KYBERNETIKA — VOLUME 41 (2005), NUMBER 4, PAGES 469–496

MULTIPLICATION, DISTRIBUTIVITY AND FUZZY-INTEGRAL II[1]

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Based on results of generalized additions and generalized multiplications, proven in Part I, we first show a structure theorem on two generalized additions which do not coincide. Then we prove structure and representation theorems for generalized multiplications which are connected by a strong and weak distributivity law, respectively. Finally – as a last preparation for the introduction of a framework for a fuzzy integral – we introduce generalized differences with respect to t-conorms (which are not necessarily Archimedean) and prove their essential properties.

Keywords: fuzzy measures, distributivity law, restricted domain, pseudo-addition, pseudo-multiplication, Choquet integral, Sugeno integral

AMS Subject Classification: 28A25, 20M30

7. INTRODUCTION

We assume that the reader is familiar with the notations and results in Part I of this paper where we have introduced generalized additions and multiplications which we called pseudo-additions and pseudo-multiplications, respectively together with a strong and a weak distributivity law.

If we now weaken appropriately the existence of a unit element then we can show that under weak assumptions the structure of the ordinal sum of $\triangle$ is 'finer' than the corresponding structure of $\amalg$, which means, that

Archimedean t-conorms of $\amalg$ are also Archimedean t-conorms of $\triangle$.

In addition, strict t-conorms of $\amalg$ are also strict t-conorms of $\triangle$.

We start with the definition of an 'individual unit'.

Definition 5. Let $\diamond$ be a pseudo-multiplication.

(RU*) For all $a \in (A, B]$ there is $e(a) \in (A, B]$ such that: $a \diamond e(a) = a$.
(individual right unit)

[1]This paper is a continuation of our paper Multiplication, Distributivity and Fuzzy-Integral I in Kybernetika No. 3/2005. We continue the enumeration of formulas, definitions, lemmas and theorems.

(LU*) For all $a \in (A,B]$ there is $\tilde{e}(a) \in (A,B]$ such that: $\tilde{e}(a) \diamond a = a$ (individual left unit).

In the case of (RU*) we define $E(a) := \sup\{e \in [A,B] : a \diamond e = a\}$ (maximal right unit for a).
In the case of (LU*) we define $\tilde{E}(a) := \sup\{\tilde{e} \in [A,B] : \tilde{e} \diamond a = a\}$ (maximal left unit for a).

It is easy to show that $E(a)$ and $\tilde{E}(a)$ are individual right units and individual left units for a, respectively.

Moreover, there is the following connection with boundary conditions:

If $\diamond$ satisfies (Z) and (CRZ), then:

(RU*) $\iff a \diamond B \geq a$ for all $a \in (A,B]$.

If $\diamond$ satisfies (Z) and (CLZ), then:

(LU*) $\iff B \diamond \dot{a} \geq a$ for all $a \in (A,B]$.

We prove only the first statement. If (RU*) is valid then we get $a \diamond B \geq a \diamond e(a) = a$.
If $a = B$ then $B \diamond B \geq B$ implies $B \diamond B = B$.
If $a \in (A,B]$ then we have that $a \diamond A = A < a \leq a \diamond B$, and by the intermediate value theorem there is $e(a) \in (A,B]$ with $a \diamond e(a) = a$.

Some often used results are contained in the following Lemma.

Lemma 4. Let $\triangle, \amalg$ be pseudo-additions, and let $\diamond$ be a pseudomultiplication satisfying (DL*) and (RU*).
Let $m \in K_\triangle$ and let $b \in (a_m^\triangle, b_m^\triangle]$ be $\amalg$−idempotent. Then we have:

(a) $\bigwedge_{a \in (a_m^\triangle, b_m^\triangle]} \bigwedge_{x \in (E(b),B]} a \diamond x > b.$

(b) $b_m^\triangle \diamond E(b) = b.$

(c) $E(b) < B \Rightarrow \bigwedge_{a \in (a_m^\triangle, b_m^\triangle]} a \diamond E(b) = b.$

(d) $b \in (a_m^\triangle, b_m^\triangle) \Rightarrow E(b) < E(b_m^\triangle) \wedge \bigwedge_{a \in (a_m^\triangle, b_m^\triangle]} a \diamond E(b) = b.$

Proof. (a) Let us assume $\bigvee_{a \in (a_m^\triangle, b_m^\triangle]} \bigvee_{x \in (E(b),B]} a \diamond x \leq b$. Then Lemma 2 (d) and (RU*) imply

$$b = b \diamond E(b) \leq b \diamond x \leq b_m^\triangle \diamond x \leq \amalg_{i=1}^{\infty}(a \diamond x) \leq \lim_{n \to \infty}(\amalg_{i=1}^{n} b) = b,$$

so that $b \diamond x = b$ contradicts the maximality of $E(b)$.

(b) Again, using Lemma 2 (d) and (RU*) the following inequalities imply the desired result:

$$b = b \diamond E(b) \leq b_m^{\triangle} \diamond E(b) \leq \amalg_{i=1}^{\infty}(b \diamond E(b)) = \lim_{n\to\infty}(\amalg_{i=1}^{n} b) = b.$$

(c) W.l.o.g. let $a \in (a_m^{\triangle}, b_m^{\triangle})$ (see (b)). Then we get by (a) and (b)

$$b = b_m^{\triangle} \diamond E(b) \geq a \diamond E(b) = \lim_{x \to E(b)+}(a \diamond x) \geq b.$$

(d) Because of (c) we have only to show that $E(b) < E(b_m^{\triangle})$. We assume $E(b_m^{\triangle}) \leq E(b)$ and get the contradiction (using (b) and (RU*))

$$b_m^{\triangle} \diamond E(b_m^{\triangle}) \leq b_m^{\triangle} \diamond E(b) = b < b_m^{\triangle} \diamond E(b_m^{\triangle}).$$

This proves Lemma 4. □

Let us now present the following structure theorem for two pseudo-additions, which do not coincide, but which have at least the same structure of ordinal sums. Again, the proof for this result is very technically, but it seems to be a completely unknown result.

Theorem 5. Let $\triangle$ and $\amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (DL*) and (RU*).

(I) If $l \in K_{\amalg}$ and if we have

$$(a_l^{\amalg} > A \vee [(\mathrm{Z}) \wedge (\mathrm{CLZ})]) \wedge (b_l^{\amalg} < B \vee [(\mathrm{DL}) \wedge (\mathrm{CRB})]) \tag{78}$$

then

(a) $\triangle$ and $\amalg$ have the same structure on $[a_l^{\amalg}, b_l^{\amalg}]^2$, that is, $\triangle]_{[a_l^{\amalg}, b_l^{\amalg}]^2}$ is Archimedean, and $[a_l^{\triangle}, b_l^{\triangle}] = [a_l^{\amalg}, b_l^{\amalg}] = [a_l, b_l]$.

(b) $\bigwedge_{x\in(A,B]} a_l \diamond x = a_l$.

(c) $\bigwedge_{x\in(A,B]} b_l \diamond x = b_l$.

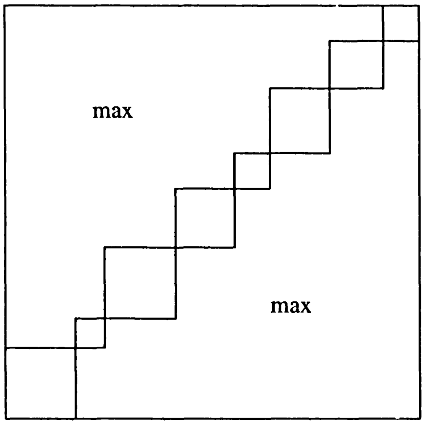
(d) $\bigwedge_{a\in[a_l,b_l]} \bigwedge_{x\in(A,B]} a \diamond x \in [a_l, b_l]$.

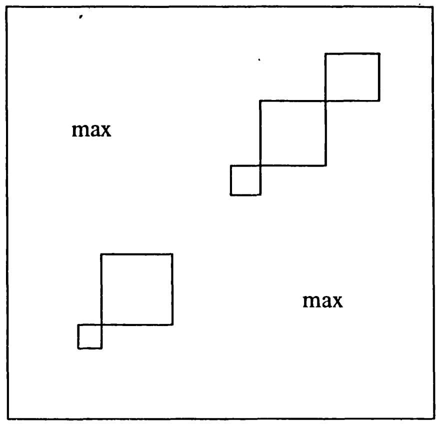
(e) If $\amalg|_{[a_l,b_l]^2}$ is strict, then $\triangle|_{[a_l,b_l]^2}$ is also strict.

(II) Let

$$\Big(\Big[\bigwedge_{l\in K_{\amalg}} a_l^{\amalg} > A\Big] \vee \Big[(\mathrm{Z}) \wedge (\mathrm{CLZ})\Big]\Big) \wedge \Big(\Big[\bigwedge_{l\in K_{\amalg}} b_l^{\amalg} < B\Big] \vee \Big[(\mathrm{DL}) \wedge (\mathrm{CRB})\Big]\Big).$$

Then the structure of the ordinal sum of $\triangle$ is 'finer' than the corresponding structure of $\amalg$, which means, that





Archimedean t-conorms of $\amalg$ are also Archimedean t-conorms of $\triangle$.

In addition, strict t-conorms of $\amalg$ are also strict t-conorms of $\triangle$.

The above two pictures, where the left one represents $\triangle$ and the right one represents $\amalg$, give an interpretation of the relation 'finer'.

Note also, that the condition (78) in (I) of Theorem 5 and the corresponding condition in (II) of Theorem 5 are rather weak assumptions.

Now we present the proof of Theorem 5.

P r o o f o f T h e o r e m 5 . (a) To prove (a) we show 3 statements:

(a1) $\bigwedge_{a\in[A,B]}[a\triangle\text{−idempotent} \Rightarrow a\,\amalg\text{ −idempotent}]$.

(a2) If $m \in K_\triangle, a_m^\triangle = A$ then we have: (Z) $\wedge$(CLZ) $\Rightarrow \amalg|_{[a_m^\triangle,b_m^\triangle]^2}$ is Archimedean.

(a3) If $m \in K_\triangle, a_m^\triangle > A$ then we have:

$$b_m^\triangle < B \vee [(\text{DL}) \wedge (\text{CRB})] \Rightarrow \Big(\amalg|_{[a_m^\triangle,b_m^\triangle]^2} \text{ is Archimedean}\Big) \vee \Big(\amalg|_{[a_m^\triangle,b_m^\triangle]^2} = \vee\Big).$$

Proof of (a1). Again, w.l.o.g. we may assume $a \in (A, B)$ (since $A\triangle A = A = A\amalg A, B\triangle B = B = B\amalg B$). But then the first statement of Lemma 2 (a) yields that $a = a \diamond E(a)$ is $\amalg$−idempotent.

Proof of (a2). Let us assume that $\amalg|_{[a_m^\triangle,b_m^\triangle]^2}$ is not Archimedean. Then there is a $\amalg$−idempotent element $b \in (a_m^\triangle, b_m^\triangle)$ and Lemma 4 (d), (Z) and (CLZ) yield the contradiction $b = \lim_{a\to a_m^\triangle+}(a \diamond E(b)) = a_m^\triangle \diamond E(b) = A \diamond E(b) = A$.

Proof of (a3). Let $(\triangle|_{[a_m^\triangle,b_m^\triangle]^2}$ Archimedean) $\wedge(\amalg_{[a_m^\triangle,b_m^\triangle]^2} \neq \vee)$.

Using (a1) we get: $\bigvee_{l\in K_\amalg}[a_m^\amalg, b_m^\amalg] \subset [a_m^\triangle, b_m^\triangle]$. We are done, if we show

(α) $a_l^\amalg = a_m^\triangle$ and $b_l^\amalg = b_m^\triangle$.

At first we prove:

(β) $a_m^\triangle > A \Rightarrow b_l^{\amalg} = b_m^\triangle$.

If we assume in contrary that $b_l^{\amalg} \in (a_m^\triangle, b_m^\triangle)$, then Lemma 4 (d) and (RU*) show that $E(b_l^{\amalg}) \in (A, B)$ and thus $a_m^\triangle \diamond E(b_l^{\amalg}) = \lim_{a \to a_m^\triangle +}[a \diamond E(b_l^{\amalg})] = b_l^{\amalg} > a_m^\triangle = a_m^\triangle \diamond E(a_m^\triangle)$.

We choose now $a := a_m^\triangle \in (A, B)$ as $\triangle-$idempotent element, $x_0 := E(b_l^{\amalg}), x := E(a_m^\triangle) \in (A, B]$. Then the last inequality together with the monotonicity of $\diamond$ leads to $(a \diamond x_0 = b_l^{\amalg} > a \diamond x) \wedge (x < x_0)$.

This contradicts the third statement of Lemma 3 (a) and proves (β).

To prove (α) we still show:

(γ) $b_m^\triangle < B \vee [(\mathrm{DL}) \wedge (\mathrm{CRB})] \Rightarrow a_l^{\amalg} = a_m^\triangle$.

Again we assume that $a_l^{\amalg} \in (a_m^\triangle, b_m^\triangle)$ and get by Lemma 4 (d) and (RU*) $b_m^\triangle \diamond E(a_l^{\amalg}) = a_l^{\amalg} < b_m^\triangle = b_m^\triangle \diamond E(b_m^\triangle)$.

Choosing now $a := b_m^\triangle$ as $\triangle-$idempotent element, $x_0 := E(a_l^{\amalg}) \in (A, B], x := E(b_m^\triangle)$ the last inequality yields $(a \diamond x_0 = a_l^{\amalg} < a \diamond x) \wedge (x > x_0)$.

This contradicts the second statement of Lemma 3 (a) (if $b_m^\triangle < B$) and the second statement of Lemma 1 (if (DL) $\wedge$ (CRB)), respectively. Thus (γ) is proven.

Now we can show (a).

Let $\amalg|_{[a_l^{\amalg}, b_l^{\amalg}]^2}$ be Archimedean. Then (a1) implies $\bigvee_{m \in K_\triangle} [a_m^{\amalg}, b_m^{\amalg}] \subset [a_m^\triangle, b_m^\triangle]$.

