots < a_n \le B := a_{n+1}$ and where $A_i \in \mathcal{A}$ are pairwise disjoint and $B_i := \bigcup_{j=i}^n A_j, B_{n+1} := \emptyset, 1 \le i \le n$.

Since $f|_M$ is constant there is $k \in \{1,2,\ldots,n\}$ satisfying $a = a_k$ and $M \subset A_k$.

Since the case $b = a$ is trivial we may assume that $b > a_k$. Then there exists exactly one $l \in \{k, k+1, \dots, n\}$ with $a_l < b \le a_{l+1}$. Thus we have $f + (b-a)1_M = \sum_{i=1}^{k-1} a_i 1_{A_i} + a_k 1_{A_k \setminus M} + \sum_{i=k+1}^{l} a_i 1_{A_i} + b 1_M + \sum_{i=l+1}^{n} a_i 1_{A_i}$.

By Definition 7 and $(A_k \setminus M) \cup \bigcup_{i=k+1}^{l} A_i \cup M \cup \bigcup_{i=l+1}^{n} A_i = B_k$ we get

$$\begin{aligned}(\diamond)\int [f + (b-a)1_M]\,\mathrm{d}\mu &= (\amalg_{i=1}^{k}[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)])\, \amalg \\ &\amalg(\amalg_{i=k+1}^{l}[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i \cup M)])\, \amalg\, [(b -_\triangle a_l) \diamond \mu(B_{l+1} \cup M)]\, \amalg \\ &\amalg[(a_{l+1} -_\triangle b) \diamond \mu(B_{l+1})]\, \amalg\, (\amalg_{i=l+2}^{n}[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)])\end{aligned}$$

(if $k = l$ or $l \ge n-1$ then the corresponding "empty sums" are defined to be 0. If $l = n$ note that $\mu(B_{l+1}) = \mu(\emptyset) = 0$).

Using the last equality, the monotonicity of μ and $\diamond$ and Theorem 11 (a) we arrive at

$$\begin{aligned}&(\diamond)\int [f + (b-a)1_M]\,\mathrm{d}\mu \ge (\amalg_{i=1}^{k}[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)])\, \amalg \\ &\amalg(\amalg_{i=k+1}^{l}[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)])\, \amalg \\ &\quad \amalg[(b -_\triangle a_l) \diamond \mu(B_{l+1})]\, \amalg\, [(a_{l+1} -_\triangle b) \diamond \mu(B_{l+1})]\, \amalg\, (\amalg_{i=l+2}^{n}[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)]) \\ &\ge (\amalg_{i=1}^{k}[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)])\, \amalg\, (\amalg_{i=k+1}^{l}[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)])\, \amalg \\ &\quad \amalg[(a_{l+1} -_\triangle a_l) \diamond \mu(B_{l+1})]\, \amalg\, (\amalg_{i=l+2}^{n}[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)]) \\ &= (\diamond)\int f\,\mathrm{d}\mu.\end{aligned}$$

In a next step we show that we have for all a_i, $b_i \in [0, B]$ satisfying $a_i \le b_i$ and for all $A_i \in \mathcal{A}$, $1 \le i \le n, n \in \mathbb{N}$:

$$\bigwedge_{m \in \{1,2,\dots,n\}} (\diamond)\int \Big[\sum_{i=1}^{n} a_i 1_{A_i}\Big]\,\mathrm{d}\mu \le (\diamond)\int \Big[\sum_{i=1}^{m} b_i 1_{A_i} + \sum_{i=m+1}^{n} a_i 1_{A_i}\Big]\,\mathrm{d}\mu.$$

Denoting the last statement by A(m) we prove by induction on $m \in \mathbb{N}$.

To prove A(1) we use (1) with $f := \sum_{i=1}^{n} a_i 1_{A_i}, M := A_1, a := a_1, b := b_1$ to get

$$(\diamond)\int \sum_{i=1}^{n} a_i 1_{A_i} \mathrm{d}\mu \le (\diamond)\int \Big[\sum_{i=1}^{n} a_i 1_{A_i} + (b_1 - a_1)1_{A_1}\Big]\,\mathrm{d}\mu = (\diamond)\int \Big[b_1 1_{A_1} + \sum_{i=2}^{n} a_i 1_{A_i}\Big]\,\mathrm{d}\mu.$$

For the step $A(m) \Rightarrow A(m+1)$ we use again (1) with $f = \sum_{i=1}^{m} b_i 1_{A_i} + \sum_{i=m+1}^{n} a_i 1_{A_i}, M := A_{m+1}, a := a_{m+1}$ and $b := b_{m+1}$ and arrive at

$$\begin{aligned}(\diamond)\int \Big[\sum_{i=1}^{n} a_i 1_{A_i}\Big]\,\mathrm{d}\mu &\le (\diamond)\int \Big[\sum_{i=1}^{m} b_i 1_{A_i} + \sum_{i=m+1}^{n} a_i 1_{A_i}\Big]\mathrm{d}\mu \\ \le\ &(\diamond)\int \Big[\sum_{i=1}^{n} b_i 1_{A_i} + \sum_{i=m+1}^{n} a_i 1_{A_i} + (b_{m+1} - a_{m+1})1_{A_{m+1}}\Big]\mathrm{d}\mu \\ =\ &(\diamond)\int \Big[\sum_{i=1}^{m+1} b_i 1_{A_i} + \sum_{i=m+2}^{n} a_i 1_{A_i}\Big]\mathrm{d}\mu.\end{aligned}$$

Finally let f, g be arbitrary simple functions with $f \leq g$. Then there is (using that the collection of the intersections of two measurable partitions of X is again a measurable partition of X) a representation

$$f = \sum_{i=1}^{n} a_i 1_{A_i}, \quad g = \sum_{i=1}^{n} b_i 1_{A_i},$$

where the $A_i \in \mathcal{A}$, $1 \leq i \leq n$ are pairwise disjoint. Because of $f \leq g$ we have $a_i \leq b_i$, $1 \leq i \leq n$. Using A(n) we get $(\diamond) \int f \, d\mu \leq (\diamond) \int g \, d\mu$.

Proof of (d):

$$(\diamond) \int \tilde{e}\, 1_M d\mu = \tilde{e} \diamond \mu(M) = \mu(M).$$

This finishes the proof of Lemma 7. □

Now we define the fuzzy integral for arbitrary measurable functions in the usual way (see Section 3).

Definition 8. Let $\triangle, \amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL*) and (Z). Then the fuzzy-integral of a measurable function f with respect to μ on $U \subset X, U \in \mathcal{A}$ is defined by

$$\int_U f \, d\mu = (\diamond) \int_U f \, d\mu := \sup \left\{ (\diamond) \int_U s \, d\mu : s \leq f, s \, \text{simple} \right\}. \tag{99}$$

It is clear that (because of Lemma 7 (c)) the integral for measurable functions extends the integral for simple functions. Using Definition 8 we get again (95) – (97), but now for measurable functions (only (DL*) and (Z) are needed). Note that for the definition of the fuzzy integral only a fuzzy measure is needed. Additional properties of a fuzzy measure are necessary if additional properties of the integral are proven.

Our first nontrivial result is the Theorem on monotone convergence if μ is continuous from below.

Theorem 12. Let $\triangle, \amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL*) and (Z). Let $\mu : \mathcal{A} \to [0, B]$ be a fuzzy measure which is continuous from below, and let (f_n) be a sequence of measurable functions $f_n : X \to [0, B]$ satisfying $f_n \leq f_{n+1}, n \in \mathbb{N}$ and $f := \lim f_n$. Then we have

$$\lim_{n \to \infty} (\diamond) \int f_n \, d\mu = (\diamond) \int f \, d\mu.$$

P r o o f. The inequality $\lim_{n \to \infty} (\diamond) \int f_n \, d\mu \leq (\diamond) \int f \, d\mu$ follows from the property (97) for measurable functions.