Case 1: If $a_m^\triangle > A$ then (β) yields $(b_m^\triangle = b_l^{\amalg}) \wedge (b_m^\triangle < B \vee [(\mathrm{DL}) \wedge (\mathrm{CRB})])$. Moreover, ($\gamma$) now gives $(a_m^\triangle = a_l^{\amalg})$.

Case 2: If $a_m^\triangle = A \wedge [(\mathrm{Z}) \wedge (\mathrm{CLZ})]$ then (a2) shows that $\amalg|_{[a_m^\triangle, b_m^\triangle]^2}$ is Archimedean.

Since $\amalg|_{[a_l^{\amalg}, b_l^{\amalg}]^2}$ is Archimedean, (a3) leads to (α), and (a) is proven.

(b) To prove (b) we consider 2 cases. If $a_l = A \wedge [(\mathrm{Z}) \wedge (\mathrm{CLZ}]$ then (b) is obviously satisfied.

Let us now consider the case $a_l > A$. In this case we show at first 4 statements:

(b1) $a_l > A \Rightarrow E(a_l) = B$,

(b2) $a_l > A \vee [(\mathrm{Z}) \wedge (\mathrm{CLZ})] \Rightarrow E(b_l) = B$,

(b3) $b_l < B \vee [(\mathrm{DL}) \wedge (\mathrm{CRB})] \Rightarrow \bigwedge_{x \in (A, E(b_l))} b_l \diamond x = b_l$,

(b4) $(a_l > A) \wedge (b_l < B \vee [(\mathrm{DL}) \wedge (\mathrm{CRB})]) \Rightarrow \bigwedge_{x \in (A, E(a_l))} a_l \diamond x = a_l$.

To prove (b1) it is sufficient to prove $a_l \diamond B = a_l$. We put $a := a_l \in (A, B)$ as $\triangle-$idempotent element, $x_0 := E(a_l) \in (A, B]$ and apply the second statement of Lemma 3 (a) to get $\bigwedge_{x \in [E(a_l), B]} a_l \diamond x = a_l$.

Proof of (b2). We assume that $E(b_l) < B$. Then Lemma 4 (c) gives $\bigwedge_{a\in(a_l,b_l]} a \diamond E(b_l) = b_l$.

If $a_l > A$ then we arrive at the contradiction (using (b1)) $b_l = \lim_{a\to a_l+}(a \diamond E(b_l)) = a_l \diamond E(b_l) \le a_l \diamond B = a_l$.

If $a_l = A \wedge (\text{Z}) \wedge (\text{CLZ})$ then (b1) again leads to the contradiction $b_l = \lim_{a\to a_l+}(a \diamond E(b_l)) = a_l \diamond E(b_l) = a_l$.

(b3) To show (b3) we choose $a := b_l \in (A, B]$ as $\triangle-$idempotent element and put $x_0 := E(b_l) \in (A, B]$. Then the third statement of Lemma 3 (a) (if $b_l < B$) and the third statement of Lemma 1 (a) (if (DL) $\wedge$ (CRB)) yield $\bigwedge_{x\in(A,E(b_l)]} b_l \diamond x = b_l$.

The proof for (b4) is similar to the proof of statement (IV) in the proof of Lemma 3. Assume that (b4) is not true, then we have $\bigvee_{x\in(A,E(a_l))} a_l \diamond x \ne a_l$.

Using (RU*) we get $a_l \diamond x < a_l \diamond E(a_l) = a_l$. Because of the monotonicity of $\diamond$ we may assume that $x \in (A, E(b_l))$ Using (b3) we get $a_l \diamond x < a_l < b_l = b_l \diamond x$, and by the intermediate value theorem there exists $a \in (a_l, b_l)$ with $a \diamond x = a_l$. Now we again apply Lemma 2 (d) to get the contradiction
$a_l < b_l \diamond x \le \amalg_{i=1}^{\infty}(a \diamond x) = \lim_{n\to\infty}(\amalg_{i=1}^{n} a_l) = a_l$.

Now we prove the second case of (b), where $a_l > A$. But then (b1) and (b4) result in (b).

(c) Statement (c) follows directly from (b2) and (b3).

(d) We use the following implication:

$$\begin{aligned} &(a_l^{\amalg} > A \vee [(\text{Z}) \wedge (\text{CLZ})]) \wedge (b_l^{\amalg} < B \vee [(\text{DL}) \wedge (\text{CRB})]) \\ \Rightarrow\ &[\,a_l > A \wedge (b_l < B \vee [\,(\text{DL}) \wedge (\text{CRB})])\,] \vee [a_l = A \wedge (\text{Z}) \wedge (\text{CLZ})]. \end{aligned}$$

To prove (d) we get in the case $a_l > A \wedge (b_l < B \vee [\,(\text{DL}) \wedge (\text{CRB})]\,)$:
$a_l = a_l \diamond x \le a \diamond x \le b_l \diamond x = b_l$ (here we have used (b1) and (b4) in the first equality and (b2) and (b3) in the last equality).

In the case $a_l = A \wedge (\text{Z}) \wedge (\text{CLZ})$ we get by (b2): $a_l = A \le a \diamond x \le b_l \diamond B = b_l$.

To prove (e), we first show:

(e1) $(\triangle|_{[a_l,b_l]^2}$ not strict$) \wedge (b_l < B \vee [\,(\text{DL}) \wedge (\text{CRB})]\,) \Rightarrow \amalg|_{[a_l,b_l]^2}$ not strict.

We assume that $\amalg|_{[a_l,b_l]^2}$ is strict. Let $a \in (a_l, b_l)$ be fixed. Then (55) gives: $\bigwedge_{n\in\mathbb{N}} \amalg_{i=1}^{n} a < b_l$. Now we use the assumption that $\triangle|_{[a_l,b_l]^2}$ is not strict by applying Lemma 2 (e) to get: $\bigvee_{s\in\mathbb{N}} b_l \diamond E(a) \le \amalg_{i=1}^{s}(a \diamond E(a)) = \amalg_{i=1}^{s} a$.

Thus we arrive on the one hand at $b_l \diamond E(a) \le \amalg_{i=1}^{s} a < b_l = b_l \diamond E(b_l)$. By monotonicity we may assume that $E(a) \in (A, E(b_l))$ and so we get on the other hand (using (b3)) $b_l \diamond E(a) = b_l$, which is impossible.

Now (e) follows from (e1) by contraposition and (f) is a consequence of of (I), (a) and (e). Thus Theorem 5 is proven. □

We remark that the properties of $\diamond$ are also valid if (RU*) is replaced by (RU). Moreover, the following problem is unsolved:

Present assumptions for the implication: $\triangle|_{[a_l,b_l]^2}$ strict $\Longrightarrow$ $\amalg|_{[a_l,b_l]^2}$ strict.

We believe that the assumptions must be so strong, that already $\triangle = \amalg$ holds.

8. PSEUDO-MULTIPLICATIONS

After these considerations concerning pseudo-additions we now show a general representation theorem for pseudo-multiplications. The only assumption is the weak distributivity, but no unit element, no zero element is required.

We here give the left distributivity version of the result, but there is also a right distributivity version.

Theorem 6. Let $\triangle$ and $\amalg$ be pseudo-additions.

Let $\diamond$ satisfy the weak left distributivity law (DL*).

Then there exists for all $m \in K_\triangle$ and for all $l \in K_\amalg$ a monotone increasing, continuous function $g_{m,l} : (A,B] \to [0,\infty]$ satisfying

$$\bigwedge_{a\in(a_m^\triangle,b_m^\triangle]} \bigwedge_{x\in(A,B]} [a\diamond x \in [a_l^\amalg, b_l^\amalg] \Longrightarrow a\diamond x = h_l^{(-1)}(k_m(a)\cdot g_{m,l}(x))\ ]. \tag{79}$$

Proof. We choose an arbitrary but fixed $m \in K_\triangle$ and denote temporarily k_m and $(a_m^\triangle, b_m^\triangle)$ (cf. (50)) by $k = k_m$ and $(a_m^\triangle, b_m^\triangle) = (a^\triangle, b^\triangle)$. In a first step we show

$$\bigwedge_{l\in K_\amalg} \bigvee_{g_l:(A,B)\to[0,\infty],\, g_l\uparrow,\, g_l \text{ continuous}} \bigwedge_{a\in(a^\triangle,b^\triangle)} \bigwedge_{x\in(A,B)} \Big[a\diamond x\in[a_l^\amalg,b_l^\amalg] \Longrightarrow a\diamond x = h_l^{(-1)}(k(a)\cdot g_l(x))\Big]. \tag{80}$$

(1) This first result (80) will be proven in several steps. We define

$$\bigwedge_{l\in K_\amalg} \bigwedge_{x\in(A,B)} g_l(x) := \begin{cases} 0 & \text{if } \bigwedge_{a\in(a^\triangle,b^\triangle)} a\diamond x \le a_l^\amalg \\ \frac{h_l(a\diamond x)}{k(a)} & \text{if } \bigvee_{a\in(a^\triangle,b^\triangle)} a\diamond x \in (a_l^\amalg, b_l^\amalg) \\ \infty & \text{if } \bigwedge_{a\in(a^\triangle,b^\triangle)} a\diamond x \ge b_l^\amalg. \end{cases}$$

(2) We remark that in (1) the subcases are complete, for otherwise, if we assume that none of the three cases occur, then there exist $a_1, a_2 \in (a^\triangle, b^\triangle)$ satisfying $a_1 \diamond x \le a_l^\amalg < \frac{a_l^\amalg + b_l^\amalg}{2} < b_l^\amalg \le a_2 \diamond x$. But then the intermediate value theorem shows that there is $a \in (a_1, a_2)$ with $a \diamond x = \frac{a_l^\amalg + b_l^\amalg}{2}$, and the second case occurs, which is a contradiction.

(3) Now we investigate the second case in (1) and show, that it is well-defined. Let us consider from now on an arbitrary, but fixed $l \in K_\amalg$.

Let $x \in (A,B)$, and the second case in (1) occurs, that is $\bigvee_{a\in(a^\triangle,b^\triangle)} a\diamond x \in (a_l^\amalg, b_l^\amalg)$. Let us define

(γ) $M_x := \{a \in (a^\triangle, b^\triangle) | a \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})\} \neq \emptyset,\ a_x := \sup M_x \in (a^\triangle, b^\triangle]$.

(3a) Obviously $\bigwedge_{a \in M_x} \frac{h_l(a \diamond x)}{k(a)} \in (0, \infty)$, since $a \in (a^\triangle, b^\triangle), a \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$, and thus $k(a), h_l(a \diamond x) \in (0, \infty)$.

(3b) We prove $\bigwedge_{a \in (a^\triangle, a_x)} a \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$.

Using the definition of a_x and $a < a_x$ we get $\bigvee_{b \in (a, b^\triangle)} b \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$. The monotonicity of $\diamond$ gives $a \diamond x < b_l^{\mathrm{II}}$. We still have to show $a \diamond x > a_l^{\mathrm{II}}$. In contrary, we assume $a \diamond x \leq a_l^{\mathrm{II}}$. But then the assumptions of Lemma 2 are satisfied and Lemma 2 (b) gives $\bigvee_{s \in \mathbb{N}} b \diamond x \leq \mathrm{II}_{i=1}^s (a \diamond x)$. so we arrive at the contradiction $a_l^{\mathrm{II}} < b \diamond x \leq \mathrm{II}_{i=1}^s (a \diamond x) \leq \mathrm{II}_{i=1}^s a_l^{\mathrm{II}} = a_l^{\mathrm{II}}$.

(3c) We show $\bigvee_{\alpha \in (0,\infty)} \bigwedge_{a \in (a^\triangle, a_x] \cap (a^\triangle, b^\triangle)} \frac{h_l(a \diamond x)}{k(a)} = \alpha$.

Let $u, v \in (0, \infty)$ with $u + v \in (0, k(a_x))$. Using $k(a^\triangle) = 0 < u, v < u + v < k(a_x)$ and the intermediate value theorem we get: $\bigvee_{a,b \in (a^\triangle, a_x)} (k(a) = u) \wedge (k(b) = v)$. Thus $k(a) + k(b) \in (0, k(a_x))$ and $a^\triangle = k^{(-1)}(0) < a \triangle b = k^{(-1)}(k(a) + k(b)) < a_x \leq b^\triangle$, so that (DL**) and (3b) imply $b_l^{\mathrm{II}} > (a \triangle b) \diamond x = (a \diamond x) \,\mathrm{II}\, (b \diamond x) = h_l^{(-1)}(h_l[a \diamond x] + h_l[b \diamond x]) > a_l^{\mathrm{II}}$ and so $h_l[(a \triangle b) \diamond x] = h_l[a \diamond x] + h_l[b \diamond x]$ (note that here $h_l^{(-1)} = h_l^{-1}$). This implies $h_l[k^{(-1)}(k(a) + k(b)) \diamond x] = h_l[k^{(-1)}k(a) \diamond x] + h_l[k^{(-1)}k(b) \diamond x]$, or $h_l[k^{(-1)}(u + v) \diamond x] = h_l[k^{(-1)}u \diamond x] + h_l[k^{(-1)}v \diamond x]$. This means, that the function $h_l[k^{(-1)}(\cdot) \diamond x]$ is additive on the restricted domain $\{(u, v) \in (0, \infty)^2 | u + v < k(a_x)\}$ satisfying $\bigwedge_{u \in (0,\infty)} h_l[k^{(-1)}(u) \diamond x] \geq 0$, the solution of which is given by $\bigvee_{\alpha \in [0,\infty]} \bigwedge_{u \in (0, k(a_x))} h_l[k^{(-1)}(u) \diamond x] = \alpha \cdot u$ (see [1], p. 48).

Because k is strictly monotonic increasing and satisfies $k(a^\triangle) = 0$ we get $\bigwedge_{a \in (a^\triangle, a_x)} h_l[k^{(-1)}k(a) \diamond x] = \alpha \cdot k(a)$ and using the continuity of $(\cdot) \diamond x$ we arrive at $\bigwedge_{a \in (a^\triangle, a_x] \cap (a^\triangle, b^\triangle)} h_l[a \diamond x] = \alpha \cdot k(a)$. Because of $k(a) \in (0, \infty)$ and (3a) we get $\bigwedge_{a \in (a^\triangle, a_x] \cap (a^\triangle, b^\triangle)} \frac{h_l(a \diamond x)}{k(a)} = \alpha \in (0, \infty)$.

(3d) Thus we have shown that the second case in (1) is well-defined, and $\bigwedge_{a \in (a^\triangle, a_x] \cap (a^\triangle, b^\triangle)} \frac{h_l(a \diamond x)}{k(a)} = g_l(x)$ (see (3c) and $M_x \subset (a^\triangle, a_x] \cap (a^\triangle, b^\triangle)$).

Before we show (80) we still prove:

(3e) $a_x < b^\triangle \Rightarrow a_x \diamond x = b_l^{\mathrm{II}}$.

By (γ) there exists a sequence $(a_n) \subset (a^\triangle, a_x)$ with $a_n \uparrow a_x$, and (3b) implies $a_x \diamond x = \sup_{n \in \mathbb{N}} (a_n \diamond x) \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}}]$. If we suppose that $a_x \diamond x < b_l^{\mathrm{II}}$ then by the continuity of $(\cdot) \diamond x$ there is $a \in (a_x, b^\triangle)$ such that $b_l^{\mathrm{II}} > a \diamond x \geq a_x \diamond x > a_l^{\mathrm{II}}$, which is a contradiction to the definition of $a_x = \sup M_x$.

(4) Now we prove (80) (but first without the properties of g_l).

We distinguish 3 subcases.

Case 1. $\bigwedge_{a\in(a^{\triangle},b^{\triangle})} a \diamond x \leq a_l^{\text{II}}$:

Let $a \in (a^{\triangle}, b^{\triangle})$ be arbitrary. Because of $a \diamond x \in [a_l^{\text{II}}, b_l^{\text{II}}]$ we have of course $a \diamond x = a_l^{\text{II}}$, and thus (1) implies $h_l^{(-1)}(k(a)g_l(x)) = h_l^{(-1)}(k(a)\cdot 0) = a_l = a\diamond x$.

Case 2. $\bigvee_{a\in(a^{\triangle},b^{\triangle})} a \diamond x \in (a_l^{\text{II}}, b_l^{\text{II}})$:

Case 2a: If $a \in (a^{\triangle}, a_x)$, then $h_l^{(-1)}(k(a)g_l(x)) = h_l^{(-1)}[k(a)\cdot \frac{h_l(a\diamond x)}{k(a)}] = a \diamond x$.

Case 2b: If $a \in [a_x, b^{\triangle})$, then 3e) gives (because of $a \diamond x \in [a_l^{\text{II}}, b_l^{\text{II}}]$) $b_l^{\text{II}} \geq a \diamond x \geq a_x \diamond x = b_l^{\text{II}}$. Thus we obtain, using (3d):

$$h_l^{(-1)}(k(a)g_l(x)) = h_l^{(-1)}(k(a)\frac{h_l(a_x \diamond x)}{k(a_x)}) = h_l^{(-1)}(k(a)\frac{h_l(b_l^{\text{II}})}{k(a_x)}) = b_l^{\text{II}} = a \diamond x.$$

Here we have used that $k(a)\frac{h_l(b_l^{\text{II}})}{k(a_x)} \geq h_l(b_l^{\text{II}})$.

Case 3. $\bigwedge_{a\in(a^{\triangle},b^{\triangle})} a \diamond x \geq b_l^{\text{II}}$.

If $a \in (a^{\triangle}, b^{\triangle})$ is arbitrary, then now (because of $a \diamond x \in [a_l^{\text{II}}, b_l^{\text{II}}]$) we have $a \diamond x = b_l^{\text{II}}$, and thus we obtain from (1) $h_l^{(-1)}(k(a)g_l(x)) = h_l^{(-1)}(k(a)\cdot\infty) = h_l^{(-1)}(\infty) = b_l^{\text{II}} = a \diamond x$.

To prove the monotonicity of g_l, we further fix $l \in K_{\text{II}}$ and introduce J_l, the set of all x, for which the second case in the definition of g_l (see (1)) is valid.

(5) $J_l := \{x \in (A, B) | \bigvee_{a\in(a^{\triangle},b^{\triangle})} a \diamond x \in (a_l^{\text{II}}, b_l^{\text{II}})\}$.

(6) We show: $x, y \in J_l \wedge x \leq y \Rightarrow a_y \leq a_x$.

Assume that $a_x < a_y$. By the definition of a_y there is $a_1 \in (a_x, a_y)$ such that $a_1 \diamond y \in (a_l^{\text{II}}, b_l^{\text{II}})$. Because of $x \in J_l$ we get: $\bigvee_{a_2\in(a^{\triangle},b^{\triangle})} a_2 \diamond x \in (a_l^{\text{II}}, b_l^{\text{II}})$. But then we obtain $a_2 \leq a_x < a_1$ and $a_l^{\text{II}} < a_2 \diamond x < a_1 \diamond x < a_1 \diamond y < b_l^{\text{II}}$. But this means that $a_1 \in M_x$, which contradicts $a_x < a_1$.

(7) We prove that g_l is monotonic increasing.

Let $x, y \in (A, B)$ and let $x \leq y$.

Case 1: If $\bigwedge_{a\in(A,B)} a \diamond y \leq a_l^{\text{II}}$ then the monotonicity of $\diamond$ yields $a \diamond x \leq a \diamond y \leq a_l^{\text{II}}$ and (1) leads to $g_l(x) = 0$.

Case 2: If $\bigwedge_{a\in(A,B)} a \diamond x \geq b_l^{\text{II}}$ then we get in the same manner (using again (1)) $g_l(y) = \infty$.

Case 3: Let $x, y \in J_l$ (note that because of (2) all possible cases are covered). By $y \in J_l$ we obtain $\bigvee_{a_1\in(a^{\triangle},b^{\triangle})} a_1 \diamond y \in (a_l^{\text{II}}, b_l^{\text{II}})$ and $a_1 \in (a^{\triangle}, a_y] \cap (a^{\triangle}, b^{\triangle})$. But (6) and (3d) imply $g_l(x) = \frac{h_l(a_1\diamond x)}{k(a_1)} \leq \frac{h_l(a_1\diamond y)}{k(a_1)} = g_l(y)$.

The continuity of $g_l, l \in K_{\text{II}}$ will be proved in several steps.

(8) $J_l = \emptyset \Rightarrow [\bigwedge_{x\in(a^\triangle,b^\triangle)} g_l(x) = \infty] \vee [\bigwedge_{x\in(a^\triangle,b^\triangle)} g_l(x) = 0]$, so that g_l is obviously continuous. Let us suppose $\bigvee_{x,y\in(a^\triangle,b^\triangle)}(g_l(x) < \infty) \wedge (g_l(y) > 0)$.

Using (1) and $J_l = \emptyset$ we get $\bigwedge_{a\in(a^\triangle,b^\triangle)} a \diamond x \le a_l^{\mathrm{II}} < b_l^{\mathrm{II}} \le a \diamond y$.
By the intermediate value theorem we have $\bigwedge_{x\in(a^\triangle,b^\triangle)} \bigvee_{x(a)\in(A,B)} a\diamond x(a) \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$, which gives the contradiction $J_l \neq \emptyset$.

(9) Let now $l \in K_{\mathrm{II}}$ be fixed with $J_l \neq \emptyset$. We define $x_m := \inf J_l \in [A,B)$ and $x_M := \sup J_l \in (A,B]$.

(10) We show $x_m < x_M$. Indeed, because of $J_l \neq \emptyset$ we know: $\bigvee_{x_1\in(A,B)} \bigvee_{a\in(a^\triangle,b^\triangle)} a\diamond x_1 \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$. The continuity of $a \diamond (\cdot)$ implies $\bigvee_{x_2\in(x_1,B)} a \diamond x_2 \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$ so that $\{x_1, x_2\} \subset J_l$.

(11) Let us prove $\bigwedge_{x\in(x_m,x_M)} x \in J_l$.

Let $x_m < x < x_M$, so that by (9): $\bigvee_{y,z\in J_l} y < x < z$. Choose now $a \in (a^\triangle, a_z) \subset (a^\triangle, a_y)$ (see (6)) to get by (3b): $a \diamond y, a \diamond z \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$. But this implies $a_l < a \diamond y \le a \diamond x \le a \diamond z < b_l^{\mathrm{II}}$ so that by definition $x \in J_l$.

(12) We show: $z \in (x_m, x_M) \Rightarrow \bigwedge_{a\in(a^\triangle,a_z]\cap(a^\triangle,b^\triangle)} \bigwedge_{x\in(x_m,z]} g_l(x) = \frac{h_l(a\diamond x)}{k(a)}$.

Note, that $z \in J_l$ (see 11)) so that $a_z > a^\triangle$ is valid.

Now let $a \in (a^\triangle, a_z] \cap (a^\triangle, b^\triangle)$ and $x \in (x_m, z]$ be arbitrarily chosen. Then (11), (6) and (3d) imply $x \in J_l, (a_z \le a_x \Rightarrow a \in (a^\triangle, a_x] \cap (a^\triangle, b^\triangle))$ and $g_l(x) = \frac{h_l(a\diamond x)}{k(a)}$.

(13) Now we conclude that g_l is continuous on (x_m, x_M).

Let $z \in (x_m, x_M)$ so that (12) shows: $\bigvee_{a\in(a^\triangle,a_z)} \bigwedge_{x\in(x_m,z]} g_l(x) = \frac{h_l(a\diamond x)}{k(a)}$. But $a \diamond (\cdot)$ and h_l are continuous, so that g_l is continuous at first on $(x_m, z]$ and then also on $(x_m, x_M) = \bigcup_{z\in(x_m,x_M)}(x_m, z]$.

(14) We prove: $x_m > A \Rightarrow \bigwedge_{x\in(a^\triangle,x_m)} g_l(x) = 0$.

Let $x \in (a^\triangle, x_m)$. By $J_l \neq \emptyset$ we see: $\bigvee_{x_1\in J_l} \bigvee_{a\in(a^\triangle,b^\triangle)} a \diamond x_1 \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$. Because of $x < x_m \le x_1$ we have $a \diamond x < b_l^{\mathrm{II}}$. Since $x \notin J_l$ and since the third case in (1) doesn't occur we get $g_l(x) = 0$.

In completely the same manner we can show:

(15) $x_m < B \Rightarrow \bigwedge_{x\in(x_m,b^\triangle)} g_l(x) = \infty$.

(16) Let us prove: $x_m > A \Rightarrow \lim_{x\to x_m+} g_l(x) = 0$.

Choose $z \in (x_m, x_M)$ so that again $\bigvee_{a\in(a^\triangle,a_z)} \bigwedge_{x\in(x_m,z]} g_l(x) = \frac{h_l(a\diamond x)}{k(a)}$ (see (12)).

But (6) and (3b) give $a < a_z \le a_x$ and $\bigwedge_{x\in(x_m,z]} a \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$. Thus $a \diamond x_m \ge a_l^{\mathrm{II}}$ and 14) implies $\bigwedge_{x\in(a^\triangle,x_m)} g_l(x) = 0$. By (1) and the continuity of $a \diamond (\cdot)$ we get $\bigwedge_{x\in(a^{\triangle},x_m)} a \diamond x \le a_l^{\mathrm{II}}$ and $a \diamond x_m \le a_l^{\mathrm{II}}$ so that we arrive at $\lim_{x\to x_m+} g_l(x) = \lim_{x\to x_m+} \frac{h_l(a\diamond x)}{k(a)} = \frac{h_l(a\diamond x_m)}{k(a)} = \frac{h_l(a_l^{\mathrm{II}})}{k(a)} = 0$.

(17) We show: $x_M < B \Rightarrow \lim_{x\to x_M+} g_l(x) = \infty$.

Let us assume in contrary $\lim_{x\to x_M+} g_l(x) < \infty$ (by (7) this limit exists). We prove first:

(17a) $\lim_{x\to x_M+} a_x = a^{\triangle}$ (the a_x exist by (11) for $x \in (x_m, x_M)$, the limit exists by (6)). We assume in contrary that $\lim_{x\to x_M+} a_x > a^{\triangle}$, so that we have $\bigvee_{\epsilon>0} \bigwedge_{x\in(x_m,x_M)} a_x > a^{\triangle} + \epsilon$. Now (3b) and the continuity of $a \diamond (\cdot)$ imply $\bigwedge_{x\in(x_m,x_M)} (a^{\triangle}+\epsilon) \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$ and $(a^{\triangle}+\epsilon) \diamond x_M \le b_l^{\mathrm{II}}$. By (18) we have $\bigwedge_{x\in(x_M,b^{\triangle})} g_l(x) = \infty$. Thus (1) and the continuity of $a \diamond (\cdot)$ yield $\bigwedge_{x\in(x_M,b^{\triangle})} (a^{\triangle}+\frac{\epsilon}{2}) \diamond x \ge b_l^{\mathrm{II}}$ and

$$(*) \quad b_l^{\mathrm{II}} \le (a^{\triangle}+\tfrac{\epsilon}{2}) \diamond x_M \le (a^{\triangle}+\epsilon) \diamond x_M \le b_l^{\mathrm{II}}.$$

Using (3d) we get $\bigwedge_{x\in(x_m,x_M)} \frac{h_l((a^{\triangle}+\epsilon)\diamond x)}{k(a^{\triangle}+\epsilon)} = \frac{h_l((a^{\triangle}+\frac{\epsilon}{2})\diamond x)}{k(a^{\triangle}+\frac{\epsilon}{2})} = g_l(x)$ which leads to (using our above assumption) $\frac{h_l((a^{\triangle}+\epsilon)\diamond x_M)}{k(a^{\triangle}+\epsilon)} = \frac{h_l((a^{\triangle}+\frac{\epsilon}{2})\diamond x_M)}{k(a^{\triangle}+\frac{\epsilon}{2})} = \lim_{x\to x_M+} g_l(x) < \infty$, that is (see $(*)$) $\frac{h_l(b_l^{\mathrm{II}})}{k(a^{\triangle}+\epsilon)} = \frac{h_l(b_l^{\mathrm{II}})}{k(a^{\triangle}+\frac{\epsilon}{2})} < \infty$. But $h_l(b_l^{\mathrm{II}}) > 0$ implies $k(a^{\triangle}+\epsilon) = k(a^{\triangle}+\frac{\epsilon}{2})$, which contradicts the strict monotonicity of k. Thus (17a) is proven.