(1) To prove the reverse inequality we assume:

$$L := \lim_{n\to\infty} (\diamond) \int f_n \mathrm{d}\mu < (\diamond) \int f \,\mathrm{d}\mu.$$

By Definition 8 there is a simple function s satisfying

$$s \le f, \quad (\diamond) \int s \,\mathrm{d}\mu > L. \tag{100}$$

By Definition 7 we have $s \neq 0$ and thus s has a representation

$$s = \sum_{i=1}^{n} a_i 1_{A_i}, \quad a_0 := 0 < a_1 \le a_2 \le \cdots \le a_m \le B,$$

$$A_i \in \mathcal{A} \text{ pairwise disjoint}, \quad B_i := \bigcup_{j=i}^{m} A_j, \quad 1 \le i \le m. \tag{101}$$

(2) We now define a sequence of simple functions (s_n) by

$$\bigwedge_{n\in\mathbb{N}} \bigwedge_{i\in\{1,2,\ldots,m\}} a_{n,i} := \left[\left(a_i - \frac{1}{n}\right) \vee \frac{a_i}{2}\right] \wedge n, \quad \bigwedge_{n\in\mathbb{N}} s_n := \sum_{i=1}^{m} a_{n,i} 1_{A_i} \tag{102}$$

and get: $\bigwedge_{n\in\mathbb{N}} a_{n,0} := 0 < a_{n,1} \le a_{n,2} \le \cdots \le a_{n,m} \le B$, $a_{n,i} \uparrow a_i$ and thus $s_n \uparrow s$.

Let us now prove

$$(*) \quad \bigwedge_{n\in\mathbb{N}} \bigwedge_{x\in X} [f(x) > 0 \Rightarrow s_n(x) < f(x)] :$$

Case 1. If $\bigvee_{i\in\{1,2,\ldots,m\}} x \in A_i$ then (using (102), (101), (100)) we get $s_n(x) = [(a_i - \frac{1}{n}) \vee \frac{a_i}{2}] \wedge n < a_i = s(x) \le f(x)$.

Case 2. If $\bigwedge_{i\in\{1,2,\ldots,m\}} x \notin A_i$ then (using (102)) $s_n(x) = 0 < f(x)$.

(3) We show $\lim_{n\to\infty} (\diamond) \int s_n \mathrm{d}\mu > L$:

The function $\amalg_{i=1}^{m} : [0,B]^m \to [0,B]$ is continuous and isotonic in each place, and thus $\amalg_{i=1}^{m}$ is continuous on $[0,B]^m$ (see [10]). Using this fact and the properties of $(a_{n,i})$, Definition 7, (102), the monotonicity properties of $-_\triangle$ and the fact that

$$\left(\sup_{m\in M} \lambda(m)\right) \circledcirc c = \sup_{m\in M} (\lambda(m) \circledcirc c), \text{ where } \circledcirc \in \{\diamond, -_\triangle\}$$

(here M is a set, $\lambda : M \to [0,B]$ is an arbitrary function; see also (87)), we arrive at

$$\lim_{n\to\infty} (\diamond) \int s_n \mathrm{d}\mu$$

$$\begin{aligned}
&= \lim_{n\to\infty} (\amalg_{i=1}^{m}[(a_{n,i} -_{\triangle} a_{n,i-1}) \diamond \mu(B_i)]) \\
&\geq \lim_{n\to\infty} (\amalg_{i=1}^{m}[(a_{n,i} -_{\triangle} a_{i-1}) \diamond \mu(B_i)]) \\
&= \amalg_{i=1}^{m}\Big(\lim_{n\to\infty} [(a_{n,i} -_{\triangle} a_{i-1}) \diamond \mu(B_i)]\Big) \\
&= \amalg_{i=1}^{m}\Big(\sup_{n\in\mathbb{N}} [(a_{n,i} -_{\triangle} a_{i-1}) \diamond \mu(B_i)]\Big) \\
&= \amalg_{i=1}^{m}\Big(\Big[\sup_{n\in\mathbb{N}} (a_{n,i} -_{\triangle} a_{i-1})\Big] \diamond \mu(B_i)\Big) \\
&= \amalg_{i=1}^{m}\Big(\Big[\Big(\sup_{n\in\mathbb{N}} a_{n,i}\Big) -_{\triangle} a_{i-1}\Big] \diamond \mu(B_i)\Big) \\
&= \amalg_{i=1}^{m}([a_i -_{\triangle} a_{i-1}] \diamond \mu(B_i)) \\
&= (\diamond) \int s \, \mathrm{d}\mu > L.
\end{aligned}$$

(4) Because of (3) there is $N \in \mathbb{N}$ with $(\diamond) \int s_N \, \mathrm{d}\mu > L$.

Let us define $\bigwedge_{n\in\mathbb{N}} U_n := \{x \in X : f_n(x) > s_N(x)\}$. Then $U_n \in \mathcal{A}, n \in \mathbb{N}$ because we get for all $x \in X$:

$$\begin{aligned}
f_n(x) > s_N(x) \;&\Leftrightarrow\; (f_n(x) \vee s_N(x)) > s_N(x) \\
&\Leftrightarrow\; h_n(x) := (f_n(x) \vee s_N(x)) - s_N(x) > 0.
\end{aligned}$$

Thus h_n is measurable, $U_n = h_n^{-1}(0, B] \in \mathcal{A}$ and $U_n \subset U_{n+1},\ n \in \mathbb{N}$.

Moreover, we prove $\bigwedge_{i\in\{1,2,\ldots,m\}} \bigcup_{n\in\mathbb{N}}(B_i \cap U_n) = B_i$:

The inclusion $\subset$ is obvious. To prove the inclusion $\supset$, let $x \in B_i$. Because of $B_i = \bigcup_{j=i}^{m} A_j$ there is $j \in \{i, i+1, \ldots, m\}$ such that $x \in A_j$. Thus we have $0 < a_j = s(x) \leq f(x)$ and using $(*)$ $s_N(x) < f(x) = \lim_{n\to\infty} f_n(x)$ which implies: $\bigvee_{n\in\mathbb{N}} f_n(x) > s_N(x)$. Therefore x is in U_n and we get $x \in \bigcup_{n\in\mathbb{N}}(B_i \cap U_n)$. Using that μ is continuous from below results in: $\bigwedge_{i\in\{1,2,\ldots,m\}} \sup_{n\in\mathbb{N}} \mu(B_i \cap U_n) = \mu(B_i)$.

Finally the above considerations in (1) - (4) and the fact that

$$c \diamond \Big(\sup_{m\in M} \lambda(m)\Big) = \sup_{m\in M} (c \diamond \lambda(m)),$$

(where M is a set and $\lambda : M \to [0, B]$ is an arbitrary function) leads to the contradiction

$$\begin{aligned}
L &= \lim_{n\to\infty} (\diamond) \int f_n \mathrm{d}\mu \geq \lim_{n\to\infty} (\diamond) \int 1_{U_n} f_n \, \mathrm{d}\mu \\
&\geq \lim_{n\to\infty} (\diamond) \int 1_{U_n} s_N \mathrm{d}\mu \geq \lim_{n\to\infty} (\diamond) \int_{U_n} s_N \, \mathrm{d}\mu \\
&= \lim_{n\to\infty} (\amalg_{i=1}^{m}[(a_{N,i} -_{\triangle} a_{N,i-1}) \diamond \mu(B_i \cap U_n)]) \\
&= \amalg_{i=1}^{m}\Big(\lim_{n\to\infty} [(a_{N,i} -_{\triangle} a_{N,i-1}) \diamond \mu(B_i \cap U_n)]\Big)
\end{aligned}$$

$$= \amalg_{i=1}^m \Big(\sup_{n\in\mathbb{N}} [(a_{N,i} -_\triangle a_{N,i-1}) \diamond \mu(B_i \cap U_n)] \Big)$$
$$= \amalg_{i=1}^m \Big((a_{N,i} -_\triangle a_{N,i-1}) \diamond \Big[\sup_{n\in\mathbb{N}} \mu(B_i \cap U_n) \Big] \Big)$$
$$= \amalg_{i=1}^m ((a_{N,i} -_\triangle a_{N,i-1}) \diamond \mu(B_i)) = (\diamond) \int s_N \mathrm{d}\mu > L.$$

This finishes the proof of Theorem 12. □

We remark that in [15] the t-conorm-integral was defined by (23) and (24), but (24) is only well-defined if μ is continuous from below. In this case their definition coincides with our definition (because of Theorem 12).

For further nice properties of the fuzzy integral so-called decomposable fuzzy measures are needed. So we present in the following additional integral representations. As a new result we show that we can replace the usual strong distributivity by the weak distributivity, if we require that the fuzzy measure is not only decomposable but also subtractive.

Definition 9. Let $\perp$ be a pseudo-addition, and let $-_\perp$ be a $\perp$-pseudo-difference. The fuzzy measure μ is called $\perp$-decomposable iff

$$U \cap V = \emptyset, \quad U, V \in \mathcal{A} \Longrightarrow \mu(U \cup V) = \mu(U) \perp \mu(V). \tag{103}$$

The fuzzy measure μ is called $\perp$-subtractive iff

$$U \subset V, \quad U, V \in \mathcal{A} \Longrightarrow \mu(U) = \mu(V) -_\perp \mu(V \setminus U). \tag{104}$$

There are two connections between these two notions:

(a) μ $\perp$-subtractive $\Longrightarrow$ μ $\perp$-decomposable. (Let $U, V \in \mathcal{A}$ with $U \cap V = \emptyset$. Because of Lemma 5 (h) we have $\mu(U \cup V) = (\mu(U \cup V) -_\perp \mu(V)) \perp \mu(V)$.)