If we now show

(17b) $\lim_{x\to x_M+} g_l(x) = \infty$, then this is a contradiction to our above assumption, and (17) is shown.

Let $(x_n) \subset (x_m, x_M)$ be an arbitrary sequence satisfying $\lim_{n\to\infty} x_n = x_M$. By (17a) we get $\bigvee_{n_0\in\mathbb{N}} \bigwedge_{n_0\le n\in\mathbb{N}} a_{x_n} < b^{\triangle}$, and (3d), (3e), (17a) and $k(a^{\triangle}) = 0$ imply

$$\lim_{n\to\infty} g_l(x_n) = \lim_{n\to\infty} \frac{h_l(a_{x_n} \diamond x_n)}{k(a_{x_n})} = \lim_{n\to\infty} \frac{h_l((b_l^{\mathrm{II}})}{k(a_{x_n})} = \frac{h_l((b_l^{\mathrm{II}})}{k(\lim_{n\to\infty} a_{x_n})} = \infty.$$

Now the continuity of g_l on the open interval (A, B) is shown:

— (14), (16) and (7) imply: $x_m > A \Rightarrow \lim_{x\to x_m} g_l(x) = 0 = g_l(x_m)$.

— (15), (17) and (7) imply: $x_M < B \Rightarrow \lim_{x\to x_M} g_l(x) = \infty = g_l(x_M)$.

— By (14), (13) and (15) we get: $J_l \ne \emptyset \Rightarrow g_l$ is continuous.

Thus (80) is proven.

Now we let vary $m \in K_{\triangle}$ and get (79) for all $a \in (a_m^{\triangle}, b_m^{\triangle})$ and for all $x \in (A, B)$. In the next step we show

(18) $\bigwedge_{a\in(a_m^{\triangle},b_m^{\triangle}]} \bigwedge_{x\in(A,B)} (\, a \diamond x \in [a_l^{\mathrm{II}}, b_l^{\mathrm{II}}] \Rightarrow a \diamond x = h_l^{(-1)}(k_m(a) \cdot g_{m,l}(x))\,)$.

Let $a = b_m^{\triangle}, x \in (A, B)$ and $a \diamond x \in [a_l^{\mathrm{II}}, b_l^{\mathrm{II}}]$. By (80) we have only to show that $b_m^{\triangle} \diamond x = h_l^{(-1)}(k_m(b_m^{\triangle}) \cdot g_{m,l}(x))$.

Case 1: Let $b_m^{\triangle} \diamond x = a_l^{\mathrm{II}}$. Then we get $g_l(x) = 0$ by (1) which results in $h_l^{(-1)}(k_m(b_m^{\triangle})g_{m,l}(x)) = h_l^{(-1)}(0) = a_l^{\mathrm{II}} = b_m^{\triangle} \diamond x$.

Case 2: $b_m^{\triangle} \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}}]$: By the continuity of $(\cdot) \diamond x$ there exists a sequence $(a_n) \subset (a_m^{\triangle}, b_m^{\triangle})$ satisfying $a_n \uparrow b_m^{\triangle}$ and $\bigwedge_{n\in\mathbb{N}} a_n \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}}]$ so that (using (80)) $b_m^{\triangle} \diamond x = \lim_{n\to\infty}(a_n \diamond x) = \lim_{n\to\infty} h_l^{(-1)}(k_m(a) \cdot g_{m,l}(x)) = h_l^{(-1)}(k_m(b_m^{\triangle})g_{m,l}(x))$.

(19) Now we extend $g_{m,l}$ by $g_{m,l}(B) := \lim_{x\to B-} (g_{m,l}(x))$.

This limit exists, since $g_{m,l}$ is monotonic increasing. Finally we prove

(20) $\bigwedge_{a\in(a_m^{\triangle}, b_m^{\triangle}]} \bigwedge_{x\in(A,B]} (\ a \diamond x \in [a_l^{\mathrm{II}}, b_l^{\mathrm{II}}] \Rightarrow a \diamond x = h_l^{(-1)}(k_m(a) \cdot g_{m,l}(x))\ )$.

Let $a \in (a_m^{\triangle}, b_m^{\triangle}], x = B$ and let $a \diamond x \in [a_l^{\mathrm{II}}, b_l^{\mathrm{II}}]$. We have to show $a \diamond B = h_l^{(-1)}(k_m(a) \cdot g_{m,l}(B))$.

Case 1. Let $a \diamond B = a_l^{\mathrm{II}}$. Then Lemma 2 (d) implies $b_l^{\mathrm{II}} \diamond B \leq \mathrm{II}_{i=1}^{\infty}(a \diamond B) = \lim_{n\to\infty}(\mathrm{II}_{i=1}^{n} a_l^{\mathrm{II}}) = a_l^{\mathrm{II}}$, and the monotonicity of $\diamond$ yields

$$\bigwedge_{a\in(a_m^{\triangle}, b_m^{\triangle})} \bigwedge_{x\in(A,B)} a \diamond x \leq a_l^{\mathrm{II}}.$$

Thus (1) and (19) lead to $\bigwedge_{x\in(A,B)} g_{m,l}(x) = 0$ and $g_{m,l}(B) = 0$ so that $h_l^{(-1)}(k_m(a) \cdot g_{m,l}(B)) = h_l^{(-1)}(0) = a_l^{\mathrm{II}} = a \diamond B$.

Case 2. Let $a \diamond B \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}}]$. Since $a \diamond (\cdot)$ is continuous on $(A, B]$ there is a sequence $(x_n) \subset (A, B)$ satisfying $x_n \uparrow B$ and $\bigwedge_{n\in\mathbb{N}} a \diamond x_n \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}}]$.

Thus we obtain (using (18) and (19))

$$a \diamond B = \lim_{n\to\infty}(a \diamond x_n) = \lim_{n\to\infty} h_l^{(-1)}(k_m(a) \cdot g_{m,l}(x_n)) = h_l^{(-1)}(k_m(a) \cdot g_{m,l}(B)).$$

Thus Theorem 6 is proven. □

Let us add some remarks and examples.

(I) In the proof of Theorem 6 we have actually shown a more general result:

For the representation of $\diamond$ on the open intervals in (80) only the following property of $\diamond$ was used: $(\cdot) \diamond x$ is continuous and monotonic increasing on $(a^{\triangle}, b^{\triangle})$ for all $x \in (A, B)$.

(II) If in addition (Z) is supposed, then we can extend $g_{m,l}$ by $g_{m,l}(A) = 0$. This extension is monotonic increasing, but not necessarily continuous.Moreover the representation (79) is valid also for $x = A$:

$$a \diamond A \in [a_l^{\mathrm{II}}, b_l^{\mathrm{II}}] \Rightarrow a \diamond A = A = a_l^{\mathrm{II}} = h_l^{(-1)}(0) = h_l^{(-1)}(k_m(a) \cdot g_{m,l}(A)).$$

(III) If the ordinal sum for II contains at least two Archimedean parts, then an extension of the representation (79) to $a = a_m$ is not possible:

Consider $\triangle = \amalg, b_{l-1}^{\amalg} = a_l^{\amalg}$ and let e be a right unit for $\diamond$. Then

$$a_m^{\triangle} \diamond e = a_l^{\amalg} = b_{l-1}^{\amalg} \in [a_{l-1}^{\amalg}, b_{l-1}^{\amalg}],$$

but

$$h_{l-1}^{(-1)}(k_m(a_m^{\triangle})g_{m,l-1}(e)) = h_{l-1}^{(-1)}(0) = a_{l-1}^{\amalg}.$$

(IV) Using the right distributivity version of Theorem 6 we can deduce the following result, which is contained in [20] (see Theorem 5.1):

Let $\perp = \amalg = \hat{+} : [0,\infty]^2 \to [0,\infty]$ be a pseudo-addition with generator set $\{g_k : [a_k, b_k] \to [0,\infty] \,|\, k \in K_{\hat{+}}\}$, and let $\hat{\cdot} : [0,\infty]^2 \to [0,\infty]$ be a pseudo-multiplication, which satisfies the right distributivity law (DR) (with respect to $(\hat{+}, \hat{+})$ and (CLB), (LU), (Z) and (CRZ). Then there exist monotonic increasing and continuous functions $H_k : (0,\infty] \to (0,\infty], k \in K_{\hat{+}}$ with $H_k(0,\infty] \subset (0,\infty]$ satisfying

$$H_k(\tilde{e}) = 1 \quad \text{and} \quad \bigwedge_{a\in(0,\infty]} \bigwedge_{x\in[a_k,b_k]} a \hat{\cdot} x = g_k^{(-1)}(H_k(a) \cdot g_k(x)).$$

The following example will be needed for our next main result and gives the representation of a pseudo-multiplication, if in Theorem 6 the two pseudo-additions coincide and if the pseudo-additions are Archimedean t-conorms.

Corollary 1. Let $\triangle = \amalg$ be continuous, Archimedean t-conorms (on $[A,B]^2$), and let $\diamond$ be a pseudo-multiplication satisfying the weak left distributivity law (DL*) and (LU). Then the following is valid:

(a) $\triangle$ strict $\Rightarrow \tilde{e} < B$.

(b) There exists exactly one generator k of $\triangle$ with $k(\tilde{e}) = 1$ and

$$\bigwedge_{a,x\in[A,B]} [\,(\mathrm{Z}) \vee (a,x) \notin \{(A,B),(B,A)\} \vee (\triangle \text{ not strict}) \Rightarrow a \diamond x = k^{(-1)}(k(a)k(x))\,].$$

(c) If $[\,(\mathrm{Z}) \wedge (\triangle \text{ strict }) \,] \vee [\tilde{e} = B]$ then we have:

$$\bigwedge_{a,x\in[A,B]} a \diamond x = k^{-1}(k(a)k(x)).$$

Thus, if $\tilde{e} = B$, then $\diamond$ is a strict t-norm on $[A,B]^2$ with multiplicative generator k.

Proof. Let $k : [A,B] \to [0,\infty]$ be a generator of $\triangle = \amalg$. By Theorem 6 there is a continuous, monotonic increasing function $g : (A,B] \to [0,\infty]$ with $\bigwedge_{a,x\in(A,B]} a \diamond x = k^{(-1)}(k(a)g(x))$.

To prove (a) let us assume that $\triangle$ is strict, but $\tilde{e} = B$. By (55) we have $k(\tilde{e}) = k(B) = \infty$, and we get the contradiction

$$\bigwedge_{x\in(A,B]} x = \tilde{e}\diamond x = k^{(-1)}(k(\tilde{e})g(x)) \in \{A, B\}$$

(since $k(\tilde{e})g(x) \in \{0,\infty\}$). Thus $\tilde{e} < B$.

(b) By (a) we have $k(\tilde{e}) < \infty$, and since a generator is determined up to a multiplicative constant we may choose $k(\tilde{e}) = 1$. But then we obtain $\bigwedge_{x\in(A,B)} x = \tilde{e}\diamond x = k^{(-1)}(k(\tilde{e})g(x)) = k^{-1}(g(x)\wedge k(B))$. Thus the strict monotonicity of k implies $\bigwedge_{x\in(A,B)} k(B) > k(x) = g(x)\wedge k(B)$, or $k(x) = g(x)$ for all $x \in (A,B)$. Since k, g are continuous we have $k = g$ on $(A,B]$ and thus $a\diamond x = k^{(-1)}(k(a)k(x))$ for all $a, x \in (A,B]$.

To prove (b) we still have to show that the last equality holds for $(a = A)\vee(x = A)$.

Case 1: (Z): This case is obvious because of $k(A) = 0$.

Case 2: $(a,x) \notin \{(A,B),(B,A)\} \vee (\triangle$ not strict): By (55) we have $k(a), k(x) \in [0,\infty)$.

Case 2a: If $(a = A)\wedge x \in (A,B]$ then we obtain: $A \le A\diamond x \le \lim_{b\to A+}(b\diamond x) = \lim_{b\to A+} k^{(-1)}(k(b)k(x)) = k^{(-1)}(k(A)k(x)) = A$.

Case 2b: The case $a \in (A,B]\wedge(x = A)$ is similar to Case 2a.

Case 2c: If $(a = A)\wedge(x = A)$ the case 2a implies
$A \le A\diamond A \le \lim_{y\to A+}(A\diamond y) = A = k^{(-1)}(k(A)k(A))$.

(c) If (Z) $\wedge$ ($\triangle$ strict) then $k(B) = \infty$ and $k^{(-1)} = k^{-1}$, and (b) implies the representation of $\diamond$. Finally, if $\tilde{e} = B$, then $\triangle$ is not strict (see (a)).

Thus (b) gives $\bigwedge_{a,x\in[A,B]} a\diamond x = k^{(-1)}(k(a)k(x))$. But now $k(a)k(x)$ remains in the range of k because of $k(a) \le k(B) = k(\tilde{e}) = 1$.

This finishes the proof of Corollary 1. □

If the pseudo-addition has a unit element then we combine Theorem 6 with Theorem 3, Theorem 4 and Corollary 1 to get more information on the pseudo-additions and pseudo-multiplications under consideration. Thus the following result looks a little bit complicated for the first moment, but it is helpful, since it covers and generalizes many recent results of the literature (see for example the last statement (h) of the following Theorem 7).

Theorem 7. Let $\triangle$ and $\amalg$ be pseudo-additions.

Let $\diamond$ satisfy the weak left distributivity law (DL*) and (RU) and (LU) (existence of a unit e).

We assume that $\triangle = \amalg = \vee$ is not valid. Then we have:

(a) $\triangle = \amalg |K_\triangle| = 1$, $e \in (a_1, b_1]$, $a\diamond x \in [a_1, b_1]$ for all $a, x \in [a_1, b_1]$ (multiplication is compatible with the structure of $\triangle$).

(b) There exists exactly one generator k of $\triangle$ with $k(e) = 1$ and $\bigwedge_{a,x\in[a_1,b_1]}[(a,x) \notin \{(a_1,b_1),(b_1,a_1)\} \Longrightarrow a \diamond x = k^{(-1)}(k(a)\cdot k(x))]$.

(c) $[a_1 = A \wedge (\mathrm{Z})] \vee [\triangle|_{[a_1,b_1]^2}$ not strict$] \Longrightarrow \bigwedge_{a,x\in[a_1,b_1]} a\diamond x = k^{(-1)}(k(a)\cdot k(x))$.

(d) $[a_1 = A \wedge (\mathrm{Z}) \wedge \triangle|_{[a_1,b_1]^2}$ strict $] \vee [e = b_1] \Longrightarrow \bigwedge_{a,x\in[a_1,b_1]} a \diamond x = k^{-1}(k(a) \cdot k(x))$.

(e) $[b_1 < B] \vee [(\mathrm{DL}) \wedge (\mathrm{CRB})] \Longrightarrow (a_1 = A) \wedge (\triangle|_{[a_1,b_1]^2}$ strict.)

(f) $[a_1 > A] \vee [(\mathrm{Z}) \wedge (\mathrm{CLZ})] \Longrightarrow b_1 = B$.

(g) $[a_1 > A \wedge (\mathrm{CRB})] \vee [e = b_1] \Longrightarrow (b_1 = B) \wedge (\triangle|_{[a_1,b_1]^2}$ not strict.)

(h) Assuming in addition (CRB) and (Z) we get the following representation theorem.

Theorem.

(I) If $e = B$ then: $(\triangle|_{[a_1,B]^2}$ is not strict$) \wedge (\bigwedge_{a,x\in[a_1,B]^2} a \diamond x = k^{-1}(k(a)\cdot k(x)))$ (here $a_1 \in [0, B)$; the assumptions (CRB), (Z) are not needed for statement (I)).

(II) If $e < B$, then $e \in (a_1, b_1)$, and there are only 3 possibilities:

— $\triangle|_{[A,b_1]^2}$ is strict and $\diamond$ satisfies (d) with $a_1 = A$, $b_1 \in (A, B)$,

— $\triangle|_{[a_1,B]^2}$ is not strict and $\diamond$ satisfies (c) with $a_1 \in (A, B)$, $b_1 = B$, (here (Z) is not needed),

— $\triangle$ is an Archimedean pseudo-addition and $\diamond$ satisfies (c) with $a_1 = A$, $b_1 = B$.

Proof. Statement (a) follows from Theorem 4 (b), (c) (and thus Corollary 1 can be applied to $\diamond|_{[a_1,b_1]^2} : [a_1, b_1]^2 \to [a_1, b_1]$).

The statements (b) and (c) follow from Corollary 1 (c), whereas statement (d) follows from Corollary 1 (c).

If $b_1 < B$ in (e) then the result follows from Theorem 4, and if (DL) $\wedge$ (CRB) in (e) then Theorem 3 (b) gives the result.

Statement (f) results from Theorem 4.

In the case $(a_1 > A \wedge (\mathrm{CRB})$ Theorem 4 (c) implies (g). If $e = b_1$ then Corollary 1 (a) shows that $\triangle|_{[a_1,b_1]^2}$ is not strict. The contraposition of (e) yields $b_1 = B$.

To prove (h), (I) let first $e = B$. Because of $e \in (a_1, b_1]$ (see (a)) we get $b_1 = B = e$. Now (g) implies: $\triangle|_{[a_1,b_1]^2}$ is not strict. Finally (b) and (d) imply the representation of k in (h).

(II) Now let $e < B$. If we assume that $e = b_1$ then (g) gives the contradiction $b_1 = B = e$. Thus we obtain $e \in (a_1, b_1)$.

We distinguish the two cases $(a_1 > A) \vee (b_1 < B)$ and $(a_1 = A) \wedge (b_1 = B)$.

If $b_1 < B$ then (e) implies that $a_1 = A$ and $\triangle|_{[A,b_1]^2}$ is strict.

If $a_1 > A$ then (g) shows that $b_1 = B$ and $\triangle|_{[a_1,B]^2}$ is not strict (only here (CRB) is used).

If $(a_1 > A) \wedge (b_1 < B)$ then (f) implies the contradiction $b_1 = B$.

Finally, the statement in the case $(a_1 = A) \wedge (b_1 = B)$ is obvious.

This finishes the proof of Theorem 7. □

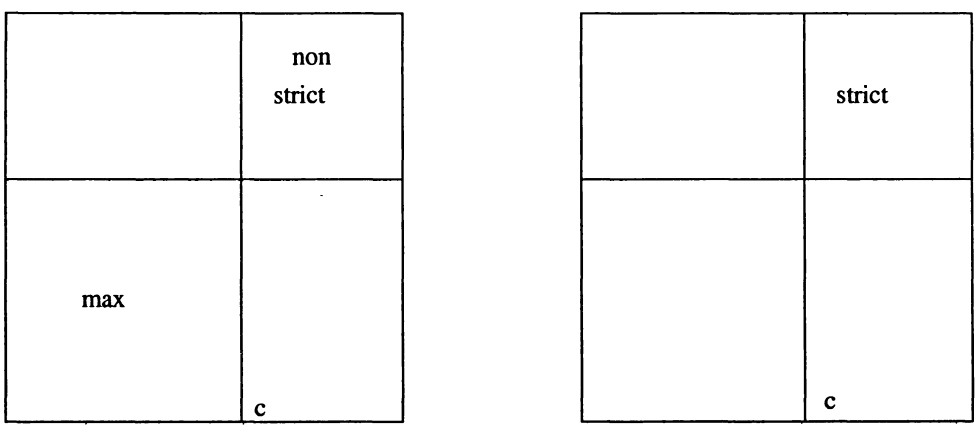
Example 3. We remark that Example 2 with $B = 2$ shows that the associativity of $\diamond$ doesn't follow in general from the representation $a \diamond x = k^{(-1)}(k(a) \cdot k(x))$, a, $x \in [A, B]$: In the example we have $k(x) = x$ for all $x \in [0, 2]$, but $(2 \diamond 2) \diamond \frac{1}{2} = 2 \diamond \frac{1}{2} = 1 \neq 2 = 2 \diamond 1 = 2 \diamond (2 \diamond \frac{1}{2})$.

Theorem 7 means that, if we assume only a weak distributivity law (which is for example satisfied if the conditionally distributivity (74) is assumed) and the existence of a unit element e, then the pseudo-multiplication $\diamond$ is completely determined on the only Archimedean square $(a_1, b_1]^2$ by means of the uniquely determined generator k (with $k(e) = 1$) of the pseudo-addition $\triangle$ ($\triangle \neq \vee$). If (CRB) and (Z) are satisfied then $\diamond$ is completely determined on $[a_1, b_1]^2$.

Especially the representation result (h) of Theorem 7 can be applied to each pseudo-addition $\triangle$ and to each pseudo-multiplication $\diamond$ (satisfying (DL*), (LU), (RU), (CRB) and (Z)).

For example, statement (h) (I) implies immediately the following result, which is Theorem 5.21 in [11].

Theorem 8. Let $T : [0, 1]^2 \to [0, 1]$ be a continuous t-norm, and let $S : [0, 1]^2 \to [0, 1]$ be a continuous t-conorm with $S \neq \vee$, satisfying the conditionally distributivity (74). Then there exists $c \in [0, 1)$ such that $S|_{[c,1]^2}$ is Archimedean and non strict. For the additive generator s of $S|_{[c,1]^2}$ with $s(1) = 1$ we have: $\bigwedge_{a,x \in [c,1]} T(a, x) = s^{-1}(s(a) \cdot s(x))$, that is, $T|_{[c,1]^2}$ is a strict t-norm on $[c, 1]^2$.



S T

An additional advantage of Theorem 7 is that in many cases Theorem 7 implies the associativity and commutativity of the pseudo-multiplication:

Theorem 9. Let $\triangle$ and $\amalg$ be pseudo-additions.

Let $\diamond$ satisfy the left distributivity law (DL), and in addition (CRB), (CLZ), (RU), (LU) and (Z). We assume that $\triangle = \amalg = \vee$ is not valid.

Then $\triangle = \amalg$ is a strict (Archimedean) t-conorm on $[A, B]^2$, and there is exactly one generator k of $\triangle$ satisfying $k(e) = 1, k(B) = \infty$ and

$$\bigwedge_{a,x\in[A,B]} a \diamond x = k^{-1}(k(a) \cdot k(x)). \tag{81}$$

Moreover, $\diamond$ is strictly monotone increasing in each place on $(A, B)^2$, and $\diamond$ is associative and commutative.

Proof. The proof is an immediate application of Theorem 7. Theorem 7 (a) implies ($\triangle = \amalg$) and $|K_\triangle| = 1$.

Theorem 7 (f) gives $b_1 = B$ and Theorem 7 (e) implies $a_1 = A$ and $\triangle|_{[A,B]^2}$ is strict.

Theorem 7 (d) implies (81).

This last result is an extension of a result of [13] where more assumptions were needed ($\triangle = \amalg$, the condition $a \diamond x = A \Rightarrow (a = A) \vee (x = A)$, associativity and commutativity of $\diamond$, and thus both distributivity laws (DL) and (DR)). □

The following result gives the representation of pseudo-multiplications if we require both weak left distributivity (DL*) and weak right distributivity (DR*).

Theorem 10. Let $\triangle, \perp$ and $\amalg$ be pseudo-additions.

Let $\diamond$ be a pseudo-multiplication satisfying (DL*) and (DR*). Moreover let $m \in K_\triangle$ and $k \in K_\perp$.

Then there are only two possibilities:

(I)

$$\bigvee_{d \text{ idempotent of } \amalg} \bigwedge_{a\in(a_m^\triangle, b_m^\triangle]} \bigwedge_{x\in(a_k^\perp, b_k^\perp]} a \diamond x = d. \tag{82}$$

(II) There exist exactly one $l \in K_\amalg$ and $c_{m,k} \in (0, \infty)$ satisfying

$$\bigwedge_{a\in(a_m^\triangle, b_m^\triangle]} \bigwedge_{x\in(a_k^\perp, b_k^\perp]} a \diamond x = h_l^{(-1)}(c_{m,k} \cdot k_m(a) \cdot g_k(x)) \in (a_l^\amalg, b_l^\amalg]. \tag{83}$$

Proof. We start the proof by proving first 2 preliminary statements (1) and (2).

Let us fix $m \in K_\triangle$ and $k \in K_\perp$. Then we show: If $a_0 \in (a_m^\triangle, b_m^\triangle], x \in (A, B]$ and if $d \in (A, B]$ is $\amalg$-idempotent then we have:

$$(1)\quad \Big(a_0 \diamond x \le d \Rightarrow \bigwedge_{a\in(a_m^{\triangle},b_m^{\triangle}]} a\diamond x \le d\Big) \wedge \Big(a_0 \diamond x > d \Rightarrow \bigwedge_{a\in(a_m^{\triangle},b_m^{\triangle}]} a\diamond x > d\Big).$$

The first statement in (1) follows from Lemma 2 (d): $a\diamond x \le b_m^{\triangle}\diamond x \le \amalg_{i=1}^{\infty}(a_0\diamond x) \le \lim_{n\to\infty}\amalg_{i=1}^{n} d = d$.

Let us assume $\bigvee_{a\in(a_m^{\triangle},b_m^{\triangle}]} a\diamond x \le d$. Then Lemma 2 (d) leads to the contradiction $d < a_0\diamond x \le b_m^{\triangle}\diamond x \le \amalg_{i=1}^{\infty}(a_0\diamond x) \le \lim_{n\to\infty}\amalg_{i=1}^{n} d = d$.

In exactly the same manner we prove:

If $x_0 \in (a_k^{\perp}, b_k^{\perp}]$, $a\in(A,B]$ and if $d\in(A,B]$ is $\amalg$−idempotent then we have:

$$(2)\quad \Big(a \diamond x_0 \le d \Rightarrow \bigwedge_{x\in(a_k^{\perp},b_k^{\perp}]} a\diamond x \le d\Big) \wedge \Big(a \diamond x_0 > d \Rightarrow \bigwedge_{x\in(a_k^{\perp},b_k^{\perp}]} a\diamond x > d\Big).$$

Now we distinguish two cases which will lead to (I) and (II) of Theorem 10.

Case (I): $\bigwedge_{a\in(a_m^{\triangle},b_m^{\triangle})}\bigwedge_{x\in(a_k^{\perp},b_k^{\perp})} a\diamond x$ is $\amalg$−idempotent.

Case (II): $\bigvee_{l\in K_{\amalg}}\bigvee_{a_0\in(a_m^{\triangle},b_m^{\triangle})}\bigvee_{x_0\in(a_k^{\perp},b_k^{\perp})} a_0\diamond x_0 \in (a_l^{\amalg}, b_l^{\amalg})$.

We treat case (I) and note: [$a\diamond x$ is $\amalg$−idempotent] $\Leftrightarrow$ [$a\diamond x \in [A,B]\setminus\bigcup_{l\in K_{\amalg}}(a_l^{\amalg}, b_l^{\amalg})$].

Since $(\cdot)\diamond x$ and $a\diamond(\cdot)$ are left-continuous on $(A,B]$ we get:

$\bigwedge_{a\in(a_m^{\triangle},b_m^{\triangle}]}\bigwedge_{x\in(a_k^{\perp},b_k^{\perp}]} a\diamond x$ is $\amalg$−idempotent.

We choose $d := b_m^{\triangle}\diamond b_k^{\perp}$ as $\amalg$−idempotent element and get for arbitrary $a\in(a_m^{\triangle},b_m^{\triangle}]$ and arbitrary $x\in(a_k^{\perp},b_k^{\perp}]$ by applying (1) and (2): $a\diamond x \le b_m^{\triangle}\diamond b_k^{\perp} = d \le a\diamond b_k^{\perp} \le a\diamond x$ so that $a\diamond x = d$. Thus (I) is proven.