(b) μ $\perp$-decomposable $\Longrightarrow$ $[U \subset V, U, V \in \mathcal{A} \Rightarrow \mu(U) \geq \mu(V) -_\perp \mu(V \setminus U)]$. (Let $U, V \in \mathcal{A}$ with $U \subset V$. Then Lemma 5 (k) implies $\mu(V) -_\perp \mu(V \setminus U) = [\mu(U \cup (V \setminus U))] -_\perp \mu(V \setminus U) = [\mu(U) \perp \mu(V \setminus U)] -_\perp \mu(V \setminus U) \leq \mu(U)$.)

In (b) we have equality only in special cases: for example for $\mu(V) \notin D_\perp$ with $\mu(V) > \mu(V \setminus U)$, (see Lemma 5 (q)).

Example 5. Independently of the pseudo-addition $\perp$ there are fuzzy measures which are $\perp$-decomposable, but not $\perp$-subtractive:

Let $\perp$ be arbitrary, choose $X := \{0,1\}, \mu(\emptyset) := 0, \mu\{0\} := a \in (0, B]$, $\mu\{1\} := B$, $\mu\{0,1\} := B$. Then μ is a fuzzy measure which is $\perp$-decomposable ($\mu\{0\} \perp \mu\{1\} = a \perp B = \mu\{0,1\}$) but not $\perp$-subtractive ($\mu\{0,1\} -_\perp \mu\{1\} = B -_\perp B = 0 < a = \mu\{0\}$, see Lemma 5 (e)).

As announced, we present in the following two results integral representations for simple and measurable functions, respectively. Here only a fuzzy measure space is required (but the fuzzy measure need not to be continuous from below).

Theorem 13. Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, let $\Delta, \perp$ and $\amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL*), (DR*) and (Z).

Then we have for simple functions f (see (88) and (91)):

$$(\diamond)\int f\,\mathrm{d}\mu = \amalg_{i=1}^{n}[a_i \diamond (\mu(B_i) -_\perp \mu(B_{i+1}))], \quad \text{where} \quad B_{n+1} := \emptyset. \tag{105}$$

If (μ is $\perp$-subtractive) or (if (DR) is satisfied and μ is $\perp$-decomposable) then:

$$(\diamond)\int f\,\mathrm{d}\mu = \amalg_{i=1}^{n}[a_i \diamond \mu(A_i)]. \tag{106}$$

Proof. In a first step we show by induction on i:

$$(1) \quad \bigwedge_{i\in\{1,2,\ldots,n\}} \bigwedge_{a\in[0,B]} \amalg_{k=i}^{n}[a \diamond (\mu(B_k) -_\perp \mu(B_{k+1}))] = a \diamond \mu(B_i):$$

If $i = n$ then we use Lemma 5 (a):

$$a \diamond (\mu(B_n) -_\perp \mu(B_{n+1})) = a \diamond (\mu(B_n) -_\perp 0) = a \diamond \mu(B_n).$$

For the step $i \to i-1$ we use an analogous result of Theorem 11 (b) to get

$$\amalg_{k=i-1}^{n}[a\diamond(\mu(B_k) -_\perp \mu(B_{k+1}))] = [a\diamond(\mu(B_{i-1}) -_\perp \mu(B_i))]\amalg[a\diamond\mu(B_i)] = a\diamond\mu(B_{i-1}).$$

We now apply (1) and Theorem 11 (d) and arrive at

$$
\begin{aligned}
(2) \qquad (\diamond)\int f\,\mathrm{d}\mu &= \amalg_{i=1}^{n}[(a_i -_\Delta a_{i-1}) \diamond \mu(B_i)] \\
&= \amalg_{i=1}^{n}\amalg_{k=i}^{n}[(a_i -_\Delta a_{i-1}) \diamond (\mu(B_k) -_\perp \mu(B_{k+1}))] \\
&= \amalg_{k=1}^{n}\amalg_{i=1}^{k}[(a_i -_\Delta a_{i-1}) \diamond (\mu(B_k) -_\perp \mu(B_{k+1}))] \\
&= \amalg_{k=1}^{n}[a_k \diamond (\mu(B_k) -_\perp \mu(B_{k+1}))].
\end{aligned}
$$

We first consider the case that μ is $\perp$-subtractive. Because of $B_i = A_i \cup B_{i+1}$ we get $\mu(B_i) -_\perp \mu(B_{i+1}) = \mu(A_i), 1 \le i \le n$ so that (2) goes over into (106).

Let us now assume (DR) and that μ is $\perp$-decomposable. Then the associativity of $\amalg$ and Theorem 11 (d) imply

$$
\begin{aligned}
&(\diamond)\int f\,\mathrm{d}\mu = \amalg_{i=1}^{n}\Big[\Big(a_i -_\Delta a_{i-1}\Big) \diamond \mu\Big(\bigcup_{j=i}^{n} A_j\Big)\Big] \\
&= \amalg_{i=1}^{n}[(a_i -_\Delta a_{i-1}) \diamond (\perp_{j=i}^{n} \mu(A_j))] = \amalg_{i=1}^{n}\amalg_{j=i}^{n}[(a_i -_\Delta a_{i-1}) \diamond \mu(A_j)] \\
&= \amalg_{j=1}^{n}\amalg_{i=1}^{j}[(a_i -_\Delta a_{i-1}) \diamond \mu(A_j)] = \amalg_{j=1}^{n}[a_j \diamond \mu(A_j)].
\end{aligned}
$$

Thus Theorem 13 is proven. □

In the next results we denote by $\mathcal{MP}$ the set of all measurable partitions of X. Thus $\mathcal{M} \in \mathcal{MP}$ means that $\mathcal{M}$ is a measurable partition of X, that is, $\mathcal{M}$ is a finite, pairwise disjoint family of measurable sets whose union is X.

We show two results: An integral representation for measurable functions, and the decomposability of the integral over the union of two disjoint measurable sets.

Theorem 14. Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, let $\Delta, \perp$ and $\amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL*), (DR*) and (Z).

If (μ is $\perp$-subtractive) or (if (DR) is satisfied and μ is $\perp$-decomposable) then:

(a) $(\diamond) \int f \,\mathrm{d}\mu = \sup_{\mathcal{M} \in \mathcal{MP}} \left(\amalg_{M \in \mathcal{M}} \left[\left(\inf_{x \in M} f(x) \right) \diamond \mu(M) \right] \right)$, (107)

(b) $U \cap V = \emptyset$, $U, V \in \mathcal{A} \Rightarrow (\diamond) \int_{U \cup V} f \,\mathrm{d}\mu = \left((\diamond) \int_U f \,\mathrm{d}\mu \right) \amalg \left((\diamond) \int_V f \,\mathrm{d}\mu \right)$ (108)

Proof. (a) "$\geq$": Let $\mathcal{M} \in \mathcal{MP}$ and choose the simple function $s := \sum_{M \in \mathcal{M}} \left(\inf_{x \in M} f(x) \right) \cdot 1_M \leq f$. Then Definition 8 and (106) imply

$$(\diamond) \int f \,\mathrm{d}\mu \geq (\diamond) \int s \,\mathrm{d}\mu = \amalg_{M \in \mathcal{M}} \left[\left(\inf_{x \in M} f(x) \right) \diamond \mu(M) \right]$$

and thus we get

$$(\diamond) \int f \,\mathrm{d}\mu \geq \sup_{\mathcal{M} \in \mathcal{MP}} \left(\amalg_{M \in \mathcal{M}} \left[\left(\inf_{x \in M} f(x) \right) \diamond \mu(M) \right] \right).$$

"$\leq$": Let s be simple with $s \leq f$. Then there is a representation $s = \sum_{i=1}^n a_i 1_{A_i}$ where $M := \{A_1, A_2, \ldots, A_n, X \setminus \bigcup_{i=1}^n A_i\} \in \mathcal{M}$. Moreover $s \leq f$ implies $a_i \leq \inf_{x \in A_i} f(x)$, and (106) yields

$$(\diamond) \int s \,\mathrm{d}\mu = \amalg_{i=1}^n [a_i \diamond \mu(A_i)] \leq \amalg_{M \in \mathcal{M}} \left[\left(\inf_{x \in M} f(x) \right) \diamond \mu(M) \right].$$