We now assume case (II). By applying (1) and (2) we obtain

$$(3)\quad \bigwedge_{a\in(a_m^{\triangle},b_m^{\triangle}]} a\diamond x_0 \in (a_l^{\amalg}, b_l^{\amalg}] \text{ and } \bigwedge_{a\in(a_m^{\triangle},b_m^{\triangle}]}\bigwedge_{x\in(a_k^{\perp},b_k^{\perp}]} a\diamond x \in (a_l^{\amalg}, b_l^{\amalg}],$$

and thus $l\in K_{\amalg}$ is uniquely determined. But now we can apply Theorem 6 and the right distributivity version of Theorem 6 to get that there are continuous and strictly increasing functions $g_{m,l}, K_{k,l} : (A,B] \to \infty$ satisfying

$$(4)\quad \bigwedge_{a\in(a_m^{\triangle},b_m^{\triangle}]}\ \bigwedge_{x\in(A,B]} [a\diamond x \in [a_l^{\amalg}, b_l^{\amalg}] \Rightarrow a\diamond x = h_l^{(-1)}(k_m(a)\cdot g_{m,l}(x))]$$

and

$$(5)\quad \bigwedge_{x\in(A,B]}\ \bigwedge_{a\in(a_k^{\perp},b_k^{\perp}]} [a\diamond x \in [a_l^{\amalg}, b_l^{\amalg}] \Rightarrow a\diamond x = h_l^{(-1)}(K_{k,l}(a)\cdot g_k(x))]$$

so that

$$(6)\quad \bigwedge_{a\in(a_m^{\triangle},b_m^{\triangle}]}\ \bigwedge_{a\in(a_k^{\perp},b_k^{\perp}]} a\diamond x = h_l^{(-1)}(k_m(a)\cdot g_{m,l}(x)) = h_l^{(-1)}(K_{k,l}(a)\cdot g_k(x)).$$

We now define

(7) $I := \{x \in (a_k^\perp, b_k^\perp) | \bigvee_{a \in (a_m^\triangle, b_m^\triangle)} a \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})\}$ and $x_M := \sup I \in (a_k^\perp, b_k^\perp]$.

Then $I \neq \emptyset$, and we prove as next step

(8) $\bigwedge_{z \in (a_k^\perp, x_M)} \bigvee_{a_z \in (a_m^\triangle, b_m^\triangle)} \bigwedge_{a \in (a_m^\triangle, a_z)} \bigwedge_{x \in (a_k^\perp, z]} a \diamond x \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$.

If $z \in (a_k^\perp, x_M)$ is arbitrary then (7) implies

$$\bigvee_{x_z \in (z, x_M)} \bigvee_{a_z \in (a_m^\triangle, b_m^\triangle)} a_z \diamond x_z \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}}).$$

Now let $a \in (a_m^\triangle, a_z)$ and $x \in (a_k^\perp, z]$ be arbitrary elements. Using (3) we arrive at $a_l^{\mathrm{II}} < a \diamond x \leq a_z \diamond x_z < b_l^{\mathrm{II}}$, and (8) is shown. But (8) and (6) imply that for all $z \in (a_k^\perp, x_M)$ there exists $a_z \in (a_m^\triangle, a_m^\triangle)$ with

$$\bigwedge_{a \in (a_m^\triangle, a_z)} \bigwedge_{x \in (a_k^\perp, z]} a \diamond x = h_l^{(-1)}(k_m(a) g_{m,l}(x)) = h_l^{(-1)}(K_{k,l}(a) g_k(x)) \in (a_l^{\mathrm{II}}, b_l^{\mathrm{II}}).$$

But this implies (since here we have $h_l^{(-1)} = h_l^{-1}$)

$$\bigwedge_{a \in (a_m^\triangle, a_z)} \bigwedge_{x \in (a_k^\perp, z]} k_m(a) g_{m,l}(x) = K_{k,l}(a) g_k(x) \in (0, \infty) \quad \text{or}$$

$$\bigwedge_{a \in (a_m^\triangle, a_z)} \bigwedge_{x \in (a_k^\perp, z]} \frac{g_{m,l}(x)}{g_k(x)} = \frac{K_{k,l}(a)}{k_m(a))} = \text{constant} \in (0, \infty).$$

These constants are equal for all $z \in (a_k^\perp, x_M)$ because of $(a_k^\perp, x_M) = \bigcup_{z \in (a_k^\perp, x_M)} (a_k^\perp, z]$. Using also the continuity of $g_{m,l}$ and g_k we arrive at:

(9) $\bigvee_{c_{m,k} \in (0,\infty)} \bigwedge_{x \in (a_k^\perp, x_M]} g_{m,l}(x) = c_{m,k} \cdot g_k(x)$.

(10) We now prove $x_M = b_k^\perp$.

Let us assume that $x_M < b_k^\perp$. Now let $x \in (x_M, b_k^\perp)$ be arbitrary. By definition of I and x_M we get first $\bigwedge_{a \in (a_m^\triangle, b_m^\triangle)} a \diamond x \notin (a_l^{\mathrm{II}}, b_l^{\mathrm{II}})$, and then (using (3) and (6)) $\bigwedge_{a \in (a_m^\triangle, b_m^\triangle)} b_l^{\mathrm{II}} = a \diamond x = h_l^{(-1)}(k_m(a) \cdot g_{m,l}(x))$.

But this means that $\bigwedge_{a \in (a_m^\triangle, b_m^\triangle)} k_m(a) \cdot g_{m,l}(x) \geq h_l(b_l^{\mathrm{II}})$.

Thus we obtain $g_{m,l}(x) = \infty$ (because of $\lim_{a \to a_m^\triangle} k_m(a) = 0$) and $g_{m,l}(x_M) = \infty$, since $g_{m,l}$ is continuous. By (9) we have the contradiction $g_k(x_M) = \infty$, that is $x_M = b_k^\perp$.

Because of (6), (9) and (10) the second statement (II) is proven. This finishes the proof. □

We remark that in case (II) of Theorem 10, $c_{m,k} \cdot k_m(a)$ is a generator of $\triangle|_{[a_m^\triangle, b_m^\triangle]^2}$.

Thus there is (in dependence of g_k and h_l) a generator $\bar{k}_m$ of $\triangle|_{[a_m^\triangle, b_m^\triangle]^2}$ fulfilling

$$\bigwedge_{a\in(a_m^\triangle, b_m^\triangle]} \quad \bigwedge_{x\in(a_k^\perp, b_k^\perp]} a \diamond x = h_l^{(-1)}(\bar{k}_m(a) \cdot g_k(x)). \tag{84}$$

Let us still mention the following Corollary, which is a generalization of Proposition 2.2 in [15].

Corollary 2. Let $\triangle, \perp$ and $\amalg$ be continuous, Archimedean t-conorms on $[A, B]^2$ with generators k, g and h, respectively.

Let the pseudo-multiplication $\diamond$ satisfy (Z) and the weak left distributivity (DL*) and the weak right distributivity (DR*).

(a) Then there are only two possible cases:

(I) $\bigwedge_{a,x\in[A,B]} a \diamond x = B$ or $\bigwedge_{a,x\in[A,B]} a \diamond x = A$.

(II) There exists a generator $\bar{g}$ of $\perp$ with

$$\bigwedge_{a,x\in[A,B]} a \diamond x = h^{(-1)}(k(a) \cdot \bar{g}(x)). \tag{85}$$

(b) If $\triangle = \perp$, then $\diamond$ is commutative.

(c) If $\triangle = \perp = \amalg$ is strict, then $\diamond$ is associative and commutative.

Proof. (a) We apply Theorem 10:

(I) There is an $\amalg$–idempotent element d with $\bigwedge_{a,x\in(A,B]} a \diamond x = d$. Since $\amalg$ is Archimedean we have $d \in \{A, B\}$.

(II) $\bigvee_{c\in(0,\infty)} \bigwedge_{a,x\in(A,B]} a \diamond x = h^{(-1)}(ck(a)g(x))$. But then $\bar{g}(x) := cg(x)$ is a generator of $\perp$. Because of (Z) and $k(A) = g(A) = h(A) = 0$ the representation is also valid for $(a = A) \vee (x = A)$.

(b) In case (I) the statement is obvious, and in case (II) we have $\bar{g} = c \cdot k$ in (5), so that commutativity is clear.

(c) In case (I) the statement is again obvious, and in case (II) we have $g = k = h$ and $k^{(-1)} = k^{-1}$. Thus the corollary is proven. □

9. PSEUDO–DIFFERENCES

We assume that the reader is familiar with the notions and results presented in Part I of this paper.

We remember the fact that it is essentially to have a "generalized difference" (which we denote from now on by pseudo-difference) for the introduction of an integral which leads in special cases to the Choquet integral (see Section 2).

In this section we introduce pseudo-differences $-_\triangle$ of pseudo-additions $\triangle$ on arbitrary intervals which generalize pseudo-differences $-_\triangle$ of Archimedean t-conorms $\triangle$ on [0, 1] (see (20)).

Definition 6. Let $\triangle$ be a pseudo-addition. Then the mapping $-_\triangle : [A,B]^2 \rightarrow [A,B]$ is called a $\triangle$-pseudo-difference (or a pseudo-difference with respect to $\triangle$) iff

$$\bigwedge_{a,b\in[A,B]} a -_\triangle b = \inf\{c \in [A,B] : b\triangle c \geq a\}. \tag{86}$$

Thus the definition is formally the same like in (20), and can be interpreted as coimplication with respect to a residuated implication $I_T(x,y) = \sup\{z : T(x,z) \leq y\}$, where T is the associated t-norm (see [7]).

Because of this correspondence, the following results have applications for fuzzy logical connectives.

Note that $a -_\triangle b \in [A,B]$ since $b\triangle B = B \geq a$.

If $\triangle = \vee$ then $a -_\triangle b = a$ if $a > b$ and $a -_\triangle b = A$ if $a \leq b$ (for all $a,b \in [A,B]$).

Moreover $-_\triangle$ is monotonic increasing in the first place and monotonic decreasing in the second place.

It is clear that some further properties of $-_\triangle$ are already known (but sometimes only in the Archimedean case of $\triangle$). Nevertheless, we give in the next Lemma a longer list of important properties because we think they are worth while to be included for handy reference in the future. The statements (a) – (o) concern pseudo-differences in general (for example (f) is the residual property and (g) the "exchange principle"), whereas the statements (p) – (s) show the connection of right boundary points of "Archimedean intervals" of a pseudo-addition with some properties of the corresponding pseudo-difference. These results become important when – like in Theorem 11 – pseudo-multiplications satisfying the weak left distributivity law (DL*) occur.

Lemma 5. Let $\triangle$ be a pseudo-addition. Then the following statements are valid:

(a) $\displaystyle\bigwedge_{a\in[A,B]} a -_\triangle A = a.$

(b) $\displaystyle\bigwedge_{a,b,c\in[A,B]} [a \geq b \Rightarrow a \geq b -_\triangle c].$

(c) $\displaystyle\bigwedge_{m\in K_\triangle} \bigwedge_{a,b\in[a_m^\triangle, b_m^\triangle]} [a > b \Rightarrow a -_\triangle b = k_m^{-1}(k_m(a) - k_m(b)) \in (a_m^\triangle, a]\,].$

(d) $\displaystyle\bigwedge_{a\in[A,B]} \Big[\neg\Big(\bigvee_{m\in K_\triangle} a,b \in (a_m^\triangle, b_m^\triangle]\Big) \wedge (a > b) \Rightarrow a -_\triangle b = a\Big].$

(e) $\displaystyle\bigwedge_{a,b\in[A,B]} [a \leq b \Leftrightarrow a -_\triangle b = A].$

(f) $\displaystyle\bigwedge_{a,b,c\in[A,B]} [c \geq a -_\triangle b \Leftrightarrow b\triangle c \geq a].$

(g) $\bigwedge_{a,b,c\in[A,B]} [(a -_\triangle b) -_\triangle c = a -_\triangle (b\triangle c) = (a -_\triangle c) -_\triangle b].$

(h) $\bigwedge_{a,b\in[A,B]} (a -_\triangle b)\triangle b = a \vee b.$

(i) $\bigwedge_{a,b,c\in[A,B]} [a \geq b \geq c \Rightarrow (a -_\triangle c) -_\triangle (b -_\triangle c) = a -_\triangle b].$

(j) $\bigwedge_{a,b,c\in[A,B]} [a \geq b \geq c \Rightarrow (a -_\triangle b)\triangle(b -_\triangle c) = a -_\triangle c].$

(k) $\bigwedge_{a,b\in[A,B]} (a\triangle b) -_\triangle b \leq a.$

(l) $\bigwedge_{a,b,c\in[A,B]} (a\triangle b) -_\triangle c \leq a\triangle(b -_\triangle c).$

(m) If $a, b \in [A, B] \wedge \neg\Big(\bigvee_{m\in K_\triangle} a, b \in (a_m^\triangle, b_m^\triangle]\Big)$ then we have:

$$(a\triangle b) -_\triangle b = a \Leftrightarrow (a > b) \vee (a = A).$$

(n) If $\bigvee_{m\in K_\triangle} a, b \in (a_m^\triangle, b_m^\triangle]$ then: $[(a\triangle b) -_\triangle b) = a \Leftrightarrow \bigwedge_{c\in[A,a)} b\triangle c < b_m^\triangle].$

(o) $\bigwedge_{a,b,c,d\in[A,B]} (a\triangle b) -_\triangle (d\triangle c) \leq (a -_\triangle d)\triangle(b -_\triangle c).$

(p) $\bigwedge_{a,b\in[A,B]} [a -_\triangle b \in D_\triangle \Rightarrow b < a -_\triangle b = a = a\triangle b \in D_\triangle].$

(q) $\bigwedge_{a,b\in[A,B]} [a\triangle b \notin D_\triangle \wedge (a\triangle b > b) \Rightarrow (a\triangle b) -_\triangle b = a].$

(r) $\bigwedge_{a,b,c\in[A,B]} [a\triangle b \notin D_\triangle \wedge (b > c) \Rightarrow (a\triangle b) -_\triangle c = a\triangle(b -_\triangle c)].$

(s) $\bigwedge_{a,b,c,d\in[A,B]} [\, a\triangle b \notin D_\triangle \wedge (a > d) \wedge (b > c) \Rightarrow (a\triangle b) -_\triangle (d\triangle c)$
$= (a -_\triangle d)\triangle(b -_\triangle c)].$

Proof.