This leads to

$$(\diamond) \int f \,\mathrm{d}\mu \leq \sup_{\mathcal{M} \in \mathcal{MP}} \left(\amalg_{M \in \mathcal{M}} \left[\left(\inf_{x \in M} f(x) \right) \diamond \mu(M) \right] \right).$$

(b) We first prove (b) for a simple function $f = \sum_{i=1}^n a_i 1_{A_i}$ where $a_i \in [0, B]$ and $A_i \in \mathcal{A}, 1 \leq i \leq n$ are pairwise disjoint. Now $U \cap V = \emptyset$ implies $1_{U \cup V} \cdot f = \sum_{i=1}^n a_i 1_{A_i \cap U} + \sum_{i=1}^n a_i 1_{A_i \cap V}$ so that we get from Theorem 13 (b)

$$\begin{aligned} & (\diamond) \int_{U \cup V} f \,\mathrm{d}\mu = (\diamond) \int_X 1_{U \cup V} f \,\mathrm{d}\mu \\ = \; & (\amalg_{i=1}^n [a_i \diamond \mu(A_i \cap U)]) \amalg (\amalg_{i=1}^n [a_i \diamond \mu(A_i \cap V)]) \\ = \; & \left((\diamond) \int_X 1_U f \,\mathrm{d}\mu \right) \amalg \left((\diamond) \int_X 1_V f \,\mathrm{d}\mu \right) \\ = \; & \left((\diamond) \int_U f \,\mathrm{d}\mu \right) \amalg \left((\diamond) \int_V f \,\mathrm{d}\mu \right). \end{aligned}$$

Now we prove (b) for an arbitrary measurable function f by showing first the following equality (where $\mathcal{E}$ denotes the set of simple functions):

$$\begin{aligned} (*) \quad & \sup_{s \leq f, s \in \mathcal{E}} \left[\left((\diamond) \int_U s \,\mathrm{d}\mu \right) \amalg \left((\diamond) \int_V s \,\mathrm{d}\mu \right) \right] \\ = \; & \left[\sup_{s \leq f, s \in \mathcal{E}} \left((\diamond) \int_U s \,\mathrm{d}\mu \right) \right] \amalg \left[\sup_{s \leq f, s \in \mathcal{E}} \left((\diamond) \int_V s \,\mathrm{d}\mu \right) \right]. \end{aligned}$$

The inequality "$\le$" follows from the monononicity of ∐ in each place. To prove the reverse inequality we assume in contrary

$$S := \sup_{s\le f, s\in\mathcal{E}} \Big[\Big((\diamond)\int_U s\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int_V f\,\mathrm{d}\mu\Big)\Big]$$
$$< \Big[\sup_{s\le f, s\in\mathcal{E}} \Big((\diamond)\int_U s\,\mathrm{d}\mu\Big)\Big] \amalg \Big[\sup_{s\le f, s\in\mathcal{E}} \Big((\diamond)\int_V s\,\mathrm{d}\mu\Big)\Big].$$

Then the monotonicity and continuity of ∐ in each place implies

$$\bigvee_{s_U\le f, s_U\in\mathcal{E}} \ \bigvee_{s_V\le f, s_V\in\mathcal{E}} S < \Big((\diamond)\int_U s_U\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int_V s_V\mathrm{d}\mu\Big).$$

We define $\bar{s} := (s_U \vee s_V) \in \mathcal{E}$. Then we have $s_U, s_V \le \bar{s} \le f$, and we get (using (97) for measurable functions) the contradiction

$$S < \Big((\diamond)\int_U s_U\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int_V s_V\,\mathrm{d}\mu\Big) \le \Big((\diamond)\int_U \bar{s}\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int_V \bar{s}\,\mathrm{d}\mu\Big)$$
$$\le \sup_{s\le f, s\in\mathcal{E}} \Big[\Big((\diamond)\int_U s\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int_V s\,\mathrm{d}\mu\Big)\Big] = S.$$

Now (b) follows from (*) and (b) for simple functions:

$$\begin{aligned}
(\diamond)\int_{U\cup V} f\,\mathrm{d}\mu &= \sup_{s\le f, s\in\mathcal{E}} (\diamond)\int_{U\cup V} s\,\mathrm{d}\mu \\
&= \sup_{s\le f, s\in\mathcal{E}} \Big[\Big((\diamond)\int_U s\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int_V s\,\mathrm{d}\mu\Big)\Big] \\
&= \Big[\sup_{s\le f, s\in\mathcal{E}} \Big((\diamond)\int_U s\,\mathrm{d}\mu\Big)\Big] \amalg \Big[\sup_{s\le f, s\in\mathcal{E}} \Big((\diamond)\int_V s\,\mathrm{d}\mu\Big)\Big] \\
&= \Big[(\diamond)\int_U f\mathrm{d}\mu\Big] \amalg \Big[(\diamond)\int_V f\mathrm{d}\mu\Big]
\end{aligned}$$

This proves Theorem 14. □

Example 6. We show that the condition (DR*) $\wedge$ (μ is$\perp$-decomposable) is not sufficient for the decomposability of the integral:

We overtake $\triangle, \perp, \amalg$ and $\diamond$ from Example 2 with $B < \infty$. Then (DL*) is satisfied, and thus (DR*), too (because of the commutativity of $\diamond$). Now we choose X and μ from Example 5, and let us put $f = \frac{1}{2}1_X \in \mathcal{E}, U := \{0\}, V := \{1\}$. Then we get (using the remarks before Lemma 7)

$$(\diamond)\int_{U\cup V} f\,\mathrm{d}\mu = \frac{1}{2}\diamond\mu\{0,1\} = \frac{1}{2}\diamond B = \frac{1}{2}B < \frac{1}{2}(a+B) = \Big(\frac{1}{2}a\Big) \amalg \Big(\frac{1}{2}B\Big)$$
$$= \Big(\frac{1}{2}\diamond\mu\{0\}\Big) \amalg \Big(\frac{1}{2}\diamond\mu\{1\}\Big) = \Big((\diamond)\int_U f\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int_V f\,\mathrm{d}\mu\Big).$$

We remark that statement (a) of Theorem 14 gives a connection with the so-called integral based on t-norms and t-conorms (see [11]).

The additivity of the classical integral can be generalized for fuzzy integrals,working with real-valued comonotone functions f, g, defined on X:

$$f, g \text{ are called comonotone on X iff } (f(y) - f(z))(g(y) - g(z)) \geq 0,\ y, z \in X.$$

We remark that the following equivalence holds:

$$f, g \text{ are comonotone on X} \iff \bigwedge_{y,z \in X}[f(y) < f(z) \Rightarrow g(y) \leq g(z)].$$

We first prove the following result, which is interesting in itself, and which will essentially be used in the proof of Theorem 16.

Theorem 15. Let $\triangle$ be a function which is monotone increasing and continuous in each place (no further assumptions). Let X be a set, and let M be a nonempty subset of X.

If $f, g : X \to [A, B]$ are comonotone, then the following two statements hold:

$$\text{(a)} \qquad \inf_{x \in M}[f(x) \triangle g(x)] = \Big[\inf_{x \in M} f(x)\Big] \triangle \Big[\inf_{x \in M} g(x)\Big]. \tag{109}$$

$$\text{(b)} \qquad \sup_{x \in M}[f(x) \triangle g(x)] = \Big[\sup_{x \in M} f(x)\Big] \triangle \Big[\sup_{x \in M} g(x)\Big]. \tag{110}$$

Proof. We only prove (a) since (b) can be proven similarly.

The inequality "$\geq$" is clear because of the monotonicity of $\triangle$.

To prove the reverse inequality we assume in contrary

$$I := \inf_{x \in M}[f(x) \triangle g(x)] > \Big[\inf_{x \in M} f(x)\Big] \triangle \Big[\inf_{x \in M} g(x)\Big].$$

But then there exist $y \in M$ and $z \in M$ such that $I > f(y) \triangle g(z)$. We define $F := \{x \in M : f(x) \leq f(y)\}$, $G := \{x \in M : g(x) \leq g(z)\}$ and get $F \cap G \neq \emptyset$ (If otherwise $F \cap G = \emptyset$ then we have $z \notin F \wedge y \notin G$ (since by definition $z \in G \wedge y \in F$). This implies $f(z) > f(y)$ and $g(y) > g(z)$ which contradicts that f, g are comonotone). But $F \cap G \neq \emptyset$ implies $\bigvee_{x \in M}(f(x) \leq f(y)) \wedge (g(x) \leq g(z))$ so that we get the contradiction:

$$I > f(y) \triangle g(z) \geq f(x) \triangle g(x) \geq \inf_{x \in M}[f(x) \triangle g(x)] = I.$$

Thus Theorem 15 is proven. □

Theorem 16. Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, let $\triangle, \perp$ and $\amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL), (DR) and (Z).

If μ is $\perp$-decomposable and if $f, g : X \to [0, B]$ are measurable and comonotone, then the fuzzy integral is comonotonic additive:

$$(\diamond)\int (f \triangle g)\, \mathrm{d}\mu = \Big((\diamond)\int f\, \mathrm{d}\mu\Big) \amalg \Big((\diamond)\int g\, \mathrm{d}\mu\Big). \tag{111}$$

Proof. (1) In a first step we prove

$$\sup_{\mathcal{M}\in\mathcal{MP}}\Big(\amalg_{M\in\mathcal{M}}\Big[\Big(\inf_{x\in M}f(x)\Big)\diamond\mu(M)\Big]\ \amalg\ \amalg_{M\in\mathcal{M}}\Big[\Big(\inf_{x\in M}g(x)\Big)\diamond\mu(M)\Big]\Big)$$
$$=\sup_{\mathcal{F}\in\mathcal{MP}}\Big(\amalg_{F\in\mathcal{F}}\Big[\Big(\inf_{x\in F}f(x)\Big)\diamond\mu(F)\Big]\ \amalg\ \sup_{\mathcal{G}\in\mathcal{MP}}\Big(\amalg_{G\in\mathcal{G}}\Big[\Big(\inf_{x\in G}g(x)\Big)\diamond\mu(G)\Big]\Big).$$

The inequality "$\geq$" is clear because of the monotonicity of $\amalg$.

To prove the reverse inequality we assume in contrary

$$S:=\sup_{\mathcal{M}\in\mathcal{MP}}\Big(\amalg_{M\in\mathcal{M}}\Big[\Big(\inf_{x\in M}f(x)\Big)\diamond\mu(M)\Big]\ \amalg\ \amalg_{M\in\mathcal{M}}\Big[\Big(\inf_{x\in M}g(x)\Big)\diamond\mu(M)\Big]\Big)$$
$$<\sup_{\mathcal{F}\in\mathcal{MP}}\Big(\amalg_{F\in\mathcal{F}}\Big[\Big(\inf_{x\in F}f(x)\Big)\diamond\mu(F)\Big]\ \amalg\ \sup_{\mathcal{G}\in\mathcal{MP}}\Big(\amalg_{G\in\mathcal{G}}\Big[\Big(\inf_{x\in G}g(x)\Big)\diamond\mu(G)\Big]\Big).$$

But then the monotonicity and continuity assumptions of $\amalg$ imply

$$\bigvee_{\mathcal{F}\in\mathcal{MP}}\ \bigvee_{\mathcal{G}\in\mathcal{MP}}\ S<\Big(\amalg_{F\in\mathcal{F}}\Big[\Big(\inf_{x\in F}f(x)\Big)\diamond\mu(F)\Big]\Big)\ \amalg\ \Big(\amalg_{G\in\mathcal{G}}\Big[\Big(\inf_{x\in G}g(x)\Big)\diamond\mu(G)\Big]\Big).$$

Now we choose the measurable partition $\mathcal{M}:=\{F\cap G: F\in\mathcal{F}, G\in\mathcal{G}\}\in\mathcal{MP}$ and use (DR), the monotonicity and associativity of $\amalg$ together with the $\perp$-decomposability of μ to get the contradiction

$$\begin{aligned}
S&<\Big(\amalg_{F\in\mathcal{F}}\Big[\Big(\inf_{x\in F}f(x)\Big)\diamond\mu\Big(\bigcup_{G\in\mathcal{G}}(F\cap G)\Big)\Big]\Big)\\
&\qquad\amalg\Big(\amalg_{G\in\mathcal{G}}\Big[\Big(\inf_{x\in G}g(x)\Big)\diamond\mu\Big(\bigcup_{F\in\mathcal{F}}(F\cap G)\Big)\Big]\Big)\\
&=\Big(\amalg_{F\in\mathcal{F}}\amalg_{G\in\mathcal{G}}\Big[\Big(\inf_{x\in F}f(x)\Big)\diamond\mu(F\cap G)\Big]\Big)\\
&\qquad\amalg\Big(\amalg_{G\in\mathcal{G}}\amalg_{F\in\mathcal{F}}\Big[\Big(\inf_{x\in G}g(x)\Big)\diamond\mu(F\cap G)\Big]\Big)\\
&\leq\Big(\amalg_{F\in\mathcal{F}}\amalg_{G\in\mathcal{G}}\Big[\Big(\inf_{x\in F\cap G}f(x)\Big)\diamond\mu(F\cap G)\Big]\Big)\\
&\qquad\amalg\Big(\amalg_{G\in\mathcal{G}}\amalg_{F\in\mathcal{F}}\Big[\Big(\inf_{x\in F\cap G}g(x)\Big)\diamond\mu(F\cap G)\Big]\Big)\\
&=\Big(\amalg_{M\in\mathcal{M}}\Big[\Big(\inf_{x\in M}f(x)\Big)\diamond\mu(M)\Big]\Big)\ \amalg\ \Big(\amalg_{M\in\mathcal{M}}\Big[\Big(\inf_{x\in M}g(x)\Big)\diamond\mu(M)\Big]\Big)\\
&\leq\sup_{\mathcal{M}\in\mathcal{MP}}\Big(\amalg_{M\in\mathcal{M}}\Big[\Big(\inf_{x\in M}f(x)\Big)\diamond\mu(M)\Big]\ \amalg\ \amalg_{M\in\mathcal{M}}\Big[\Big(\inf_{x\in M}g(x)\Big)\diamond\mu(M)\Big]\Big)=S.
\end{aligned}$$

By using the above result together with Theorem 14 and 15 we arrive at the desired result:

$$(\diamond)\int(f\triangle g)\,\mathrm{d}\mu=\sup_{\mathcal{M}\in\mathcal{MP}}\Big(\amalg_{M\in\mathcal{M}}\Big[\Big(\inf_{x\in M}[f(x)\triangle g(x)]\Big)\diamond\mu(M)\Big]\Big)$$
$$=\sup_{\mathcal{M}\in\mathcal{MP}}\Big(\amalg_{M\in\mathcal{M}}\Big[\Big(\Big[\inf_{x\in M}f(x)\Big]\triangle\Big[\inf_{x\in M}g(x)\Big]\Big)\diamond\mu(M)\Big]\Big)$$

$$
\begin{aligned}
&= \sup_{\mathcal{M}\in\mathcal{MP}} \Big(\amalg_{M\in\mathcal{M}} \Big[\Big(\inf_{x\in M} f(x)\Big) \diamond \mu(M)\Big] \amalg \Big(\amalg_{M\in\mathcal{M}} \Big[\Big(\inf_{x\in M} g(x)\Big) \diamond \mu(M)\Big]\Big) \\
&= \sup_{\mathcal{F}\in\mathcal{MP}} \Big(\amalg_{F\in\mathcal{F}} \Big[\Big(\inf_{x\in F} f(x)\Big) \diamond \mu(F)\Big]\Big) \amalg \sup_{\mathcal{G}\in\mathcal{MP}} \Big(\amalg_{G\in\mathcal{G}} \Big[\Big(\inf_{x\in G} g(x)\Big) \diamond \mu(G)\Big]\Big) \\
&= \Big((\diamond)\int f\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int g\,\mathrm{d}\mu\Big).
\end{aligned}
$$

Example 7. Let us show that the fuzzy integral is not longer comonotonic additive if we replace (DL), (DR) by (DL*), (DR*). Even if we assume, that the fuzzy measure is subtractive (instead of decomposibility), the fuzzy integral is not comonotonic additive:

We choose $B = 1, \triangle := \perp := \hat{+}$, $\diamond = \hat{\cdot}$ (see Example 2). Moreover let X := [0,1], and let $\mathcal{A}$ be the Borel sets of $[0,1]$, and let μ be the Borel measure on [0,1]. Then $\diamond$ satisfies (DL*), (DR*) (see Example 2). By Example 4 and using that μ is finite, μ is subtractive: If $U, V \in \mathcal{A}$ satisfying $U \subset V$ then $\mu(V) -_{\perp} \mu(V\setminus U) = \mu(V) - \mu(V \setminus U) = \mu(U)$. But then μ is $-_{\perp}$-decomposable (see the remark (b) after Definition 9).