(a) $a -_\triangle A = \inf\{c \in [A, B] : A\triangle c \geq a\} = \inf\{c \in [A, B] : c \geq a\} = a.$

(b) Using (a) we get $b -_\triangle c \leq b -_\triangle A \leq a.$

(c) Using that k_m, k_m^{-1} are continuous and strictly monotonic increasing, that $k_m(a) - k_m(b) \in (0, k_m(a)]$ and that $b \triangle a_m^{\triangle} = b < a$ and $b \triangle b_m^{\triangle} = b_m^{\triangle} \geq a$ we obtain:

$$\begin{aligned} a -_{\triangle} b &= \inf\{c \in [A,B] : b\triangle c \geq a\} = \inf\{c \in [a_m^{\triangle}, b_m^{\triangle}] : b\triangle c \geq a\} \\ &= \inf\{c \in [a_m^{\triangle}, b_m^{\triangle}] : k_m^{-1}[k_m(b_m^{\triangle}) \wedge (k_m(b) + k_m(c))] \geq a\} \\ &= \inf\{c \in [a_m^{\triangle}, b_m^{\triangle}] : k_m(b_m^{\triangle}) \wedge (k_m(b) + k_m(c)) \geq k_m(a)\} \\ &= \inf\{c \in [a_m^{\triangle}, b_m^{\triangle}] : k_m(b) + k_m(c) \geq k_m(a)\} \\ &= \inf\{k_m^{-1}(k_m(c)) \in [a_m^{\triangle}, b_m^{\triangle}] : k_m(c) \geq k_m(a) - k_m(b)\} \\ &= \inf\{k_m^{-1}(u) : (u \geq k_m(a) - k_m(b)) \wedge u \in [k_m(a_m^{\triangle}), k_m(b_m^{\triangle})]\} \\ &= k_m^{-1} \inf\{u \in [0, k_m(b_m^{\triangle})] : u \geq k_m(a) - k_m(b)\} \\ &= k_m^{-1}(k_m(a) - k_m(b)) \in (a_m^{\triangle}, a]. \end{aligned}$$

(d) To prove (d) we first show: (*) $\bigvee_{\tilde{a} \in (b,a)} \bigwedge_{c \in (\tilde{a},a)} \neg\Big(\bigvee_{m \in K_{\triangle}} c, b \in (a_m^{\triangle}, b_m^{\triangle}]\Big)$.

Case 1: If $\bigwedge_{m \in K_{\triangle}} b \notin (a_m^{\triangle}, b_m^{\triangle}]$ then we choose $\tilde{a} \in (b,a)$ to satisfy (*).

Case 2: If $\bigvee_{m \in K_{\triangle}} b \in (a_m^{\triangle}, b_m^{\triangle}]$ then (because of $a > b$ and thus $a > b_m^{\triangle}$) we choose $\tilde{a} \in (b_m^{\triangle}, a)$ to fulfil (*). Thus (*) is valid.

Because of (*) we have $\bigwedge_{c \in (\tilde{a},a)} b\triangle c = b \vee c < a$, and the monotonicity of $\triangle$ implies $\bigwedge_{c \in [A,a)} b\triangle c < a$, so that by definition 6 $a -_{\triangle} b \geq a$. On the other hand $b\triangle a = (b \vee a) = a$ so that $a -_{\triangle} b \leq a$. Thus (d) is shown.

(e) Using (c) and (d) we get: $a > b \Rightarrow a -_{\triangle} b > A$. By contraposition one implication is shown. Conversely, if $a \leq b$ then $b\triangle A = b \geq a$, and by definition 6 we obtain $a -_{\triangle} b = A$.

(f) If $b\triangle c \geq a$ then we get $c \geq \inf\{x \in [A,B] : b\triangle x \geq a\} = a -_{\triangle} b$. To prove the converse, we consider first the case $c = B$. But then obviously $b\triangle B = B \geq a$. If now $c < B$, let d be an arbitrary element in $(c, B]$. Then $d > c \geq a -_{\triangle} b = \inf\{x \in [A,B] : b\triangle x \geq a\}$ implies: $\bigvee_{x \in [A,d)} a \leq b\triangle x \leq b\triangle d$. Thus we have $b\triangle c = \lim_{d \to c+}(b\triangle d) \geq a$.

(g) Using (f) in the following second equality we arrive at the desired result:

$$\begin{aligned} (a -_{\triangle} b) -_{\triangle} c &= \inf\{x \in [A,B] : c\triangle x \geq a -_{\triangle} b\} = \inf\{x \in [A,B] : b\triangle(c\triangle x) \geq a\} \\ &= \inf\{x \in [A,B] : (b\triangle c)\triangle x \geq a\} = a -_{\triangle} (b\triangle c) \\ &= a -_{\triangle} (c\triangle b) = (a -_{\triangle} c) -_{\triangle} b. \end{aligned}$$

(h) If $a \leq b$ then $(a -_{\triangle} b)\triangle b = A\triangle b = b = a \vee b$ (here we have used (e)). Now we treat the case $a > b$. We apply (f) two times to obtain: $\Big(\bigwedge_{c \geq a -_{\triangle} b} b\triangle c \geq a\Big) \wedge \Big(\bigwedge_{c < a -_{\triangle} b} b\triangle c < a\Big)$. Since $\triangle$ is continuous in each place we get $b\triangle(a -_{\triangle} b) \geq a \geq b\triangle(a -_{\triangle} b)$ (note that $a -_{\triangle} b = A$ implies $a \geq b\triangle(a -_{\triangle} b)$). Thus we arrive at $(a -_{\triangle} b)\triangle b = a = a \vee b$.

(i) In the case $a = b$ we have (by (e)): $(a -_\triangle c) -_\triangle (b -_\triangle c) = A = a -_\triangle b$, whereas the case $b = c$ yields (using (e) and (a)): $(a -_\triangle c) -_\triangle (b -_\triangle c) = (a -_\triangle c) -_\triangle A = a -_\triangle c = a -_\triangle b$.
Now we treat the main case $a > b > c$ where we distinguish three subcases:

(I) $\bigvee_{m \in K_\triangle} a, b, c \in [a_m^\triangle, b_m^\triangle]$: Since k_m and k_m^{-1} are strictly monotonic increasing we get, using (c) $a -_\triangle c = k_m^{-1}(k_m(a) - k_m(c)) > k_m^{-1}(k_m(b) - k_m(c)) = b -_\triangle c$ and then: $(a -_\triangle c) -_\triangle (b -_\triangle c) = k_m^{-1}(k_m k_m^{-1}[k_m(a) - k_m(c)] - k_m k_m^{-1}[k_m(b) - k_m(c)]) = k_m^{-1}(k_m(a) - k_m(b)) = a -_\triangle b$.

(II) $\left(\bigvee_{m \in K_\triangle} b, c \in [a_m^\triangle, b_m^\triangle]\right) \wedge a \notin [a_m^\triangle, b_m^\triangle]$: (c) and (d) imply $b -_\triangle c \in (a_m^\triangle, b_m^\triangle]$ so that $(a -_\triangle c) -_\triangle (b -_\triangle c) = a -_\triangle (b -_\triangle c) = a = a -_\triangle b$.

(III) $\neg\left(\bigvee_{m \in K_\triangle} b, c \in [a_m^\triangle, b_m^\triangle]\right)$: Here $a > b > c$ yields $\neg\left(\bigvee_{m \in K_\triangle} a, c \in [a_m^\triangle, b_m^\triangle]\right)$ and thus (by (d)) $(a -_\triangle c) -_\triangle (b -_\triangle c) = a -_\triangle b$.

(j) In the case $a = b$ we get (using (e)) $(a -_\triangle b)\triangle(b -_\triangle c) = A\triangle(b -_\triangle c) = b -_\triangle c = a -_\triangle c$. The case $b = c$ leads again because of (e) to $(a -_\triangle b)\triangle(b -_\triangle c) = (a -_\triangle b)\triangle A = a -_\triangle c$. In the main case $a > b > c$ we treat the same three subcases (I) – (III) like in (i):

In case (I), (c) yields $(a -_\triangle b)\triangle(b -_\triangle c) = k_m^{-1}(k_m(a) - k_m(b))\triangle k_m^{-1}(k_m(b) - k_m(c)) = k_m^{-1}(k_m(a) - k_m(b) + k_m(b) - k_m(c)) = k_m^{-1}(k_m(a) - k_m(c)) = a -_\triangle c$.

In case (II), (c) and (d) give $b -_\triangle c \in (a_m^\triangle, b_m^\triangle]$ and thus $(a -_\triangle b)\triangle(b -_\triangle c) = a\triangle(b -_\triangle c) = a \vee (b -_\triangle c) = a = a -_\triangle c$.

In case (III) we obtain (because of (d) and (h)): $(a -_\triangle b)\triangle(b -_\triangle c) = (a -_\triangle b)\triangle b = a \vee b = a = a -_\triangle c$.

(k) $(a\triangle b) -_\triangle b = (b\triangle a) -_\triangle b = \inf\{c \in [A, B] : b\triangle c \geq b\triangle a\} \leq a$.

(l) Using (h) and (k) we obtain $(a\triangle b) -_\triangle c \leq (a\triangle[b \vee c]) -_\triangle c = (a\triangle[(b -_\triangle c)\triangle c]) -_\triangle c = ([a\triangle(b -_\triangle c)]\triangle c) -_\triangle c \leq a\triangle(b -_\triangle c)$.

(m) If $a > b$ then we get by (d): $(a\triangle b) -_\triangle b = (a \vee b) -_\triangle b = a -_\triangle b = a$.

If $a \leq b$ then (e) gives $(a\triangle b) -_\triangle b = (a \vee b) -_\triangle b = b -_\triangle b = A$.

(n) We consider the cases (α) $\bigwedge_{c \in [A,a)} b\triangle c < b_m^\triangle$ and (β) $\bigvee_{c \in [A,a)} b\triangle c \geq b_m^\triangle$.

In case (α) we take an arbitrary $c \in [a_m^\triangle, a)$ and get $b_m^\triangle > b\triangle c = k_m^{(-1)}(k_m(b) + k_m(c))$ so that $k_m(b_m^\triangle) > k_m(b) + k_m(c)$. Thus we obtain $k_m^{-1}(k_m(b) + k_m(c)) < k_m^{-1}([(k_m(b) + k_m(a)] \wedge k_m(b_m^\triangle)) = k_m^{(-1)}(k_m(b) + k_m(a)) = a\triangle b$, so that $b\triangle c = k_m^{-1}(k_m(b) + k_m(c)) < a\triangle b$. The monotonicity of $\triangle$ gives $\bigwedge_{c \in [A,a)} b\triangle c < a\triangle b$, or $(a\triangle b) -_\triangle b \geq a$. Together with (k) we obtain $(a\triangle b) -_\triangle b = a$.

In case (β) we get $b_m^\triangle \geq b\triangle a \geq b\triangle c \geq b_m^\triangle$ which means $b\triangle c = b\triangle a = a\triangle b$. Now definition (6) yields $(a\triangle b) -_\triangle b \leq c < a$. Thus (n) is proven.

(o) Using (l) twice and(g) we arrive at:

$$(a\triangle b) -_\triangle (d\triangle c) = [(a\triangle b) -_\triangle c] -_\triangle d \leq [a\triangle(b -_\triangle c)] -_\triangle d$$
$$= [(b -_\triangle c)\triangle a] -_\triangle d \leq (b -_\triangle c)\triangle(a -_\triangle d) = (a -_\triangle d)\triangle(b -_\triangle c).$$

(p) Since $a -_\triangle b \in D_\triangle$ we have $a -_\triangle b > A$, so that (l) implies $a > b$. To prove $a = a -_\triangle b$ we consider two cases. If $\neg\Big(\bigvee_{m\in K_\triangle} a, b \in (a_m^\triangle, b_m^\triangle]\Big)$ then we obtain together with $a > b$ from (d) just $a = a -_\triangle b$. In the other case $\bigvee_{m\in K_\triangle} a, b \in (a_m^\triangle, b_m^\triangle]$, (c) gives $a -_\triangle b \in (a_m^\triangle, a]$, so that $a -_\triangle b \in D_\triangle$ yields $a = b_m^\triangle = a -_\triangle b$. Finally, we get from $a = a -_\triangle b \in D_\triangle$ and $a > b$ the desired result $a\triangle b = a \vee b = a$.

(q) We consider two cases:

$$(\gamma): \neg\Big(\bigvee_{m\in K_\triangle} a, b \in (a_m^\triangle, b_m^\triangle]\Big) \quad \text{and} \quad (\delta): \bigvee_{m\in K_\triangle} a, b \in (a_m^\triangle, b_m^\triangle].$$

In case (γ) we have $a \vee b = a\triangle b > b$ so that $a > b$. Thus (m) gives $(a\triangle b) -_\triangle b = a$.

In case (δ) we know that $a\triangle b \in [a_m^\triangle, b_m^\triangle]$. But $a\triangle b \notin D_\triangle$ yields $a\triangle b < b_m^\triangle$, which implies $\bigwedge_{c\in[A,a)} b\triangle c \leq b\triangle a = a\triangle b < b_m^\triangle$. Now (n) gives $(a\triangle b) -_\triangle b = a$.

(r) We consider (similarly to (q)) two subcases (γ) and (δ):

$$(\gamma): \neg\Big(\bigvee_{m\in K_\triangle} b, c \in (a_m^\triangle, b_m^\triangle]\Big) \quad \text{and} \quad (\delta): \bigvee_{m\in K_\triangle} b, c \in (a_m^\triangle, b_m^\triangle].$$

If (γ) is supposed we get $\neg\Big(\bigvee_{m\in K_\triangle} a\triangle b, c \in (a_m^\triangle, b_m^\triangle]\Big)$, for, if this is not the case then we obtain from $c < b = A\triangle b \leq a\triangle b$ the contradiction $b \in (a_m^\triangle, b_m^\triangle]$. Thus twice application of (d) results in $(a\triangle b) -_\triangle c = a\triangle b = a\triangle(b -_\triangle c)$.

In the case (δ) we have $b -_\triangle c \in (a_m^\triangle, b_m^\triangle]$ (see (c)) and consider three subcases:

(r1): $a \leq a_m^\triangle$ (r2):, $a > b_m^\triangle$, (r3): $a \in (a_m^\triangle, b_m^\triangle]$.

Case (r1): $(a\triangle b) -_\triangle c = (a \vee b) -_\triangle c = b -_\triangle c = a\triangle(b -_\triangle c)$.

Case (r2): $(a\triangle b) -_\triangle c = a -_\triangle c = a = a\triangle(b -_\triangle c)$ (here we have used (d)).

Case (r3): The assumption $a\triangle b \notin D_\triangle$ yields $b_m^\triangle > a\triangle b = k_m^{(-1)}(k_m(a) + k_m(b))$ so that $a\triangle b = k_m^{-1}(k_m(a) + k_m(b))$. Moreover $b > c$ implies $a\triangle b > c$, so that (c) leads to $(a\triangle b) -_\triangle c = k_m^{-1}(k_m(a\triangle b) - k_m(c)) = k_m^{-1}((k_m(a) + k_m(b) - k_m(c)) = k_m^{-1}((k_m(a) + k_m(b -_\triangle c)) = a\triangle(b -_\triangle c))$.

(s) First (r) implies $(b -_\triangle c)\triangle a = a\triangle(b -_\triangle c) = (a\triangle b) -_\triangle c$. Now we show $(b -_\triangle c)\triangle a \notin D_\triangle$. If this is not the case then we get from the above equality that $a\triangle(b -_\triangle c) \in D_\triangle$, so that (p) implies the contradiction $a\triangle b \in D_\triangle$.

These two partial results together with (g) and (r) lead to $(a\triangle b) -_\triangle (d\triangle c) = [(a\triangle b) -_\triangle c] -_\triangle d = [(b -_\triangle c)\triangle a] -_\triangle d = (b -_\triangle c)\triangle(a -_\triangle d) = (a -_\triangle d)\triangle(b -_\triangle c)$.

Thus Lemma 5 is proven. □

Example 4. (1) We start with the pseudo-difference with respect to the classical addition on intervals (which extends the example in Example 2):

Let $-\infty < A < B \leq \infty$ and let $\triangle = \hat{+}$, that is

$$a\hat{+}b := A + [(a - A) + (b - A)] \wedge (B - A), \ a, b \in [A, B].$$

Then we have: $a -_\triangle b = A + (0 \vee (a - b)), \quad a, b \in [A, B]$

(If $a = b = \infty$ then we define $a -_\triangle b := 0$).

Indeed, $h(x) := x - A, \ x \in [A, B]$ is a generator of $\hat{+}$, since $h^{-1}(y) = A + y, \ y \in [0, B - A]$ so that $a\hat{+}b = h^{-1}(h(a) + h(b)), \ a, b \in [A, B]$.

Now, if $a > b$ then Lemma 5 (c) implies $a -_\triangle b = h^{-1}(h(a) - h(b)) = h^{-1}(a - b) = A + (a - b)$. If $a \leq b$ then Lemma 5 (e) leads to $a -_\triangle b = A = A + (0 \vee (a - b))$.

(2) If we choose in (1) $A = 0, \ B = 4, \ \triangle = \hat{+}$, then this example shows that we cannot omit the assumption

$a\triangle b \notin D_\triangle$ in Lemma 5 (q): $(2\triangle 3) -_\triangle 3 = 4 -_\triangle 3 = 1 < 2 = 2\triangle 0 = 2\triangle(3 -_\triangle 3)$,

$a\triangle b \notin D_\triangle$ in Lemma 5 (r): $(2\triangle 4) -_\triangle 3 = 4 -_\triangle 3 = 1 < 3 = 2\triangle 1 = 2\triangle(4 -_\triangle 3)$,

$b > c$ in Lemma 5 (r): $(1\triangle 1) -_\triangle 2 = 2 -_\triangle 2 = 0 < 1 = 1\triangle 0 = 1\triangle(1 -_\triangle 2)$,

$a\triangle b \notin D_\triangle$ in Lemma 5 (s): $(4\triangle 4) -_\triangle (2\triangle 2) = 4 -_\triangle 4 = 0 < 4 = 2\triangle 2 = (4 -_\triangle 2)\triangle(4 -_\triangle 2)$,

$b > c$ (and $a > d$, because of the commutativity of $\triangle$) in Lemma 5 (s): $(2\triangle 1) -_\triangle (1\triangle 2) = 3 -_\triangle 3 = 0 < 1 = 1\triangle 0 = (2 -_\triangle 1)\triangle(1 -_\triangle 2)$.

For our purposes it is important to know whether the weak distributivity law is compatible with a pseudo-difference. The following result shows, that the answer is positive.

Theorem 11. Let $\triangle$ and $\amalg$ be pseudo-additions, and let $\diamond$ be a pseudo-multiplication satisfying (Z) and (DL^*). Then the following holds:

(a) $\displaystyle\bigwedge_{a,b,c\in[A,B]} \bigwedge_{x\in[A,B]} [a \geq b \geq c \Rightarrow ([a -_\triangle b] \diamond x) \amalg ([b -_\triangle c] \diamond x) = [a -_\triangle c] \diamond x]$.

(b) $\displaystyle\bigwedge_{a,b\in[A,B]} \bigwedge_{x\in[A,B]} [a \geq b \Rightarrow ([a -_\triangle b] \diamond x) \amalg (b \diamond x) = a \diamond x]$.

(c) $\displaystyle\bigwedge_{a,b\in[A,B]} \bigwedge_{x\in[A,B]} (a\triangle b) \diamond x = ([(a\triangle b) -_\triangle b] \diamond x) \amalg (b \diamond x) \leq (a \diamond x) \amalg (b \diamond x)$.

(d) $(a_0 = A \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq B) \wedge x \in [A, B] \Rightarrow \amalg_{i=1}^n[(a_i -_\triangle a_{i-1}) \diamond x] = a_n \diamond x$.

Proof. (a) Because of (Z) we assume w.l.o.g.that $x \in (A, B]$. We consider 4 cases:

$$\text{(I)}: \ a = b, \qquad \text{(II)}: \ b = c, \qquad \text{(III)}: \ (a > b > c) \wedge a \notin D_\triangle,$$
$$\text{(IV)}: \ (a > b > c) \wedge \left(\bigvee_{m\in K_\triangle} a = b_m^\triangle\right).$$

Case (I): We use Lemma 5 (e) and obtain: $([a -_\triangle b] \diamond x) \amalg ([b -_\triangle c] \diamond x) = (A \diamond x) \amalg ([b -_\triangle c] \diamond x) = A \amalg ([b -_\triangle c] \diamond x) = ([b -_\triangle c] \diamond x) = ([a -_\triangle c] \diamond x)$.

Case (II) can be proven in the same manner like Case (I).

Case (III): We use (e), (j) and (p) of Lemma 5 (in that order) to get $a -_\triangle b, b -_\triangle c \in (A, B]$, $(a -_\triangle b) \triangle (b -_\triangle c) = a -_\triangle c \notin D_\triangle$, and (DL*) implies $([a -_\triangle b] \diamond x) \amalg ([b -_\triangle c] \diamond x) = ([a -_\triangle c] \diamond x)$.

In Case (IV) there exists a sequence $(a_n) \subset (b, b_m^\triangle)$ satisfying $a_n \uparrow b_m^\triangle$. In the following chain of equations we use case (III) and

$$(\sup a_n) -_\triangle b = \sup(a_n -_\triangle b) \tag{87}$$

to arrive at

$$\begin{aligned} & ([a -_\triangle b] \diamond x) \amalg ([b -_\triangle c] \diamond x) = ([(\sup a_n) -_\triangle b] \diamond x) \amalg ([b -_\triangle c] \diamond x) \\ = \;& ([\sup(a_n -_\triangle b)] \diamond x) \amalg ([b -_\triangle c] \diamond x) = (\sup[(a_n -_\triangle b) \diamond x]) \amalg ([b -_\triangle c] \diamond x) \\ = \;& \sup([(a_n -_\triangle b) \diamond x] \amalg ([b -_\triangle c] \diamond x)) = \sup[(a_n -_\triangle c) \diamond x] = [\sup(a_n -_\triangle c)] \diamond x \\ = \;& [(\sup a_n) -_\triangle c] \diamond x = (a -_\triangle c) \diamond x. \end{aligned}$$

We include still the proof for (87): $a_n -_\triangle b \leq (\sup a_n) -_\triangle b$ so that $\sup(a_n -_\triangle b) \leq (\sup a_n) -_\triangle b$. If we now put $s = \sup(a_n -_\triangle b)$ then we get $s \geq a_n -_\triangle b$ and by Lemma 5 (f) $b \triangle s \geq a_n$ for all $n \in \mathbb{N}$. Thus we have $b \triangle s \geq (\sup a_n)$ and finally (again by Lemma 5 (f)) $s \geq (\sup a_n) -_\triangle b$.

(b) We use statement (a) with $c := A$ and apply Lemma 5 (a).

(c) To prove the first equality in (c) we use simply (b). The inequality follows from Lemma 5 (k).

(d) follows by induction on $n \in \mathbb{N}$: If $n = 1$ then (d) follows from Lemma 5 (a). Suppose (d) is true for $n \in \mathbb{N}$. Then we get $\amalg_{i=1}^{n+1}[(a_i -_\triangle a_{i-1}) \diamond x] = [(a_{n+1} -_\triangle a_n) \diamond x] \amalg (\amalg_{i=1}^{n}[(a_i -_\triangle a_{i-1}) \diamond x]) = [(a_{n+1} -_\triangle a_n) \diamond x] \amalg [a_n \diamond x] = a_{n+1} \diamond x$ (in the last equality we have used (b)). Therefore Theorem 11 is proven. □

10. SUMMARY

We have shown that the concept of weak distributivity has many applications and leads by Theorem 5 to one more unexpected result (comparison of two pseudo-additions). The main results in Part II are probably Theorem 6 (representation theorem for pseudo-multiplications) and Theorem 7 (representation theorem for pseudo-additions and pseudo-multiplication under weak assumptions).

In Section 9 it is shown that the introduced pseudo-difference is compatible with the weak distributivity.

(Received October 27, 2004.)

REFERENCES

[1] J. Aczél: Lectures on Functional Equations and Their Applications. Academic Press, New York - London 1966.

[2] P. Benvenuti, R. Mesiar, and D. Vivona: Monotone set-functions-based integrals. In: Handbook of Measure Theory, Vol. II (E. Pap, ed.), Elsevier, Amsterdam 2002, pp. 1329–1379.

[3] P. Benvenuti, D. Vivona, and M. Divari: The Cauchy equation on I-semigroups. Aequationes Math. *63* (2002), 220–230.

[4] C. Bertoluzza and D. Cariolaro: On the measure of a fuzzy set based on continuous t-conorms. Fuzzy Sets and Systems *88* (1997), 355–362.

[5] L. M. deCampos and M. J. Bolaños: Characterization and comparison of Sugeno and Choquet integral. Fuzzy Sets and Systems *52* (1992), 61–67.

[6] D. Denneberg: Non-additive Measure and Integral. Kluwer, Dordrecht 1994.

[7] J. Fodor and M. Roubens: Fuzzy Preference Modelling and Multicriteria Decision Support. Kluwer Academic Publishers, Dordrecht 1994.

[8] M. Grabisch, T. Murofushi, and M. Sugeno (eds.): Fuzzy Measures and Integrals. Theory and Applications. Physica–Verlag, Heidelberg 2000.

[9] M. Grabisch, H. T. Nguyen, and E. A. Walker: Fundamentals of Uncertainty Calculi with Applications to Fuzzy Inference. Kluwer Academic Publishers, Dordrecht 1995.

[10] R. L. Kruse and J. J. Deeley: Joint continuity of monotonic functions. Amer. Math. Soc. *76* (1969), 74–76.

[11] E. P. Klement, R. Mesiar, and E. Pap: Triangular Norms. (Trends in Logic, Volume 8.) Kluwer Academic Publishers, Dordrecht 2000.

[12] C. H. Ling: Representations of associative functions. Publ. Math. Debrecen *12* (1965), 189–212.

[13] R. Mesiar: Choquet-like integrals. J. Math. Anal. Appl. *194* (1995), 477–488.

[14] P. S. Mostert and A. L. Shields: On the structure of semigroups on a compact manifold with boundary. Ann. of Math. *65* (1957), 117–143.

[15] T. Murofushi and M. Sugeno: Fuzzy t-conorm integral with respect to fuzzy measures: Generalization of Sugeno integral and Choquet integral. Fuzzy Sets and Systems *42* (1991), 57–71.

[16] E. Pap: Null-Additive Set Functions. Kluwer Academic Publishers. Dordrecht 1995.

[17] W. Sander: Associative aggregation operators. In: Aggregation Operators. New Trends and Applications (T. Calvo, G. Mayor, and R. Mesiar, eds.), Physica–Verlag, Heidelberg - New York 2002, pp. 124–158.

[18] N. Shilkret: Maxitive measures and integration. Indag. Math. *33* (1971), 109–116.

[19] J. Siedekum: Multiplikation und t-Conorm Integral. Ph.D. Thesis. Braunschweig 2002.

[20] M. Sugeno and T. Murofushi: Pseudo-additive measures and integrals. J. Math. Anal. Appl. *122* (1987), 197–222.

[21] Z. Wang and G. J. Klir: Fuzzy Measure Theory. Plenum Press, New York 1992.

[22] S. Weber: $\perp$-decomposable measures and integrals for Archimedean t-conorms $\perp$. J. Math. Anal. Appl. *101* (1984), 114–138.

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