Now let $f := g := \frac{3}{4}\,1_{[0,\frac{1}{2}]}$, so that f, g are comonotone. Using Example 2 we get

$$
\begin{aligned}
\Big((\diamond)\int f\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int g\,\mathrm{d}\mu\Big) &= \Big(\frac{3}{4} \diamond \mu\Big[0,\frac{1}{2}\Big]\Big) \amalg \Big(\frac{3}{4} \diamond \mu\Big[0,\frac{1}{2}\Big]\Big) \\
&= \Big(\frac{3}{4} \diamond \frac{1}{2}\Big) \amalg \Big(\frac{3}{4} \diamond \frac{1}{2}\Big) = \frac{3}{4}, \\
f\triangle g &= \Big(\frac{3}{4}\triangle\frac{3}{4}\Big)\, 1_{[0,\frac{1}{2}]} = 1_{[0,\frac{1}{2}]}
\end{aligned}
$$

and thus

$$
(\diamond)\int (f\triangle g)\,\mathrm{d}\mu = 1 \diamond \mu\Big[0,\frac{1}{2}\Big] = \frac{1}{2} < \Big((\diamond)\int f\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int g\,\mathrm{d}\mu\Big).
$$

Finally we want to present a characterization theorem for the fuzzy integral, which is similar to Theorem 2 in Section 3 (note that $\mathcal{F}$ is defined in Section 3).

Theorem 17. Let $(X, \mathcal{A}, \mu)$ be a fuzzy measure space, let $\triangle, \perp$ and $\amalg$ be pseudo-additions (on $[0, B]$), let $\diamond$ be a pseudo-multiplication satisfying (DL), (DR), (Z) and (LU), and let $I : \mathcal{F} \to [0, B]$ be a functional.

Then there is a fuzzy measure μ, which is $\perp$-decomposable, continuous from below, and which satisfies $I(f) = (\diamond)\int f\,\mathrm{d}\mu$ for all $f \in \mathcal{F}$ iff

$$I(a\,1_A) = a \diamond I(\tilde{e}\,1_A), a \in [0, B], A \in \mathcal{A}, \qquad \text{(weak Homogeneity)} \qquad (112)$$

$$\bigwedge_{f,g\in\mathcal{F}} [f \le g \Rightarrow I(f) \le I(g)], \qquad \text{(Monotonicity)} \qquad (113)$$

$$I(f\triangle g) = I(f) \amalg I(g), f, g \in \mathcal{F},\ f, g \qquad \text{comonotone, (Additivity)} \qquad (114)$$

$U \cap V = \emptyset,$

$$U, V \in \mathcal{A} \Rightarrow I(\tilde{e}\, 1_{U \cup V}) = I(\tilde{e}\, 1_U) \amalg I(\tilde{e}\, 1_V), \qquad \text{(Decomposibility)} \qquad (115)$$

$$((f_n) \subseteq \mathcal{F}) \wedge (f_n \uparrow f) \Rightarrow \lim_{n \to \infty} I(f_n) = I(f) \qquad \text{(Continuity from below).} \quad (116)$$

Proof. Using (DR), (LU) and the asymmetric version of Lemma 3 (b) we get $\perp = \amalg$.

The implication "$\Rightarrow$" is nearly already proven: (113), (114), (115) and (116) follow from Lemma 7 (c), Theorem 16, Theorem 14 (b) and Theorem 12, respectively. To prove (a) we use (LU):

$$(\diamond) \int (a\, 1_A)\, \mathrm{d}\mu = a \diamond \mu(A) = a \diamond [\tilde{e} \diamond \mu(A)] = a \diamond \Big((\diamond) \int \tilde{e}\, 1_A \mathrm{d}\mu \Big).$$

To prove the reverse implication "$\Leftarrow$" we put $\bigwedge_{M \in \mathcal{A}} \mu(M) := I(\tilde{e}\, 1_M)$.

We now show that μ is a $\perp$-decomposable fuzzy measure which is continuous from below:

At first $\mu(\emptyset) = I(\tilde{e}\, 1_\emptyset) = I(0\, 1_X) = 0 \diamond I(\tilde{e}\, 1_X) = 0$. Moreover μ is isotone and $\perp$-decomposable because of the definition of μ and (113) and (115), respectively. Now let $(U_n) \subset \mathcal{A}$ satisfying $\bigwedge_{n \in \mathbb{N}} U_n \subset U_{n+1}$. Then (116) implies (because of $\tilde{e}\, 1_{U_n} \uparrow \tilde{e}\, 1_{\bigcup_{n \in \mathbb{N}} U_n}$)

$$\lim_{n \to \infty} \mu(U_n) = \lim_{n \to \infty} I(\tilde{e}\, 1_{U_n}) = I(\tilde{e}\, 1_{\bigcup_{n \in \mathbb{N}} U_n}) = \mu\Big(\bigcup_{n \in \mathbb{N}} U_n \Big).$$

Thus μ is continuous from below. Now we prove

$$(*) \qquad \bigwedge_{f \in \mathcal{E}} I(f) = (\diamond) \int f\, \mathrm{d}\mu.$$

W.l.o.g we assume that $f \neq 0$ (because of $I(0) = I(0\, 1_X) = 0 \diamond I(\tilde{e}\, 1_X) = 0$). Then there is a representation $f = \sum_{i=1}^{n} a_i\, 1_{A_i}$ with $a_0 = 0 < a_1 < a_2 < \cdots < a_n \leq B$, $A_i \in \mathcal{A}$ are pairwise disjoint, $B_i = \bigcup_{j=i}^{n} A_j$, $1 \leq i \leq n$.

We now show (93) by induction on $n \in \mathbb{N}$.

$n = 1: I(a_1\, 1_{A_1}) = a_1 \diamond I(\tilde{e}\, 1_{A_1}) = a_1 \diamond \mu(A_1) = (a_1 -_\triangle a_0) \diamond \mu(A_1)$ (here we have used (112) and Lemma 5 (a)).

$n \to n+1:$ Let $\bar{f} := \sum_{i=1}^{n+1} a_i\, 1_{A_i}$ (with the usual assumptions on $a_i, A_i, 1 \leq i \leq n+1$) and put $g := \triangle_{i=2}^{n+1} [(a_i -_\triangle a_1)\, 1_{A_i}] \in \mathcal{F}$ and $h := a_1\, 1_{\bigcup_{j=1}^{n+1} A_j} \in \mathcal{F}$.

Then we have $g \triangle h = \bar{f}$. To show this we distinguish three cases.

Case 1: If $x \in A_1$ then $g(x) \triangle h(x) = 0 \triangle a_1 = a_1 = f(x)$.

Case 2: If $\bigvee_{m \in \{2,3,\ldots,n+1\}} x \in A_m$ then (by Lemma 5 (h)) $g(x) \triangle h(x)$
$= (a_m -_\triangle a_1) \triangle a_1 = a_m = f(x)$.

Case 3: If $\bigwedge_{m \in \{2,3,\ldots,n+1\}} x \notin A_m$ then $g(x) \triangle h(x) = 0 = f(x)$.

Moreover, g, h are comonotone since $h|_{\bigcup_{j=1}^{n+1}}$ is constant and we arrive at (using (114), Lemma 5 (i) and $a_1 -_\triangle a_1 = 0$)

$$I\Big(\sum_{i=1}^{n+1} a_i 1_{A_i}\Big) = I(\bar{f}) = I(g\triangle h) = I(g) \amalg I(h)$$

$$= \Big(\amalg_{i=2}^{n+1} \Big[\Big(\Big[a_i -_\triangle a_1\Big] -_\triangle \Big[\Big(\Big[a_{i-1} -_\triangle a_1\Big]\Big) \diamond \mu\Big(\bigcup_{j=i}^{n+1} A_j\Big)\Big]\Big)$$

$$\amalg\Big[\Big(a_1 -_\triangle a_0\Big) \diamond \mu\Big(\bigcup_{j=i}^{n+1} A_j\Big)\Big]$$

$$\doteq \Big(\amalg_{i=2}^{n+1} \Big[\Big(a_i -_\triangle a_{i-1}\Big) \diamond \mu(B_i)\Big]\Big) \amalg \Big(a_1 -_\triangle a_0\Big) \diamond \mu(B_1)\Big].$$

Finally we show $\bigwedge_{f\in\mathcal{F}} I(f) = (\diamond)\int f\,\mathrm{d}\mu$: Let us take a sequence $(f_n) \subset \mathcal{E}$ with $f_n \uparrow f$ so that we get by Theorem 12

$$I(f) = \lim_{n\to\infty} I(f_n) = \lim_{n\to\infty} (\diamond) \int f_n \mathrm{d}\mu = (\diamond) \int f\,\mathrm{d}\mu.$$

Thus Theorem 17 is proven. □

We mention that this result improves earlier characterization theorems of the Choquet- Sugeno- and the Choquet-like integral, for example the result in [13]. Because of the flexibility of fuzzy measures also characterization theorems for discrete fuzzy measures like in [5] are covered.

12. EXAMPLES

The fuzzy integral introduced in Section 8 covers many known integrals of the literature, for example, the generalized Sugeno integrals (cf. (25)), the Choquet integral, the Weber integral, the t-conorm integral, the integral of Sugeno and Murofushi, based on pseudo-additions and pseudo-multiplications in [20], the Pan integral of Wang and Klir in [21], the integral based on t-norms and t-conorms in [11] and the Shilkret integral in [18].

These results can be deduced from the following key Lemma which gives the representation of the fuzzy integral for Archimedean t-conorms using the Choquet integral

$$\text{(C)} \qquad \int f\,\mathrm{d}\mu := \int_0^\infty \mu(\{x \in X : f(x) > t\})\,\mathrm{d}t.$$

Lemma 8. Let $\triangle$ and $\amalg$ be continuous, Archimedian t-conorms on $[0, B]^2$ with generators $k : [0, B] \to [0, \infty]$ and $h : [0, B] \to [0, \infty]$, respectively. Moreover let $\mu : \mathcal{A} \to [0, B]$ be a fuzzy measure which is continuous from below, and let $\mathcal{F} := \{f : X \to [0, B] \mid f \text{ is a measurable function}\}$.

(a) If

$$\bigvee_{\substack{g:[0,B]\to[0,\infty],\ g(0)=0,\\ g\uparrow,\ g \text{ continuous on } (0,B]}} \bigwedge_{a,x\in[0,B]} a\diamond x = h^{(-1)}(k(a)g(x))$$

then

$$\bigwedge_{f\in\mathcal{F}} (\diamond)\int f\,\mathrm{d}\mu = h^{(-1)}\Big((C)\int k(f)\,\mathrm{d}(g\circ\mu)\Big).$$

(b)

$$\bigvee_{\substack{g:[0,B]\to[0,\infty],g(0)=0,\\ g\uparrow,\ g \text{ continuous on } (0,B]}} \bigwedge_{f\in\mathcal{F}} (\diamond)\int f\,\mathrm{d}\mu = h^{(-1)}\Big((C)\int k(f)\,\mathrm{d}(g\circ\mu)\Big).$$

Proof. (a) Let $g:[0,B]\to[0,\infty]$, $g(0)=0$, and let g be isotonic and continuous on $(0,B]$ satisfying $\bigwedge_{a,x\in[0,B]} a\diamond x = h^{(-1)}(k(a)g(x))$.

Then $g\circ\mu$ is a fuzzy measure which is continuous from below:

Obviously, $g\circ\mu(\emptyset)=g(0)=0$ and $g\circ\mu$ is isotonic.

Now consider $(U_n)\subset\mathcal{A}$ with $\bigwedge_{n\in\mathbb{N}} U_n\subset U_{n+1}$.

If $\mu(\bigcup_{n\in\mathbb{N}}U_n)=0$ then $\lim_{n\to\infty} g\circ\mu(U_n)=g\circ\mu(\bigcup_{n\in\mathbb{N}}U_n)=g(0)=0$.

If $\mu(\bigcup_{n\in\mathbb{N}}U_n)>0$ then $g[\mu(\bigcup_{n\in\mathbb{N}}U_n)] = g\Big(\lim_{n\to\infty}\mu(U_n)]\Big) = \lim_{n\to\infty} g\circ\mu(U_n)$.

Consider again a simple function f with $f\neq 0$ and $f=\sum_{i=1}^n a_i 1_{A_i}$ with $a_0=0<a_1<a_2<\cdots<a_n\le B$, where $A_i\in\mathcal{A}$ are pairwise disjoint, and $B_i=\bigcup_{j=i}^n A_j$, $1\le i\le n$.

Because of $\mu(f>t)=\sum_{i=1}^n \mu(\bigcup_{j=i}^n A_j)\,1_{[a_{i-1},a_i)}(t)$ we get

$$(C)\quad \int f\mathrm{d}\mu = \int_0^\infty \mu(f>t)\,\mathrm{d}t = \sum_{i=1}^n \mu\Big(\bigcup_{j=i}^n A_j\Big)\int_0^\infty 1_{[a_{i-1},a_i)}(t)\,\mathrm{d}t$$
$$= \sum_{i=1}^n \mu(B_i)(a_i-a_{i-1}).$$

Let us now prove:

$$(\alpha)\qquad \bigwedge_{c_1,c_2,\dots,c_n\in[0,B]} h^{(-1)}\Big(\sum_{i=1}^n hh^{(-1)}(c_i)\Big) = h^{(-1)}\Big(\sum_{i=1}^n c_i\Big).$$

Case 1. If $\bigwedge_{i\in\{1,2,\dots,n\}} c_i < h(B)$ then we have $\bigwedge_{i\in\{1,2,\dots,n\}} h^{(-1)}(c_i)=h^{-1}(c_i)$.

Case 2. If $\bigvee_{i\in\{1,2,\dots,n\}} c_i \ge h(B)$ then both sums in (α) are greater or equal to $h(B)$ so that both sides of the equality in (α) result in B.

Now we prove the statement in (a) for a simple function f using (53) and Lemma 5 (c):

$$\begin{aligned}(\diamond)\int f\,\mathrm{d}\mu &= \amalg_{i=1}^n[(a_i -_\triangle a_{i-1})\diamond\mu(B_i)] = \amalg_{i=1}^n[k^{-1}(k(a_i)-k(a_{i-1}))\diamond\mu(B_i)]\\ &= \amalg_{i=1}^n[h^{(-1)}([k(a_i)-k(a_{i-1})]\,g[\mu(B_i)])]\\ &= h^{(-1)}\Big(\sum_{i=1}^n hh^{(-1)}([k(a_i)-k(a_{i-1})]\,g[\mu(B_i)])\Big)\\ &= h^{(-1)}\Big(\sum_{i=1}^n([k(a_i)-k(a_{i-1})]\,g[\mu(B_i)])\Big) = h^{(-1)}\Big((C)\int k(f)\,\mathrm{d}(g\circ\mu)\Big).\end{aligned}$$

Now let $f\in\mathcal{F}$ and choose a sequence $(f_n)\in\mathcal{E}$ with $f_n\uparrow f$. Then $k(f_n)\uparrow k(f)$ and we arrive at (using Theorem 12, that the Choquet integral is continuous from below, and that $h^{(-1)}$ is continuous):

$$\begin{aligned}(\diamond)\int f\,\mathrm{d}\mu &= \lim_{n\to\infty}(\diamond)\int f_n\,\mathrm{d}\mu = \lim_{n\to\infty} h^{(-1)}\Big((C)\int k(f_n)\,\mathrm{d}(g\circ\mu)\Big)\\ &= h^{(-1)}\Big(\lim_{n\to\infty}(C)\int k(f_n)\,\mathrm{d}(g\circ\mu)\Big) = h^{(-1)}\Big((C)\int k(f)\,\mathrm{d}(g\circ\mu)\Big).\end{aligned}$$

(b) Because of remark (IV) following Theorem 6 there is an isotonic and continuous function $g:(0,B]\to[0,\infty]$ with $\bigwedge_{a,x\in(0,B]} a\diamond x = h^{(-1)}(k(a)g(x))$.

We extend g by $g(0)=0$. Using (Z) we get $\bigwedge_{a,x\in[0,B]} a\diamond x = h^{(-1)}(k(a)g(x))$. Applying now part (a) we get (b). This proves Lemma 8. □

We give three examples. The first example shows that the t-conorm integral is a generalization of the Choquet integral.

Example 8. Let $B:=\infty, \triangle := \amalg := \hat{+}, \diamond = \hat{\cdot}$ (see Example 2).

If $\mu:\mathcal{A}\to[0,\infty]$ is a fuzzy measure which is continuous from below then $\bigwedge_{f\in\mathcal{F}}(\diamond)\int f\,\mathrm{d}\mu = (C)\int f\,\mathrm{d}\mu$.

Proof. A generator of $\hat{+}:[0,B]^2\to[0,B]$ is given by $k(x)=x$ (see Example 4). Because of $B:=\infty$ we have $\bigwedge_{x\in[0,\infty]} k^{(-1)}(x) = k^{-1}(x) = x$ and thus $\bigwedge_{a,x\in[0,B]} a\diamond x = a\hat{\cdot}x = k^{(-1)}(k(a)k(x))$. Thus Lemma 8 results in $\bigwedge_{f\int\mathcal{F}}(\diamond)\int f\,\mathrm{d}\mu = k^{(-1)}\Big((C)\int k(f)\,\mathrm{d}(k\circ\mu)\Big) = (C)\int f\,\mathrm{d}\mu$. □

The next example leads to the Weber integral (see [22]).

Example 9.

1. Let $B:=1$, let $\perp$ be a continuous Archimedean t-conorm with generator $g:[0,1]\to[0,\infty]$, and let μ be a $\perp$-decomposable fuzzy measure which is continuous from below. Then there are 3 possibilities: (S), (NSA) and (NSP) (see [22] and the explicite description after (26) in Section 3).

2. We define $\bigwedge_{a,x\in[0,B]} a\diamond x := g^{-1}(a\cdot g(x))$.

   Then $\diamond$ is a pseudo-multiplication which satisfies (Z) and the weak left-right distributivity law with respect to $(\hat{+},\perp,\perp)$. By Lemma 8 we get

$$\bigwedge_{f\in\mathcal{F}}(\diamond)\int f\,\mathrm{d}\mu = g^{(-1)}\Big((C)\int f\mathrm{d}(g\circ\mu)\Big).$$

   In the cases (S) and (NSA) we arrive at a Weber integral:

$$\bigwedge_{f\in\mathcal{F}}(\diamond)\int f\,\mathrm{d}\mu = g^{(-1)}\Big(\int f\,\mathrm{d}(g\circ\mu)\Big).$$

3. In the case (NSP) Sugeno and Murofushi already investigated the connections of the t-conorm integral with the Weber integral (see [15]).

The last example shows that the t-conorm integral is a generalization of the Quasi-Sugeno integral.

Example 10. Let $0 < B \le \infty, \Delta = \amalg = \vee$, let $\diamond : [0,B]^2 \to [0,B]$ be a pseudo-multiplication satisfying (Z), and let $\mu : \mathcal{A} \to [0,B]$ be a fuzzy measure (it is not required that μ is continuous from below). Then we have:

$$\bigwedge_{f\in\mathcal{F}}(\diamond)\int f\,\mathrm{d}\mu = (S)\int f\,\mathrm{d}\mu := \sup_{\alpha\in[0,B]}[\alpha\diamond\mu(f\ge\alpha)].$$

Proof. First we note that $\diamond$ is a pseudo-multiplication which satisfies the left distributivity law with respect to $(\vee,\vee)$. By Definition 8 we have to show:

$$\sup_{s\le f,\, s\in\mathcal{E}}(\diamond)\int s\,\mathrm{d}\mu = \sup_{\alpha\in[0,B]}[\alpha\diamond\mu(f\ge\alpha)].$$

To prove "$\ge$" let $\alpha \in [0,B]$ be arbitrary and choose the simple function $s := \alpha\,1_{(f\ge\alpha)} \le f$. Then we first get

$$\alpha\diamond\mu(f\ge\alpha) = (\diamond)\int s\,\mathrm{d}\mu \le \sup_{s\le f,\, s\in\mathcal{E}}(\diamond)\int s\,\mathrm{d}\mu$$

and then

$$\sup_{\alpha\in[0,B]}[\alpha\diamond\mu(f\ge\alpha)] \le \sup_{s\le f,\, s\in\mathcal{E}}(\diamond)\int s\,\mathrm{d}\mu.$$

Now we show the inequality "$\le$" and take $s\in\mathcal{E}, s\le f$.
Again, let $s\ne 0$ with a representation $s = \sum_{i=1}^{n} a_i\,1_{A_i}$ with $a_0 = 0 < a_1 < a_2 <$

$\cdots < a_n \leq B$, where $A_i \in \mathcal{A}$ are pairwise disjoint, and $B_i = \bigcup_{j=i}^n A_j$, $1 \leq i \leq n$. But then Lemma 5 (d) yields

$$(\diamond)\int s\,\mathrm{d}\mu = \amalg_{i=1}^n[(a_i -_\triangle a_{i-1}) \diamond \mu(B_i)]$$

$$= \bigvee_{i=1}^n [a_i \diamond \mu(f \geq a_i)] \leq \sup_{\alpha\in[0,B]} [\alpha \diamond \mu(f \geq \alpha)]$$

so that

$$\sup_{s\leq f\,,\,s\in\mathcal{E}} (\diamond)\int s\,\mathrm{d}\mu \leq \sup_{\alpha\in[0,B]} [\alpha \diamond \mu(f \geq \alpha)].$$ □

Looking at the the many examples, given in [2], we see that, starting with a concrete ordinal sum representation of the pseudo-addition $\triangle$ the corresponding pseudo-multiplication $\diamond$ can be presented by using the generators of the pseudo-addition. And then the corresponding integral can be presented with the aid of the generators of the pseudo-addition, too.

We remark that all pseudo-multiplications, which are presented in [2] have the form (79) of Theorem 6. Since we have only required the weak condition (DL*) in Theorem 6, we get (of course) the representation of the pseudo-multiplication $\diamond$ only on 'Archimedean intervals'. By additional assumptions we have more information and get more 'complete results'. By applying Lemma 8 we can evaluate the fuzzy integral on "Archimedean intervals".

Finally we here want to give an outline of a more general result, which is based on the paper [20].

We say that a fuzzy measure μ satisfies ccc (countable chain condition) if it is continuous from below and if there are at most countably many, pairwise disjoint measurable sets of positive μ-measure. We refer to [20] for further explanations and characterizations.

For a $\perp$-decomposable fuzzy measure μ satisfying ccc there exist disjoint sets $W_I, W_k \in \mathcal{A}$, $k \in K_\perp$ satisfying

$$\text{(a)} \qquad \mu\Big(X \setminus \Big[\bigcup_{k\in K_\perp} W_k \cup W_I\Big]\Big) = 0, \tag{117}$$

$$\text{(b)} \qquad \bigwedge_{k\in K_\perp} \bigwedge_{M\in\mathcal{A}, M\subseteq W_k} \mu(M) \in \{0\} \cup (a_k^\perp, b_k^\perp] \quad \text{and} \tag{118}$$

$$\text{(c)} \qquad \bigwedge_{M\in\mathcal{A}, M\subseteq W_k} \mu(M) \quad \text{idempotent of } \perp. \tag{119}$$

Using the theorem of monotone convergence and Theorem 14 we get, supposing (DR), the following decomposition theorem:

$$(\diamond)\int_X f\mathrm{d}\mu = \Big(\amalg_{k\in K_\perp} (\diamond)\int_{W_k} f\,\mathrm{d}\mu\Big) \amalg \Big((\diamond)\int_{W_I} f\mathrm{d}\mu\Big). \tag{120}$$

Using (106) in Theorem 13 we get exactly the generalized Sugeno integral (see Example 10)

$$(\diamond)\int_{W_I} f\,\mathrm{d}\mu = (S)\int_{W_I} f\,\mathrm{d}\mu = \sup_{\alpha\in[0,B]} [\alpha \diamond \mu((f \geq \alpha) \cap W_I)], \tag{121}$$

and if $\triangle$ and $\amalg$ are Archimedean then we get a representation loke in Lemma 8 (a).

$$(\diamond)\int_{W_k} f\,\mathrm{d}\mu = h^{(-1)}\Big((C)\int_{W_k} k(f)\,\mathrm{d}(\bar{g}\circ\mu)\Big), \tag{122}$$

where $\bar{g}_k(x) := g_k(x), \bar{g}_k(0) := 0$.

For the representation of non-Archimedean $\triangle$ and $\amalg$, Theorem 10 can be applied.

For another decomposition theorem we refer to [4].

13. SUMMARY

We have presented here our idea of a fuzzy integral, but perhaps the time is ready for collecting all results on fuzzy integrals in a unified framework for at least two reasons. First, to have them in a handy form, and not distributed in publications coming from all over the world. And second, to get a starting point for new ideas and new results.

In our three papers we have undertaken some steps into this direction, but sometimes it is still difficult to compare different results in the literature. For example, results of Section 3 can sometimes not be compared directly with our results: in Theorem 2 we have the assumptions $\triangle = \amalg$ and the continuity from below of μ is assumed, whereas in Theorem 17 the decomposibility of μ is required.

We have the feeling that the introduction of weak distributivity seems to be of some advantage because we get rather general results.

Of course, still many problems are left. Let us mention only two problems:

- The extension of the integral with values in the interval $[-B, B]$. We refer to the remarks in [2].
- It would be nice to have further results like Radon-Nikodym-like theorems (see for example [20]).

Finally we would like to thank Sugeno and Murofushi for their pioneering work in [15] and [20]. Based on their papers, we were able to contribute to this topic.

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